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Asymptotic Consistency of Least Squares Estimators in Fuzzy Regression Model†

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Abstract

This paper deals with the properties of the fuzzy least squares estimators for fuzzy linear regression model. Especially fuzzy triangular input-output model including error term is proposed. The error term is considered as a fuzzy random variable. The asymptotic unbiasedness and the consistency of the estimators are proved using a suitable metric.

Keywords: Fuzzy least squares estimators; asymptotic unbiasedness; asymptotic consistency.

1. Introduction

The least squares method is the most widely used statistical technique to find the unknown parameters of regression model. But there are many situations where observations cannot be described accurately. To record these data, we need some approach to handle the uncertainty. Zadeh (1965) first introduced the concept of fuzzy sets to explain such uncertainty or vagueness. Tanaka *et al.* (1982) introduced fuzzy concept to regression analysis. Diamond (1988) introduced fuzzy least squares estimations for triangular fuzzy numbers. He considered two types of fuzzy linear regression models: the fuzzy input-output regression model and the crisp input, fuzzy output model. After that many authors have addressed and attempted to resolve the fuzzy least squares problems. But many studies have emphasized the fuzziness of the response alone, so they deal with crisp input, fuzzy output model. Some authors have discussed the situation in which both the response and the explanatory variables are fuzzy (Celminš, 1987; Sakawa and Yano, 1992; Yang and Lin, 2002; Yang and Liu, 2003). A common characteristic of these studies is that the regression coefficients were treated as fuzzy numbers. But this approach has a weakness because the spread of the estimated responses widens as the magnitude of the explanatory variables increases, even though the spreads of the observed responses remain roughly constant, or even decrease. So some authors have been studying the fuzzy input-output model with crisp parameters, not fuzzy parameters, of the model (Diamond, 1988; Diamond and Körner, 1997; Kao and Chyu, 1989;

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Kao, and Chyu, 2003; Choi *et al.*, 2000; Ming *et al.*, 1997). Two situations also arise, which have been studied in such models. The first situation involves the fuzzy regression model without error structure. The other is that an error structure exists in the models. When the data have an error structure which is assumed in the model, Diamond (1989) and Körner and Näther (1997, 1998) introduced fuzzy BLUE. Kim *et al.* (2008) proved some asymptotic properties using suitable assumptions for error terms. Chang and Lee (1994), Kao and Chyu (1989, 2003) also introduced error term in the fuzzy regression model. Here is the fuzzy linear regression model with error term which is considered as a fuzzy random variable.

$$Y_i = \beta_0 + \beta_1 X_{i1} + \cdots + \beta_p X_{ip} + \Phi_i, \qquad i = 1, \ldots, n, \tag{1.1}$$

where X_{ip}, $Y_i (i = 0, 1, \ldots, n)$ are triangular input-output fuzzy numbers respectively, $\beta_i (i = 0, 1, \ldots, n)$ are crisp parameters and error terms $\Phi_i (i = 1, \ldots, n)$ are assumed to be fuzzy random variables. We restrict the model into following simple case.

$$Y_i = \beta_0 + \beta_1 X_i + \Phi_i, \qquad i = 1, \ldots, n. \tag{1.2}$$

2. Mathematical Preliminaries

In this section, we introduce some definitions and theorems that will be needed in our study.

Definition 2.1 Let U be a set. A fuzzy subset X in U is a set of ordered pairs:

$$X = \{ (x, \mu_X(x)) \,|\, x \in U \},$$

where μ_X is a function from U to the closed interval $[0, 1]$. Here, μ_X is said to be the grade function. $\mu_X(x)$ is called the grade of x in X.

A more general and even more useful notion is that of an α-level set.

Definition 2.2 The (crisp) set of elements that belong to the fuzzy subset X in U at least to the grade α is called the α-level set of X:

$$X^\alpha = \{ x \in U \,|\, \mu_X(x) \geq \alpha \}.$$

Definition 2.3 A fuzzy number X, denoted by $X = \langle m_X, l_X, r_X \rangle_{LR}$, is called LR-fuzzy number, if it has the following grade function:

$$\mu_X(x) = \begin{cases} L\left(\dfrac{m_X - x}{l_X}\right), & \text{if } m_X - l_X \leq x < m_X, \\ R\left(\dfrac{x - m_X}{r_X}\right), & \text{if } m_X \leq x < m_X + r_X, \\ 0, & \text{otherwise,} \end{cases} \qquad \text{for } x \in \mathbb{R}, \tag{2.1}$$

where $m_X \in \mathbb{R}$, l_X, $r_X \geq 0$ and L, R are fixed left-continuous and non-increasing functions $L, R : [0, 1] \to [0, 1]$ with $R(0) = L(0) = 1$ and $R(1) = L(1) = 0$. Here, L and R are called left and right shape functions of X, respectively.

The point m_X is said to be mode and l_X, r_X are left, right spread of X, respectively. We denote the space of LR-fuzzy numbers as $\mathcal{F}_{LR}(\mathbb{R})$.

Definition 2.4 An LR-fuzzy number X is a triangular fuzzy number if L, R are of the form

$$T(x) = \begin{cases} 1 - |x|, & \text{if } 0 \le x \le 1, \\ 0, & \text{otherwise,} \end{cases} \quad \text{for } x \in \mathbb{R} \tag{2.2}$$

and we denote $X = \langle m_X,\, l_X,\, r_X \rangle_T$, or briefly $X = \langle m_X,\, l_X,\, r_X \rangle$ is said to be triangular. We denote $\mathcal{T}(\mathbb{R})$ is the space of triangular fuzzy numbers.

We introduce the extension principle by Zadeh (1965).

Definition 2.5 Let U be a Cartesian product $U_1 \times \cdots \times U_r$ of universes $U_1, \ldots, U_r$ and let $X_1, \ldots, X_r$ be r fuzzy subsets in $U_1, \ldots, U_r$, respectively. f is a function from U to a unverse V, $y = f(x_1, \ldots, x_r)$. Then the extension principle allows us to define a fuzzy subset Y in V by

$$Y = \{\, (y,\, \mu_Y(y)) \mid y = f(x_1, \ldots, x_r),\, (x_1, \ldots, x_r) \in U \,\}$$

and

$$\mu_Y(y) = \begin{cases} \sup\limits_{(x_1,\ldots,x_r) \in f^{-1}(y)} \min\{\, \mu_{X_1}(x_1), \ldots, \mu_{X_r}(x_r) \,\}, & \text{if } f^{-1}(y) \neq \emptyset, \\ 0, & \text{otherwise,} \end{cases}$$

where f^{-1} is the inverse of f.

Using the extension principle, we define operations on the fuzzy number space.

Definition 2.6 Let X and Y be fuzzy numbers with grade functions μ_X and μ_Y, respectively. Addition $X + Y$ of X and Y is defined and its grade function is defined by

$$\mu_{X+Y}(z) = \sup_{x+y=z} \min\{\mu_X(x),\, \mu_Y(y)\}. \tag{2.3}$$

According to the preceding definition for addition of fuzzy numbers, we can show that $\mu_{X+Y}(z) = \sup_{t \in \mathbb{R}} \min\{\mu_X(t),\, \mu_Y(z-t)\}$. We can regard $2X$ as $X + X$. So, $\mu_{2X}(z) = \sup_{t \in \mathbb{R}} \min\{\mu_X(t),\, \mu_X(z-t)\}$. We can easily show that $\mu_{2X}(z) = \mu_X(z/2)$. The positive integer scalar product of X is led by $\mu_{cX}(z) = \mu_X(z/c)$. So, the following definition is well defined.

Definition 2.7 Let X be a fuzzy number with grade function μ_X and c be a non-zero scalar. Scalar multiplication cX of c and X is defined and its grade function is defined by

$$\mu_{cX}(z) = \mu_X\left(\frac{z}{c}\right). \tag{2.4}$$

For LR-fuzzy numbers, the LR-shape is preserved by addition and scalar multiplication operations. That is, let $X_i = \langle m_i,\, l_i,\, r_i \rangle_{LR}$, $i = 1, 2, \ldots, n$, be LR-fuzzy numbers and c be a scalar. Then

$$X_1 + X_2 + \cdots + X_n = \left\langle \sum m_i,\, \sum l_i,\, \sum r_i \right\rangle_{LR}, \tag{2.5}$$

$$cX_i = \begin{cases} \langle cm_i,\ cl_i,\ cr_i\rangle_{LR}, & \text{if } c > 0, \\ \langle cm_i,\ -cr_i,\ -cl_i\rangle_{RL}, & \text{if } c < 0, \\ \langle 0,\ 0,\ 0\rangle_{LR} = I_{\{0\}}, & \text{if } c = 0. \end{cases} \tag{2.6}$$

Here I_A is the indicator function of a set A. So, for $c_1 \geq 0$ and $c_2 \geq 0$, we get

$$c_1\langle m_1,\ l_1,\ r_1\rangle_{LR} + c_2\langle m_2,\ l_2,\ r_2\rangle_{LR} = \langle c_1m_1 + c_2m_2,\ c_1l_1 + c_2l_2,\ c_1r_1 + c_2r_2\rangle_{LR}. \tag{2.7}$$

Definition 2.8 Let x be a point in $\mathbb{R}^n$ and A be a nonempty subset of $\mathbb{R}^n$. We define distance $d(x, A)$ from x to A by

$$d(x, A) = \inf\{\|x - a\| : a \in A\}.$$

Definition 2.9 Let A and B be subsets of $\mathbb{R}^n$. We define *Hausdorff separation* of B from A by

$$d^*_H(B, A) = \sup\{d(b, A) : b \in B\}.$$

Let A and B be subsets of $\mathbb{R}^n$. *Hausdorff Metric* d_H is defined by

$$d_H = \max\{d^*_H(A, B),\ d^*_H(B, A)\},$$

where nonempty A and B are in $\mathbb{R}^n$. Or equivalently,

$$d_H = \max\left(\sup_{a\in A}\inf_{b\in B}\|a - b\|,\ \sup_{b\in B}\inf_{a\in A}\|a - b\|\right).$$

Let $\mathcal{F}_0(\mathbb{R}^n)$ denote the set of all fuzzy subsets in $\mathbb{R}^n$.

Definition 2.10 We define d_p on $\mathcal{F}_0(\mathbb{R}^n)$ by

$$d_p = \left[\int_0^1 d_H(A^\alpha,\ B^\alpha)^p d\alpha\right]^{\frac{1}{p}},$$

for all $A, B \in \mathcal{F}_0(\mathbb{R}^n)$, $1 \leq p < \infty$.

Diamond and Kloeden (1994) introduced the concept of support function. Let $X \in \mathbb{R}^n$. The support function $s : S^{n-1} \times I \to \mathbb{R}$ is defined by

$$s(\lambda, A) = \sup\left\{< \lambda,\ x >:\ \lambda \in S^{n-1},\ x \in A\right\},$$

where S^{n-1} is the $(n-1)$-dimensional unit sphere of $\mathbb{R}^n$ and $< \cdot, \cdot >$ is the inner product of the Euclidean space $\mathbb{R}^n$. Then an element $A \in \mathbb{R}^n$ is uniquely characterized by its support function.

Support functions may be used to define an following L_2-metric.

Definition 2.11 For $A, B \in \mathbb{R}^n$, we define an extended L_2-metric on $\mathbb{R}^n$ as follows:

$$D_2^2(A, B) = \int_{S^{n-1}} \left[s(\lambda, A) - s_Y(\lambda, B)\right]^2 d\lambda. \tag{2.8}$$

For compact intervals, the metric D_2 takes an especially simple form because the support function is defined at just two points, -1 and 1.

Diamond (1988), Diamond and Kloeden (1994) introduce following metric in $\mathcal{T}(\mathbb{R})$.

Definition 2.12 For $A, B \in \mathcal{T}(\mathbb{R})$, define

$$d^2(X,Y) = D_2^2\,(\text{supp}X,\ \text{supp}Y) + [m(X) - m(Y)]^2\,, \tag{2.9}$$

where supp$X$ denotes the compact interval of support of X and $m(X)$ its mode.

If $X = \langle x,\, \xi^l,\, \xi^r\rangle$, $Y = \langle y,\, \eta^l,\, \eta^r\rangle$, then

$$d^2(X,\,Y) = \left[x - y - (\xi^l - \eta^l)\right]^2 + \left[x - y + (\xi^r - \eta^r)\right]^2 + (x-y)^2. \tag{2.10}$$

In this paper, we use the metric $d(\cdot,\cdot)$ in Definition 2.12.

One of the most general definition of a fuzzy random variable is made by Puri and Ralescu (1986). We digest expected value and variance of a fuzzy random variable which are defined by Puri and Ralescu (1986) and Körner (1997), respectively.

Let $(\Omega,\ \mathcal{A},\ P)$ be a probability space and let U be a random variable defined on the space. Consider a partition $\{W_i : i \in J\}$ of $\mathbb{R}$, where W_i is an interval of the real line and J is a finite or countable index set. Then for each $\omega \in \Omega$, there is only one $i \in J$ such that $U(\omega) \in W_i$.

We define the perception of a fuzzy random variable X as the mapping $X : \Omega \to \mathcal{F}_0(\mathbb{R})$ given by $\omega \longmapsto X_\omega$. Here $X_\omega = I_{W_i}$ if and only if $U(\omega) \in W_i$, where I_{W_i} is the indicate function. The mapping $X : \Omega \to \mathcal{F}_0(\mathbb{R})$ characterizes a special type of fuzzy random variables. The random variable U is called an original of the fuzzy random variable X. Corresponding to a given fuzzy random variable, there may exist a lot of originals.

Now, we generalize the concept of the fuzzy random variable. The definition was given by Puri and Ralescu (1986).

Definition 2.13 Let $(\Omega,\ \mathcal{A},\ P)$ be a probability space. A *fuzzy random variable* (*f.r.v.*) is a function $X : \Omega \to \mathcal{F}_0(\mathbb{R}^n)$ such that

$$\{(\omega,\ x) : x \in X^\alpha(\omega)\} \in \mathcal{A} \times \mathcal{B}, \tag{2.11}$$

for every $\alpha \in [0,\ 1]$, where $\mathcal{B}$ denotes the Borel subsets of $\mathbb{R}^n$ and $X^\alpha : \Omega \to \mathcal{P}(\mathbb{R}^n)$ is defined by

$$X^\alpha(\omega) = \{x \in \mathbb{R}^n : \mu_{X(\omega)}(x) \geq \alpha\}. \tag{2.12}$$

A set valued function $F : \Omega \to \mathcal{P}(\mathbb{R}^n)$ is called *integrably bounded* if there exists a function $h : \Omega \to \mathbb{R}$, $h \in L^1(P)$ such that $||x|| \leq h(\omega)$ for all x, ω with $x \in F(\omega)$. Here $L^1(P)$ denotes a family of all functions $h : \Omega \to \mathbb{R}$ which are integrable with respect to the probability measure P. A fuzzy random variable X is called *integrably bounded* if X^α is integrably bounded for all $\alpha \in [0,1]$, that is, if for any $\alpha \in [0,1]$, there exists $h_\alpha \in L^1(P)$ such that $||x|| \leq h_\alpha(\omega)$ for each x, ω with $x \in X^\alpha(\omega)$.

For a fuzzy random variable $X : \Omega \to \mathcal{F}_0(\mathbb{R})$, we define mappings x_α^- and x_α^+ as $x_\alpha^-(\omega) = \inf X^\alpha(\omega)$ and $x_\alpha^+(\omega) = \sup X^\alpha(\omega)$ for all $\alpha \in (0,1)$ and all $\omega \in \Omega$. Then they are $\mathcal{A} \to \mathcal{B}$ measurable functions. That is, these mappings x_α^- and x_α^+ are random variables.

On the other hand, let x^-, x^m and x^+ be crisp random variables such that $x^-(\omega) \le x^m(\omega) \le x^+(\omega)$ for each $\omega \in \Omega$. Then $X : \Omega \to \mathcal{F}_{LR}(\mathbb{R})$ with $X(\omega) = \langle x^m(\omega), x^m(\omega) - x^-(\omega), x^+(\omega) - x^m(\omega)\rangle_{LR}$ is a fuzzy random variable. $X(\omega)$ is a LR-fuzzy number. These fuzzy random variables are said to be *LR-fuzzy random variables.* Let $(\Omega, \mathcal{A}, P)$ be a probability space where the probability measure P is assumed to be non-atomic.

A *set-valued function* is a function $F : \Omega \to \mathcal{P}(\mathbb{R}^n)$ such that $F(\omega) \neq \varnothing$ for every $\omega \in \Omega$. By $L^1(P)$ we denote the space of P-integrable function $f : \Omega \to \mathbb{R}^n$. We denote by $S(F)$ the set of all $L^1(P)$ selection of F, that is,

$$S(F) = \{f \in L^1(P) : f(\omega) \in F(\omega) \ a.e.\}.$$

The *Aumann integral* of F is defined by

$$(A)\int F = \left\{\int_\Omega f dP : f \in S(F)\right\}.$$

We define the expected value $E(X)$ of a fuzzy random variable $X : \Omega \to \mathcal{F}_0(\mathbb{R}^n)$ in such a way that the following conditions are satisfied:

$$E(X) \in \mathcal{F}_0(\mathbb{R}^n),$$

$$\{x \in \mathbb{R} : \mu_{E(X)}(x) \ge \alpha\} = (A)\int X^\alpha, \qquad \text{for each } \alpha \in [0,1].$$

Puri and Ralescu (1986) showed that under certain assumption, there is a unique fuzzy subset satisfying these requirements.

We use the preceding proposition to define the expected value of a fuzzy random variable $X : \Omega \to \mathcal{F}_0(\mathbb{R}^n)$ which is integrably bounded.

Definition 2.14 The *expected value* of X, denoted by $E(X)$ is the fuzzy subset $v \in \mathcal{F}_0(\mathbb{R}^n)$ such that $\{x \in \mathbb{R} : \mu_v(x) \ge \alpha\} = (A)\int X^\alpha$ for every $\alpha \in [0,1]$.

If m_X, l_X, r_X are crisp random variables satisfying that l_X, r_X are non-negative random variables, then we can easily check that the LR-fuzzy number variable $X = \langle m_X, l_X, r_X\rangle_{LR}$ is a fuzzy random variable. On the contrary, let $X : \Omega \to \mathcal{F}_{LR}(\mathbb{R})$ be a LR-fuzzy random variable, denoted by $X = \langle m_X, l_X, r_X\rangle_{LR}$. Consider the set $\{(\omega, x) : x \in X^1(\omega)\}$, where $X^1(\omega)$ is 1-level set of the fuzzy set $X(\omega)$. Since X is a fuzzy random variable, the preceding set is $\mathcal{A}\times\mathcal{B}$-measurable set, that is $\{(\omega, x) : x \in X^1(\omega)\} \in \mathcal{A}\times\mathcal{B}$. We know that for each $\omega \in \Omega$, $X^1(\omega)$ is singleton because X is a LR-fuzzy number. So, $m_X(\omega) = x_\omega$, where $x_\omega \in X^1(\omega)$. Then m_X is a $\mathcal{B}$-measurable function, *i.e.* it is a random variable from Ω to $\mathbb{R}$. According to the same method we can easily show that l_X and r_X are random variables.

Consequently, every LR-fuzzy random variable can be represented by a LR-fuzzy number function such that the mode, left and right spreads are constituted by crisp random variables. We can say again that a LR-fuzzy number function $X = \langle m_X, l_X, r_X\rangle_{LR}$ is a fuzzy random variable if and only if m_X, l_X and r_X are crisp random variables and l_X

and r_X are non-negative. Using the definition for the expected value of a fuzzy random variable, we lead some properties of the expected value of a LR-fuzzy random variable X.

Theorem 2.1 If $X : \Omega \to \mathcal{F}_{LR}(\mathbb{R})$ is a LR-fuzzy random variable, denoted by $X = \langle m_X, l_X, r_X \rangle_{LR}$, then the expected value of X is that

$$EX = \left\langle Em_X, El_X, Er_X \right\rangle_{LR}. \tag{2.13}$$

Proof: To show it, we claim that the α-level set $[E(X)]^\alpha$ of EX is equal to the α-level set of the LR-fuzzy number $\langle Em_X, El_X, Er_X \rangle_{LR}$.

First we remark that the α-level set of the LR-fuzzy number $\langle Em_X, El_X, Er_X \rangle_{LR}$ is represented by the following closed interval:

$$\left[E(m_X) - E(l_X)L^{-1}(\alpha), \ E(m_X) + E(r_X)R^{-1}(\alpha) \right].$$

Now, we lead α-level set $[E(X)]^\alpha$ of $E(X)$. By the definition, $[E(X)]^\alpha = (A) \int X^\alpha dP$ for each α. Since $X(\omega) = \left\langle m_X(\omega), l_X(\omega), r_X(\omega) \right\rangle_{LR}$,

$$X^\alpha(\omega) = \left[m_X(\omega) - l_X(\omega)L^{-1}(\alpha), \ m_X(\omega) + r_X(\omega)R^{-1}(\alpha) \right]. \tag{2.14}$$

By equation (2.14), we know that

$$(A) \int X^\alpha = \left\{ \int_\Omega f dP : f \in S(X^\alpha) \right\}.$$

So, $(A) \int X^\alpha$ must be a closed interval by the equation. The left bound of the close interval is that $E(m_X - l_X L^{-1}(\alpha))$ and the right bound is that $E(m_X + r_X R^{-1}(\alpha))$. Thus we can conclude that

$$[E(X)]^\alpha = \left[E(m_X) - E(l_X)L^{-1}(\alpha), \ E(m_X) + E(r_X)R^{-1}(\alpha)\right].$$

Our proof is completed. □

Definition 2.15 Let $X : \Omega \to \mathcal{F}_C(\mathbb{R}^n)$ be a fuzzy random variable with expected value $E(X)$, where $\mathcal{F}_C(\mathbb{R}^n)$ be the family of all normal compact convex fuzzy subsets of $\mathbb{R}^n$. We define a *variance* of the fuzzy random variable X with respect to an metric D_2 as follows:

$$\mathrm{Var}(X) = ED_2(X, E(X))^2, \tag{2.15}$$

where $D_2(\cdot, \cdot)$ is a metric in $\mathcal{F}_C(\mathbb{R}^n)$.

Definition 2.16 Let $X_1, X_2, \ldots$ be a sequence of fuzzy random variables. A sequence of fuzzy estimates $T_n(X_1, X_2, \ldots, X_n) = T_n$ will be called *weak consistent* for a fuzzy parameter Θ if

$$T_n \xrightarrow{P} \Theta \quad \text{as } n \to \infty. \tag{2.16}$$

We recall that

$$T_n \xrightarrow{P} \Theta \qquad \text{if and only if} \quad P\left[d(T_n, \Theta) > \epsilon \right] \to 0 \quad \text{as } n \to \infty, \tag{2.17}$$

for any non-negative real number ϵ.

Definition 2.17 A fuzzy estimate $\hat{\Theta}$ of a fuzzy parameter Θ is said to be *unbiased* for Θ if

$$E_{\Theta}\left(\hat{\Theta}\right) = \Theta. \tag{2.18}$$

Theorem 2.2 (Weak Law of Large Numbers: WLLN) Let $X_1, X_2, \ldots, X_n$ be uncorrelated random variables which satisfy $EX_i = \mu$ and $\mathrm{Var}X_i \le c < \infty$ $(i = 1, \ldots, n)$ for some positive constant c. And let $S_n = X_1 + \cdots + X_n$. Then

$$\frac{S_n - ES_n}{n} \xrightarrow{P} 0 \quad \text{in } L^2 \quad as\ n \to \infty. \tag{2.19}$$

i.e.,

$$\frac{1}{n}S_n \xrightarrow{P} \mu \quad \text{in } L^2 \quad as\ n \to \infty.$$

Proof: See Rohatgi (1979). □

3. Asymptotic Optimal Properties

We assume that X_i, Y_i are triangular fuzzy numbers for $i = 1, \ldots, n$. We represent that $Y_i = \langle y_i, \eta_i^l, \eta_i^r \rangle$, where y_i is the mode, η_i^l is the left spread and η_i^r is the right spread of Y_i. And we also represent that $X_i = \langle x_i, \xi_i^l, \xi_i^r \rangle$, where x_i, ξ_i^l, ξ_i^r are the mode, left and right spreads of X_i respectively. $\Phi_i\,(i = 1, \ldots, n)$ which are assumed to be the fuzzy random variables are the fuzzy errors for expressing randomness. So, we express as that $\Phi_i = \langle \epsilon_i, \theta_i^l, \theta_i^r \rangle$. And ϵ_i, θ_i^l, θ_i^r are also represent mode, left and right spreads of Φ_i respectively. We regard ϵ_i, θ_i^l, θ_i^r as crisp random variables. Throughout this paper, we will use following assumptions.

Assumption 3.1

(A1) ϵ_i are *i.i.d. r.v.*'s with $E[\epsilon_i] = 0$ and $\mathrm{Var}[\epsilon_i] = \sigma_\epsilon^2(< \infty)$.

(A2) θ_i^r, θ_i^l are nonnegative *r.v.*'s with $E[\theta_i^r] = m_r$, $E[\theta_i^l] = m_l$ and $\mathrm{Var}[\theta_i^r] = \sigma_r^2(< \infty)$, $\mathrm{Var}[\theta_i^l] = \sigma_l^2(< \infty)$, $\mathrm{Cov}(\theta_i^r, \theta_j^r) = 0$ and $\mathrm{Cov}(\theta_i^l, \theta_j^l) = 0$ for $i \neq j$.

(A3) ϵ_i, θ_i^r and θ_i^l are mutually uncorrelated.

3.1. Estimating the parameters

To find the estimators, we want to minimize following objective function:

$$Q(\beta_0, \beta_1) = \sum_{i=1}^{n} d^2(\beta_0 + \beta_1 X_i, Y_i). \tag{3.1}$$

And two cases must be considered:

Case I: $\beta_1 > 0$.

In this case, $Q(\beta_0, \beta_1)$ is given by

$$Q(\beta_0, \beta_1) = \sum_{i=1}^{n} \left\{\beta_0 + \beta_1 x_i - y_i - \left(\beta_1 \xi_i^l - \eta_i^l\right)\right\}^2$$

$$+\left\{\beta_0+\beta_1 x_i-y_i+(\beta_1\xi_i^r-\eta_i^r)\right\}^2+(\beta_0+\beta_1 x_i-y_i)^2. \tag{3.2}$$

And the unique solution, $\hat{\beta_0^+}$, $\hat{\beta_1^+}$, which minimize above expression are founded by solving $\partial Q/\partial\beta_0=0$ and $\partial Q/\partial\beta_1=0$. Namely,

$$\begin{cases} 3n\beta_0+\beta_1\sum_{i=1}^{n}\left(3x_i+\xi_i^r-\xi_i^l\right)=\sum_{i=1}^{n}(3y_i+\eta_i^r-\eta_i^l), \\ \beta_0\sum_{i=1}^{n}\left(3x_i+\xi_i^r-\xi_i^l\right)+\beta_1\sum_{i=1}^{n}\left\{x_i^2+(x_i+\xi_i^r)^2+\left(x_i-\xi_i^l\right)^2\right\} \\ =\sum_{i=1}^{n}\left\{\left(x_i-\xi_i^l\right)\left(y_i-\eta_i^l\right)+\left(x_i+\xi_i^r\right)\left(y_i+\eta_i^r\right)+x_iy_i\right\}. \end{cases} \tag{3.3}$$

The solution to above system is denoted by $(\hat{\beta_1^+}, \hat{\beta_0^+})$. And we can easily find that:

$$\hat{\beta_0^+}=\frac{AD-BC}{A^2-3nC} \quad\text{and}\quad \hat{\beta_1^+}=\frac{AB-3nD}{A^2-3nC}, \tag{3.4}$$

where $A=\sum_{i=1}^n(3x_i+\xi_i^r-\xi_i^l)$, $B=\sum_{i=1}^n(3y_i+\eta_i^r-\eta_i^l)$, $C=\sum_{i=1}^n(3x_i^2+2x_i\xi_i^r-2x_i\xi_i^l+\xi_i^{r2}+\xi_i^{l2})$ and $D=\sum_{i=1}^n[(x_i-\xi_i^l)(y_i-\eta_i^l)+(x_i+\xi_i^r)(y_i+\eta_i^r)+x_iy_i]$.

Case II: $\beta_1<0$.

In this case, $Q(\beta_0,\beta_1)$ is given by

$$\begin{aligned} Q(\beta_0,\beta_1)= &\left(\beta_0+\beta_1x_i-y_i+\beta_1\xi_i^r+\eta_i^l\right)^2 \\ &+\left\{\beta_0+\beta_1x_i-y_i-\left(\beta_1\xi_i^l+\eta_i^r\right)\right\}^2+(\beta_0+\beta_1x_i-y_i)^2. \end{aligned} \tag{3.5}$$

And the unique minimizing solution is derived by following linear system.

$$\begin{cases} 3n\beta_0+\beta_1\sum_{i=1}^{n}(3x_i+\xi_i^r-\xi_i^l)=\sum_{i=1}^{n}\left(3y_i+\eta_i^r-\eta_i^l\right), \\ \beta_0\sum_{i=1}^{n}(3x_i+\xi_i^r-\xi_i^l)+\beta_1\sum_{i=1}^{n}\left\{x_i^2+(x_i+\xi_i^r)^2+\left(x_i-\xi_i^l\right)^2\right\} \\ =\sum_{i=1}^{n}\left\{\left(x_i+\xi_i^r\right)\left(y_i-\eta_i^l\right)+\left(x_i-\xi_i^l\right)\left(y_i+\eta_i^r\right)+x_iy_i\right\}. \end{cases} \tag{3.6}$$

The solution to above system is denoted by $(\hat{\beta_0^-}, \hat{\beta_1^-})$. We can also easily find that:

$$\hat{\beta_0^-}=\frac{AD'-BC}{A^2-3nC} \quad\text{and}\quad \hat{\beta_1^-}=\frac{AB-3nD'}{A^2-3nC}, \tag{3.7}$$

where $D'=\sum_{i=1}^n[(x_i+\xi_i^r)(y_i-\eta_i^l)+(x_i-\xi_i^l)(y_i+\eta_i^r)+x_iy_i]$.

If the solution to this system satisfies $\beta_1^->0$, then it is not acceptable. *i.e.*, it is not a solution to the minimize problem.

Proposition 3.1 The numbers $\hat{\beta_1^+}$ and $\hat{\beta_1^-}$ defined previously, satisfy $\hat{\beta_1^+}\geq\hat{\beta_1^-}$.

Proof: See Diamond (1988). □

3.2. Asymptotic unbiasedness

To prove asymptotic properties, we use following Assumption 3.2 in addition to Assumption 3.1.

Assumption 3.2

(B1) $\sum_{i=1}^{n}(x_i\xi_i^r - x_i\xi_i^l) \to 0$ and $\sum_{i=1}^{n}(\xi_i^r - \xi_i^l) \to 0$ as $n \to \infty$.

(B2) $\overline{x^2} - \overline{x}^2$ converges to some positive constant as $n \to \infty$.

(B3) $(m_r\overline{\xi^{r2}} - m_l\overline{\xi^{l2}}) \to 0$ and $(m_l\overline{\xi^{r2}} - m_r\overline{\xi^{l2}}) \to 0$ as $n \to \infty$.

(B4) $\overline{\xi^r} > C_1 > 0$ and $\overline{\xi^l} > C_2 > 0$ for some positive constants C_1 and C_2.

(B5) $\overline{\xi^r}/\overline{\xi^{r2}} \to 0$ and $\overline{\xi^l}/\overline{\xi^{l2}} \to 0$ as $n \to \infty$. The overline notation means that $\overline{x} = 1/n\sum_{i=1}^{n} x_i$, $\overline{\xi^r} = 1/n\sum_{i=1}^{n}\xi_i^r$ and so on.

Theorem 3.1 Under the Assumptions 3.1 and 3.2, the expectations of fuzzy least squares estimators are given by

$$E\left(\hat{\beta_0^+}\right) = \frac{AF - CE}{A^2 - 3nC}, \qquad E\left(\hat{\beta_1^+}\right) = \frac{AE - 3nF}{A^2 - 3nC} \tag{3.8}$$

and

$$E\left(\hat{\beta_0^-}\right) = \frac{AF' - CE}{A^2 - 3nC}, \qquad E\left(\hat{\beta_1^-}\right) = \frac{AE - 3nF'}{A^2 - 3nC}, \tag{3.9}$$

where $E = n[3\beta_0^+ + \beta_1^+(3\overline{x} + I) + L]$, $F = n[\beta_0^+(3\overline{x} + I) + \beta_1^+(3\overline{x^2} + 2K + J) + L\overline{x} + m_r\overline{\xi^r} + m_l\overline{\xi^l}]$, $F' = n[\beta_0^-(3\overline{x} + I) + \beta_1^-(3\overline{x^2} + 2K + J) + L\overline{x} - (m_r\overline{\xi^l} + m_l\overline{\xi^r})]$, $I = \overline{\xi^r} - \overline{\xi^l}$, $J = \overline{\xi^{r2}} + \overline{\xi^{l2}}$, $K = \overline{x\xi^r} - \overline{x\xi^l}$ and $L = m_r - m_l$.

Proof: Consider the normal equation (3.3) and take expectations to the each equation of (3.3). If we take each sides of the model $Y_i = \beta_0 + \beta_1 X_i + \Phi_i$ to triangular components, the mode and spreads of Y_i can be denoted by $y_i = \beta_0^+ + \beta_1^+ x_i + \epsilon_i$, $\eta_i^l = \beta\xi_i^l + \theta_i^l$ and $\eta_i^r = \beta\xi_i^r + \theta_i^r$ in Case I. Then $E(y_i) = \beta_0^+ + \beta_1^+ x_i$ from (A1) and $E(\eta_i^l) = \beta_1^+\xi_i^l + m_l$, $E(\eta_i^r) = \beta_1^+\xi_i^r + m_r$ from (A2). After taking expectations, the left sides of the equations will do not changed except $E(\hat{\beta_0^+})$ and $E(\hat{\beta_1^+})$ and the right sides will be changed to $E = E(B)$ and $F = E(D)$ respectively. And with previous notations, the equations can be denoted by followings:

$$\begin{cases} 3nE\left(\hat{\beta_0^+}\right) + AE\left(\hat{\beta_1^+}\right) = E, \\ AE\left(\hat{\beta_0^+}\right) + CE\left(\hat{\beta_1^+}\right) = F. \end{cases} \tag{3.10}$$

If we solve above system (3.10), we can easily find (3.8). For Case II, we get $F' = E(D')$ from $E(\eta_i^l) = -\beta_1^-\xi_i^r + m_l$ and $E(\eta_i^r) = -\beta_1^-\xi_i^l + m_r$. And we can get (3.9) if we replace F with $F' = E(D')$ in (3.10). □

Theorem 3.2 Under the Assumptions 3.1 and 3.2, the fuzzy least squares estimators $\hat{\beta_0^+}$ and $\hat{\beta_1^+}$ are asymptotic unbiased estimators for β_0^+ and β_1^+. *i.e.*,

$$E\left(\hat{\beta_0^+}\right) \longrightarrow \beta_0^+ \quad \text{and} \quad E\left(\hat{\beta_1^+}\right) \longrightarrow \beta_1^+ \quad \text{as } n \to \infty.$$

Proof: Consider $A^2 - 3nC$ is $n^2(3\overline{x}+I)^2 - 3n^2(3\overline{x^2}+2K+J)$. And

$$\begin{aligned} AF - CE &= n^2(3\overline{x}+I)\left\{\beta_0^+(3\overline{x}+I) + \beta_1^+\left(3\overline{x^2}+2K+J\right) + L\overline{x} + \left(m_r\overline{\xi^r} + m_l\overline{\xi^l}\right)\right\} \\ &\quad - n^2\left(3\overline{x^2}+2K+J\right)\{3\beta_0^+ + \beta_1^+(3\overline{x}+I) + L\} \\ &= \beta_0^+(A^2 - 3nC) + n^2\Big\{L\overline{x}(3\overline{x}+I) + (3\overline{x}+I)\left(m_r\overline{\xi^r} + m_l\overline{\xi^l}\right) \\ &\quad - L\left(3\overline{x^2}+2K+J\right)\Big\}. \end{aligned}$$

After some calculations, the second term will be of the form $n^2\{L\Gamma + (m_r\overline{\xi^r}^2 + m_l\overline{\xi^l}^2)\}$ for some Γ. And $A^2 - 3nC$ will be of the form $n^2\{\Pi + (\overline{\xi^r}^2 + \overline{\xi^l}^2)\}$ for some Π. The above Γ and Π have same degrees, thus the convergence is up to $(m_r\overline{\xi^r}^2 - m_l\overline{\xi^l}^2)/(\overline{\xi^r}^2 + \overline{\xi^l}^2)$. Now, since we have (B3) and (B4), $(m_r\overline{\xi^r}^2 - m_l\overline{\xi^l}^2)/(\overline{\xi^r}^2 + \overline{\xi^l}^2)$ converges to 0 as $n \to \infty$. Therefore, $E(\hat{\beta_0^+}) \longrightarrow \beta_0^+$ when $n \to \infty$.

For $E(\hat{\beta_1^+})$, $AE - 3nF = n^2(3\overline{x}+I)\{3\beta_0^+ + \beta_1^+(3\overline{x}+I)+L\} - 3n^2\{\beta_0^+(3\overline{x}+I) + \beta_1^+(3\overline{x^2}+2K+J) + L\overline{x} + (m_r\overline{\xi^r} + m_l\overline{\xi^l})\} = \beta_1^+(A^2 - 3nC) + n^2\{LI - 3(m_r\overline{\xi^r} + m_l\overline{\xi^l})\}$. With (B1), the convergence of $E(\hat{\beta_1^+})$ is up to the convergence of $(k_1\overline{\xi^r} + k_2\overline{\xi^l})/(k_3\overline{\xi^{r2}} + k_4\overline{\xi^{l2}})$ for some constants $k_1, \ldots, k_4$. And it is converges to 0 by (B5) as $n \to \infty$. *i.e.*, $E(\hat{\beta_1^+}) \longrightarrow \beta_1^+$ as $n \to \infty$. Thus, our proof is completed. □

Theorem 3.3 Under the Assumptions 3.1 and 3.2, the fuzzy least squares estimators $\hat{\beta_0^-}$ and $\hat{\beta_1^-}$ are asymptotic unbiased estimators for β_0^- and β_1^-. *i.e.*,

$$E\left(\hat{\beta_0^-}\right) \longrightarrow \beta_0^- \quad \text{and} \quad E\left(\hat{\beta_1^-}\right) \longrightarrow \beta_1^- \quad \text{as } n \to \infty.$$

Proof: Consider $AF' - CE = n^2(3\overline{x}+I)\{\beta_0^-(3\overline{x}+I) + \beta_1^-(3\overline{x^2}+2K+J) + L\overline{x} - (m_r\overline{\xi^l} + m_l\overline{\xi^r})\} - n^2(3\overline{x^2}+2K+J)\{3\beta_0^- + \beta_1^-(3\overline{x}+I) + L\} = \beta_0^-(A^2 - 3nC) + n^2\{L\overline{x}(3\overline{x}+I) - (3\overline{x}+I)(m_r\overline{\xi^l} + m_l\overline{\xi^r}) - L(3\overline{x^2}+2K+J)\}$. Thus $E(\hat{\beta_0^-}) \longrightarrow \beta_0^-$ when $n \to \infty$ by the same assumptions (B3) and (B4) as Theorem 3.2. And $AE - 3nF' = n^2(3\overline{x}+I)\{3\beta_0^- + \beta_1^-(3\overline{x}+I) + L\} - 3n^2\{\beta_0^-(3\overline{x}+I) + \beta_1^+(3\overline{x^2}+2K+J) + L\overline{x} - (m_r\overline{\xi^l} + m_l\overline{\xi^r})\} = \beta_1^-(A^2 - 3nC) + n^2\{\mathrm{LI} + 3(m_r\overline{\xi^l} + m_l\overline{\xi^r})\}$. With (B1) and (B2), $E(\hat{\beta_1^-}) \longrightarrow \beta_1^-$ as $n \to \infty$ by the same assumption (B5) as Theorem 3.2. □

3.3. Asymptotic consistency

To prove consistency, we use Theorem 2.2, the WLLN(Weak Law of Large Numbers).

Proposition 3.2 By WLLN, following properties hold with (A1)-(A2) as $n \to \infty$.

(P1) $\frac{1}{n}\sum_{i=1}^{n}\epsilon_i \xrightarrow{P} 0.$

(P2) $\frac{1}{n}\sum_{i=1}^{n}x_i\epsilon_i \xrightarrow{P} 0.$

(P3) $\frac{1}{n}\sum_{i=1}^{n}\theta_i^r \xrightarrow{P} m_r$ and $\frac{1}{n}\sum_{i=1}^{n}\theta_i^l \xrightarrow{P} m_l.$

(P4) $\frac{1}{n}\sum_{i=1}^{n}x_i\theta_i^r \xrightarrow{P} \overline{x}m_r$ and $\frac{1}{n}\sum_{i=1}^{n}x_i\theta_i^l \xrightarrow{P} \overline{x}m_l.$

(P5) $\frac{1}{n}\sum_{i=1}^{n}\epsilon_i\xi_i^r \xrightarrow{P} 0$ and $\frac{1}{n}\sum_{i=1}^{n}\epsilon_i\xi_i^l \xrightarrow{P} 0.$

(P6) $\frac{1}{n}\sum_{i=1}^{n}\theta_i\xi_i^r \xrightarrow{P} \overline{\xi^r}m_r$ and $\frac{1}{n}\sum_{i=1}^{n}\theta_i\xi_i^l \xrightarrow{P} \overline{\xi^l}m_l.$

Proof:

(P1) By (A1) and (A2), each ϵ_i has finite variance and $\epsilon_i(i = 1, 2, \ldots, n)$ are uncorrelated. So, $\{\sum_{i=1}^n \epsilon_i - E(\sum_{i=1}^n \epsilon_i)\}/n \xrightarrow{P} 0$ as $n \to \infty$ by WLLN. *i.e.*, $1/n\sum_{i=1}^n \epsilon_i \xrightarrow{P} 1/nE(\sum_{i=1}^n \epsilon_i) = 0$ as $n \to \infty$.

(P2) Similarly, with the same assumptions as (P1), $1/n\sum_{i=1}^n x_i\epsilon_i \xrightarrow{P} 1/nE(\sum_{i=1}^n x_i\epsilon_i)$ $= 1/nx_iE\epsilon_i = 0$ as $n \to \infty$ by WLLN.

(P3) By (A2), each θ_i^r and θ_i^l has finite variance and by (A2), $\theta_i^r(i = 1, 2, \ldots, n)$ are uncorrelated and so are $\theta_i^l(i = 1, 2, \ldots, n)$. So, $1/n\sum_{i=1}^n \theta_i^r \xrightarrow{P} 1/nE(\sum_{i=1}^n \theta_i^r) =$ m_r and $1/n\sum_{i=1}^n \theta_i^l \xrightarrow{P} 1/nE(\sum_{i=1}^n \theta_i^l) = m_l$ as $n \to \infty$. by WLLN.

(P4) Similarly, with the same assumptions as (P3), $1/n\sum_{i=1}^n x_i\theta_i^r \xrightarrow{P} 1/nE(\sum_{i=1}^n x_i\theta_i^r)$ $= \overline{x}m_r$ and $1/n\sum_{i=1}^n \theta_i^l \xrightarrow{P} 1/nE(\sum_{i=1}^n x_i\theta_i^l) = \overline{x}m_l$ as $n \to \infty$ by WLLN.

(P5) With the same assumptions as (P1), $1/n\sum_{i=1}^n \epsilon_i\xi_i^r \xrightarrow{P} 1/nE(\sum_{i=1}^n \epsilon_i\xi_i^r) = 1/n$ $\sum_{i=1}^n \xi_i^r E(\epsilon_i) = \overline{\xi^r}\cdot 0 = 0$ and similarly, $1/n\sum_{i=1}^n \epsilon_i\xi_i^l \xrightarrow{P} 0$ as $n \to \infty$ by WLLN.

(P6) With the same assumptions as (P3), $1/n\sum_{i=1}^n \theta_i^r\xi_i^r \xrightarrow{P} 1/n\sum_{i=1}^n \xi_i^r E(\theta_i^r) = \overline{\xi^r}m_r$ and similarly, $1/n\sum_{i=1}^n \theta_i^l\xi_i^l \xrightarrow{P} \overline{\xi^l}m_l$ by WLLN.

□

Theorem 3.4 Under the Assumptions 3.1 and 3.2, the fuzzy least squares estimators $\hat{\beta_0^+}$ and $\hat{\beta_1^+}$ are weak consistent estimators for β_0^+ and β_1^+ respectively. *i.e.*,

$$\hat{\beta_0^+} \xrightarrow{P} \beta_0^+ \quad \text{and} \quad \hat{\beta_1^+} \xrightarrow{P} \beta_1^+ \quad \text{as } n \to \infty.$$

Proof: By (P1)–(P6), following holds.

$$\begin{aligned} AD - BC &= n^2(3\overline{x}+I)\left\{\beta_0^+(3\overline{x}+I)+\beta_1^+\left(3\overline{x^2}+2K+J\right)+3\overline{x\epsilon}+\left(\overline{\epsilon\xi^r}-\overline{\epsilon\xi^l}\right)\right. \\ &\quad \left.+\left(\overline{x\theta^r}-\overline{x\theta^l}\right)+\left(\overline{\theta^r\xi r}+\overline{\theta^l\xi l}\right)\right\} \\ &\quad -n^2\left(3\overline{x^2}+2K+J\right)\left\{3\beta_0^+ +\beta_1^+(3\overline{x}+I)+\left(3\overline{\epsilon}+\overline{\theta^r}-\overline{\theta^l}\right)\right\} \\ &\xrightarrow{P} \beta_0^+\left(A^2-3nC\right)+n^2\left\{L\overline{x}(3\overline{x}+I)+(3\overline{x}+I)\left(m_r\overline{\xi^r}+m_l\overline{\xi^l}\right)\right. \\ &\quad \left.-L\left(3\overline{x^2}+2K+J\right)\right\}. \end{aligned}$$

We already showed that the second term converges to 0 when it is divided by A^2-3nC as $n\to\infty$ in Theorem 3.2. Hence $\hat{\beta_0^+}\xrightarrow{P}\beta_0^+$. While,

$$\begin{aligned} AB-3nD &= n^2(3\overline{x}+I)\left\{3\beta_0^+ +\beta_1^+(3\overline{x}+I)+\left(3\overline{\epsilon}+\overline{\theta^r}-\overline{\theta^l}\right)\right\}-3n^2\left\{\beta_0^+(3\overline{x}+I)\right. \\ &\quad \left.+\beta_1^+\left(3\overline{x^2}+2K+J\right)+3\overline{x\epsilon}+\left(\overline{\epsilon\xi^r}-\overline{\epsilon\xi^l}\right)+\left(\overline{x\theta^r}-\overline{x\theta^l}\right)+\left(\overline{\theta^r\xi r}+\overline{\theta^l\xi l}\right)\right\} \\ &\xrightarrow{P} \beta_1^+(A^2-3nC)+n^2\left\{LI-3\left(m_r\overline{\xi^r}+m_l\overline{\xi^l}\right)\right\}. \end{aligned}$$

In Theorem 3.2, we also showed the second term converges to 0 when it is divided by A^2-3nC as $n\to\infty$. Therefore $\hat{\beta_1^+}\xrightarrow{P}\beta_1^+$. *i.e.*, $\hat{\beta_0^+}$ and $\hat{\beta_1^+}$ are weak consistent estimators for β_0^+ and β_1^+ respectively. □

Theorem 3.5 Under the Assumptions 3.1 and 3.2, the fuzzy least squares estimators $\hat{\beta_0^-}$ and $\hat{\beta_1^-}$ are weak consistent estimators for β_0^- and β_1^- respectively. *i.e.*,

$$\hat{\beta_0^-}\xrightarrow{P}\beta_0^- \quad \text{and} \quad \hat{\beta_1^-}\xrightarrow{P}\beta_1^- \quad \text{as } n\to\infty.$$

Proof: By (P1)–(P6), following holds.

$$\begin{aligned} AD'-BC &= n^2(3\overline{x}+I)\left\{\beta_0^-(3\overline{x}+I)+\beta_1^-\left(3\overline{x^2}+2K+J\right)+3\overline{x\epsilon}+\left(\overline{\epsilon\xi^r}-\overline{\epsilon\xi^l}\right)\right. \\ &\quad \left.+\left(\overline{x\theta^r}-\overline{x\theta^l}\right)-\left(\overline{\theta^r\xi l}+\overline{\theta^l\xi r}\right)\right\}-n^2\left(3\overline{x^2}+2K+J\right) \\ &\quad \left\{3\beta_0^- +\beta_1^-(3\overline{x}+I)+\left(3\overline{\epsilon}+\overline{\theta^r}-\overline{\theta^l}\right)\right\} \\ &\xrightarrow{P} \beta_0^-\left(A^2-3nC\right)+n^2\left\{L\overline{x}(3\overline{x}+I)-(3\overline{x}+I)\left(m_r\overline{\xi^l}+m_l\overline{\xi^r}\right)\right. \\ &\quad \left.-L\left(3\overline{x^2}+2K+J\right)\right\}. \end{aligned}$$

We showed that the second term converges to 0 when it is divided by A^2-3nC as $n\to\infty$ in Theorem 3.3. Hence $\hat{\beta_0^-}\xrightarrow{P}\beta_0^-$. While,

$$\begin{aligned} AB-3nD' &= n^2(3\overline{x}+I)\left\{3\beta_0^+ +\beta_1^+(3\overline{x}+I)+\left(3\overline{\epsilon}+\overline{\theta^r}-\overline{\theta^l}\right)\right\}-3n^2\left\{\beta_0^+(3\overline{x}+I)\right. \\ &\quad \left.+\beta_1^+\left(3\overline{x^2}+2K+J\right)+3\overline{x\epsilon}+\left(\overline{\epsilon\xi^r}-\overline{\epsilon\xi^l}\right)+\left(\overline{x\theta^r}-\overline{x\theta^l}\right)-\left(\overline{\theta^r\xi l}+\overline{\theta^l\xi r}\right)\right\} \\ &\xrightarrow{P} \beta_1^+\left(A^2-3nC\right)+n^2\left\{LI+3\left(m_r\overline{\xi^l}+m_l\overline{\xi^r}\right)\right\}. \end{aligned}$$

In Theorem 3.3, We also showed the second term converges to 0 when it is divided by $A^2 - 3nC$ as $n \to \infty$. Therefore $\hat{\beta_1^-} \xrightarrow{P} \beta_1^-$. *i.e.*, $\hat{\beta_0^-}$ and $\hat{\beta_1^-}$ are weak consistent estimators for β_0^- and β_1^- respectively. □

4. Conclusions

Fuzzy least squares problem has been worked by many authors. But researches of statistical optimal properties of the fuzzy least squares estimators are rare because the structure of fuzzy least squares estimators are complicated. This paper proved the asymptotic unbiasedness and consistency of the estimators under some assumptions with an L_2-metric in case of fuzzy input-output model.

Although only the simple linear regression model with triangular fuzzy data and suitable metric is discussed here, further research needs to be undertaken to discover the analogous results in the class of more general models, such as multiple linear regression models or autoregressive models in time series analysis, with the more complicated metrics and/or other types of fuzzy data.

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