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Median Ranked Ordering-Set Sample Test for Ordered Alternatives

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Abstract

In this paper, we consider the c-sample location problem for ordered alternatives using median ranked ordering-set samples(MROSS). We propose the test statistic using the median of samples that have the same ranked order in each cycle of ranked ordering-set sample(ROSS). We obtain the asymptotic property of the proposed test statistic and Pitman efficiency with respect to other test statistic. In simulation study, our proposed test statistic has good powers for some underlying distributions we consider.

Keywords: Median ranked ordering-set samples; ordered alternatives; Jonckheere (1954); Pitman efficiency.

1. Introduction

After McIntyre (1952) provided the original description of ranked-set sampling(RSS), numerous parametric and nonparametric procedures based on RSS have been developed. Takahasi and Wakimoto (1968) dealt with the statistical properties on RSS. Dell and Clutter (1972) considered a useful method for improving estimates of the mean under the imperfect judgment situation. Stokes (1977) studied the relation of the original and concomitant variables on RSS. Stokes and Sager (1988) considered nonparametric inference for RSS.

In the one-sample problem, Hettmansperger (1995) considered the sign test on RSS under the perfect and imperfect judgment ranking. Koti and Babu (1996) evaluated the exact null distribution of sign test on RSS. Bohn and Wolfe (1992, 1994) proposed the Mann-Whitney-Wilcoxon statistic and investigated the properties of the test procedures based on RSS for perfect and imperfect judgements. Öztürk (1999) extended one-sample sign test to two-sample sign test on RSS, comparing the confidence intervals of each one-sample test for two population. Kim *et al.* (2000) studied c-sample problem for ordered alternatives on RSS. Kim and Kim (2003) discussed the ranked ordering-set sampling as an opposed RSS. The main difference of ROSS and RSS is such that ROSS does not return observations until the sampling procedure is over but RSS does. Although ROSS is more complex than RSS due to this procedure, the efficiency of ROSS is better than that of RSS for the same sample distribution.

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$$\begin{array}{cc} \text{Set} & \text{Rank} \\ & \begin{array}{cccc} 1 & 2 & \cdots & k \end{array} \\ \begin{array}{c} 1 \\ 2 \\ \vdots \\ k \end{array} & \begin{pmatrix} X_{(11)} & X_{(12)} & \cdots & X_{(1k)} \\ X_{(21)} & X_{(22)} & \cdots & X_{(2k)} \\ \vdots & \vdots & \ddots & \vdots \\ X_{(k1)} & X_{(k2)} & \cdots & X_{(kk)} \end{pmatrix} \\ & \begin{array}{cccc} \Downarrow & \Downarrow & \Downarrow & \Downarrow \end{array} \\ \text{ROSS} & \begin{array}{cccc} X^{(1)}_{(1)1} & X^{(2)}_{(2)1} & \cdots & X^{(k)}_{(k)1} \end{array} \end{array}$$

Figure 2.1: Ranked ordering-set sample structure(first cycle)

In this paper we now consider c-sample nonparametric testing problem for ordered alternatives on MROSS. Our proposed test statistic does not use all samples, but use the median of the i^{th} samples in each cycle of ROSS. To compare our test statistic on MROSS with Page-type test on ROSS, comparative tests on RSS and SRS, we obtain the asymptotic relative efficiencies(ARE) of the proposed test statistic with respect to those on RSS and SRS. For the simulation work, we compare the empirical powers of the proposed test statistic with Page-type test on ROSS as well as those on RSS and SRS. The uniform, normal, double exponential, logistic and Cauchy distributions are considered as the underlying distributions. Through the simulation results, we show that our proposed test statistic has the best power under uniform and normal distribution and is similar or superior to the other statistics for logistic distribution. For the small sample size, when underlying distribution is heavy tailed, the power of the proposed statistic is slightly lower than that of RSS. But as the sample size k of MROSS is larger, the power of the proposed statistic is better and better and similar or superior to that of RSS.

This paper is organized as follows. In Section 2, we explain the sample structure of MROSS and propose the test statistic. Section 3 deals with the ARE and the asymptotic properties of the proposed test statistic. Simulation design and results under the underlying distributions are given in Section 4.

2. The Proposed Test Statistic

In this section, we introduce ROSS methods, MROSS and propose our test statistic.

2.1. Review of ROSS

This sampling design needs k^2 samples from a specified population. The units within each set, size k, are ranked by using a visual ordering or the ordering of a concomitant variable. We obtain k^2 ordered samples; $X_{(11)}, \ldots, X_{(1k)}, \ldots, X_{(k1)}, \ldots, X_{(kk)}$, where $X_{(ij)}$ is the j^{th} order statistic in the i^{th} set. From this samples, we next select k samples, which are denoted by $X^{(1)}_{(1)1}, X^{(2)}_{(2)1}, \ldots, X^{(k)}_{(k)1}$, where $X^{(j)}_{(j)l}$ is the j^{th} ranked sample of the j^{th} ordering observations in the l^{th} cycle. This notation like $X^{(j)}_{(j)l}$ is already used in Kim and Kim (2003). Figure 2.1 shows the ROSS.

$$
\begin{array}{cccccccc}
\text{Population} & & & \text{Cycle} & & & & \text{MROSS} \\
 & 1 & 2 & \cdots & n_j & & & \\
j & \left(\begin{array}{c} X^{(1)}_{j(1)1} \\ X^{(2)}_{j(2)1} \\ \vdots \\ X^{(k_j)}_{j(k_j)1} \end{array}\right. & \begin{array}{c} X^{(1)}_{j(1)2} \\ X^{(2)}_{j(2)2} \\ \vdots \\ X^{(k_j)}_{j(k_j)2} \end{array} & \begin{array}{c} \cdots \\ \cdots \\ \ddots \\ \cdots \end{array} & \left.\begin{array}{c} X^{(1)}_{j(1)n_j} \\ X^{(2)}_{j(2)n_j} \\ \vdots \\ X^{(k_j)}_{j(k_j)n_j} \end{array}\right) & \begin{array}{c} \Rightarrow \\ \Rightarrow \\ \vdots \\ \Rightarrow \end{array} & & \begin{array}{c} \tilde{X}^{(1)}_{j(1)} \\ \tilde{X}^{(2)}_{j(2)} \\ \vdots \\ \tilde{X}^{(k_j)}_{j(k_j)} \end{array}
\end{array}
$$

Figure 2.2: Median ranked ordering-set sample structure

2.2. The MROSS

We now introduce the MROSS, which plays an important role of our paper. Let $X^{(r)}_{j(r)i}$ be the r^{th} ranked ordering-set sample in the i^{th} cycle from the j^{th} population, $j = 1, \dots, c$; $r = 1, \dots, k_j$; $i = 1, \dots, n_j$, we take the median of ROSS having same rank. Figure 2.2 shows the structure of MROSS.

Let $X^{(1)}_{j(1)1}, \dots, X^{(1)}_{j(1)n_j}, \dots, X^{(k_j)}_{j(k_j)1}, \dots, X^{(k_j)}_{j(k_j)n_j}$ be the ROSS of size $n_j k_j$ from a continuous distribution with $F_j(x) = F(x - \theta_j)$ and *pdf* $f_j(x) = f(x - \theta_j)$, $j = 1, \dots, c$, where θ_j is a location parameter of the j^{th} population. Let k_j and n_j be the sample size and the cycle size, respectively. Then we obtain the independent $n_j k_j$ observations from $n_j k_j^2$ pre-ranking sample observations.

Let $\tilde{X}^{(r)}_{j(r)}$ be a median of $X^{(r)}_{j(r)i}$, $j = 1, \dots, c$; $r = 1, \dots, k_j$; $i = 1, \dots, n_j$. *i.e.* $\tilde{X}^{(r)}_{j(r)}$ is a median of the i^{th} ROSS having same rank. We assume that the cycle size n_j is odd for convenience of calculations. With $m_j = (n_j + 1)/2$, we can easily derive the *cdf* $F_{\tilde{X}^{(r)}_{j(r)}}(x)$ and *pdf* $f_{\tilde{X}^{(r)}_{j(r)}}(x)$ of $\tilde{X}^{(r)}_{j(r)}$, respectively,

$$
F_{\tilde{X}^{(r)}_{j(r)}}(x) = \sum_{u=m_j}^{n_j} \binom{n_j}{u} \left\{F^{(r)}_{(r)}(x)\right\}^u \left\{1 - F^{(r)}_{(r)}(x)\right\}^{n_j - u}, \tag{2.1}
$$

$$
f_{\tilde{X}^{(r)}_{j(r)}}(x) = m_j \binom{n_j}{m_j} \left\{F^{(r)}_{(r)}(x)\right\}^{m_j - 1} \left\{1 - F^{(r)}_{(r)}(x)\right\}^{n_j - m_j} f^{(r)}_{(r)}(x), \tag{2.2}
$$

where

$$
F^{(r)}_{(r)}(x) = \sum_{s=r}^{k_j} \binom{k_j}{s} \{F_{(r)}(x)\}^s \{1 - F_{(r)}(x)\}^{k_j - s}, \tag{2.3}
$$

$$
f^{(r)}_{(r)}(x) = r \binom{k_j}{r} \{F_{(r)}(x)\}^{r-1} \{1 - F_{(r)}(x)\}^{k_j - r} f_{(r)}(x), \tag{2.4}
$$

$$
F_{(r)}(x) = \sum_{q=r}^{k_j} \binom{k_j}{q} \{F(x)\}^q \{1 - F(x)\}^{k_j - q}, \tag{2.5}
$$

$$
f_{(r)}(x) = r \binom{k_j}{r} \{F(x)\}^{r-1} \{1 - F(x)\}^{k_j - r} f(x). \tag{2.6}
$$

2.3. The proposed test statistic.

We consider the testing problem for testing $H_0 : \theta_1 = \cdots = \theta_c (= \theta_0)$ against $H_1 : \theta_1 \leq \cdots \leq \theta_c$ with at least one strict inequality. Our proposed test statistic based on MROSS is

$$J_{MROSS} = \sum_{j<j'}^{c} \sum_{r=1}^{k_j} \sum_{r'=1}^{k_{j'}} \Psi\left(\tilde{X}_{j'(r')}^{(r')} - \tilde{X}_{j(r)}^{(r)}\right), \tag{2.7}$$

where $\Psi(t) = 1, 0$ as $t > 0, \leq 0$. The proposed test statistic uses a median of the i^{th} ROSS having same rank. Under the ordered alternatives H_1, the test statistic J_{MROSS} tends to be larger, so we reject H_0 for large values of J_{MROSS}.

3. Pitman Efficiency

We first calculate the mean and variance of the proposed statistic under null hypothesis. For simplicity, we assume that $n_j = n$, $k_j = k$ and n is odd. Then the mean and null variance of J_{MROSS} are

$$\begin{aligned} E_0(J_{MROSS}) &= \frac{c(c-1)k^2}{4}, \\ \mathrm{Var}_0(J_{MROSS}) &= \frac{c(c-1)}{12}\left\{3k^2 - 2(4c-11)\sum_{r=1}^{k}\sum_{r'=1}^{k}\delta_{r,r'}^2 + 4(2c-7)\sum_{r=1}^{k}\sum_{r'=1}^{k}\delta_{r,r,r'} \right. \\ &\quad \left. +12\sum_{r_1=1}^{k}\sum_{r_2=1}^{k}\sum_{r'=1}^{k}(\delta_{r_1,r_2,r'} - \delta_{r_1,r'}\delta_{r_2,r'})\right\}, \end{aligned}$$

where $\delta_{p,q} = \int F_{\tilde{X}_{(p)}^{(p)}}(x)dF_{\tilde{X}_{(q)}^{(q)}}(x)$, $\delta_{p,q,r} = \int F_{\tilde{X}_{(p)}^{(p)}}(x)F_{\tilde{X}_{(q)}^{(q)}}(x)dF_{\tilde{X}_{(r)}^{(r)}}(x)$.

We write the detailed expression of $\mathrm{Var}_0(J_{MROSS})$ in Appendix and can compute the variance using computer. When we take the specified sample structures, $c = 3, 5$, $k = 3, 5$ and $n = 3, 5$, we obtain the null variance in Table 3.1.

To use our proposed test statistic, we need to show that our test statistic is asymptotic normal. The following theorem can be obtained by Hoeffding (1948) and the Cental Limit Theorem.

Theorem 3.1 Under the assumption that $H_0 : \theta_1 = \cdots = \theta_c (= \theta_0)$ is true and the sample sizes in each cycle are all equal(*i.e.* $k_j = k$, $j = 1, \ldots, c$), the limiting $(k \to \infty)$ null distribution of

$$\frac{J_{MROSS} - E_0(J_{MROSS})}{\sqrt{\mathrm{Var}_0(J_{MROSS})}}$$

is standard normal.

We next derive the Pitman efficiency of J_{MROSS} with respect to J_{RSS}, J_{SRS} and P_{ROSS}. we consider a sequence of translation alternatives of the form, $H_{1k} : \theta_j = \theta_0 + j\triangle$, $j = 1, \ldots c$, where $\triangle = \theta/\sqrt{N}$, $N = kc$. Under the sequence of translation alternatives, the efficacy of the J_{MROSS} is obtained as follows.

Table 3.1: The Pitman AREs

c	k	n	ARE(J_{MROSS}, J_{RSS})	ARE(J_{MROSS}, P_{ROSS})	ARE(J_{MROSS}, J_{SRS})
3	3	3	4.478	6.427	10.502
3	3	5	4.417	9.274	10.693
3	5	3	9.364	9.736	34.915
3	5	5	9.102	14.205	35.945
5	3	3	3.950	5.391	9.426
5	3	5	3.791	7.751	9.562
5	5	3	8.294	8.208	31.947
5	5	5	7.920	11.886	32.642

$$E_{\triangle}(J_{MROSS}) = \sum_{j<j'}^{c} \sum_{r=1}^{k} \sum_{r'=1}^{k} \int_{-\infty}^{\infty} F_{\tilde{X}_{(r)}^{(r)}}(t + (j'-j)\triangle) f_{\tilde{X}_{(r')}^{(r')}}(t)dt. \tag{3.1}$$

The derivative of (3.1) evaluated at $\triangle = 0$ is

$$\begin{aligned}
&\frac{\partial}{\partial\triangle} E_{\triangle}(J_{MROSS}) \mid_{\triangle=0} \\
&= \sum_{j<j'}^{c} (j'-j) \sum_{r=1}^{k} \sum_{r'=1}^{k} \int_{-\infty}^{\infty} f_{\tilde{X}_{(r)}^{(r)}}(t + (j'-j)\triangle) f_{\tilde{X}_{(r')}^{(r')}}(t)dt \mid_{\triangle=0} \\
&= \sum_{j<j'}^{c} (j'-j) \sum_{r=1}^{k} \sum_{r'=1}^{k} \int_{-\infty}^{\infty} \left\{\frac{n!}{(m-1)!(n-m)!}\right\}^2 \left\{\frac{k!}{(r-1)!(k-r)!}\right\}^2 \\
&\times \left\{\frac{k!}{(r'-1)!(k-r')!}\right\}^2 \left\{F_{(r)}^{(r)}(t)\right\}^{(m-1)} \left\{(1 - F_{(r)}^{(r)}(t)\right\}^{(n-m)} \\
&\times \left\{F_{(r')}^{(r')}(t)\right\}^{(m-1)} \left\{1 - F_{(r')}^{(r')}(t)\right\}^{(n-m)} \left\{F_{(r)}(t)\right\}^{(r-1)} \\
&\times \left\{1 - F_{(r)}(t)\right\}^{(k-r)} \left\{F_{(r')}(t)\right\}^{(r'-1)} \left\{1 - F_{(r')}(t)\right\}^{(k-r')} \\
&\times \left\{F(t)\right\}^{(r+r'-1)} \left\{1 - F(t)\right\}^{(2k-r-r')} f^2(t)dt.
\end{aligned} \tag{3.2}$$

The derivatives of $E_{\triangle}J_{(RSS)}$ and $E_{\triangle}J_{(SRS)}$ evaluated at $\theta = 0$ are given in Kim *et al.* (2000).

$$\begin{aligned}
\frac{\partial}{\partial\triangle} E_{\triangle}(J_{SRS}) \mid_{\triangle=0} &= \sum_{j<j'}^{c} (j'-j) k^2 \int_{-\infty}^{\infty} \left\{\frac{n!}{(m-1)!(n-m)!}\right\}^2 \\
&\times \left\{F(t)\right\}^{2(m-1)} \left\{1 - F(t)\right\}^{2(n-m)} f^2(t)dt,
\end{aligned} \tag{3.3}$$

$$\begin{aligned}
\frac{\partial}{\partial\triangle} E_{\triangle}(J_{RSS}) \mid_{\triangle=0} &= \sum_{j<j'}^{c} (j'-j) \sum_{r=1}^{k} \sum_{r'=1}^{k} \int_{-\infty}^{\infty} \left\{\frac{n!}{(m-1)!(n-m)!}\right\}^2 \\
&\times \left\{\frac{k!}{(r-1)!(k-r)!}\right\} \left\{\frac{k!}{(r'-1)!(k-r')!}\right\} \\
&\times \left\{F_{(r)}(t)\right\}^{(m-1)} \left\{1 - F_{(r)}(t)\right\}^{(n-m)}
\end{aligned}$$

$$\times \left\{F_{(r')}(t)\right\}^{(m-1)} \left\{1 - F_{(r')}(t)\right\}^{(n-m)}$$
$$\times \left\{F(t)\right\}^{(r+r'-2)} \left\{1 - F(t)\right\}^{(2k-r-r')} f^2(t)dt, \qquad (3.4)$$

where

$$J_{SRS} = \sum_{j<j'}^{c} \sum_{r=1}^{k} \sum_{r'=1}^{k} \Psi\left(\tilde{X}_{j'r'} - \tilde{X}_{jr}\right),$$
$$J_{RSS} = \sum_{j<j'}^{c} \sum_{r=1}^{k} \sum_{r'=1}^{k} \Psi\left(\tilde{X}_{j'(r')} - \tilde{X}_{j(r)}\right).$$

From (3.2), (3.3) and (3.4), we have the following efficacies of J_{MROSS}, J_{RSS} and J_{SRS} by the definition 5.2.14 of Randles and Wolfe (1979).

$$\text{eff}(J_{MROSS}) = \lim_{k\to\infty} \frac{\frac{\partial}{\partial\triangle} E_{\triangle}(J_{MROSS}) \mid_{\triangle=0}}{\sqrt{kc\,\text{Var}_0(J_{MROSS})}}, \qquad (3.5)$$
$$\text{eff}(J_{RSS}) = \lim_{k\to\infty} \frac{\frac{\partial}{\partial\triangle} E_{\triangle}(J_{RSS}) \mid_{\triangle=0}}{\sqrt{kc\,\text{Var}_0(J_{RSS})}}, \qquad (3.6)$$
$$\text{eff}(J_{SRS}) = \lim_{k\to\infty} \frac{\frac{\partial}{\partial\triangle} E_{\triangle}(J_{SRS}) \mid_{\triangle=0}}{\sqrt{kc\,\text{Var}_0(J_{SRS})}}, \qquad (3.7)$$

where $\text{Var}_0(J_{SRS}) = ck^2(c-1)(2kc+2k+3)/72$ and $\text{Var}_0(J_{RSS})$ is given in Kim *et al.* (2000).

The efficacy of P_{ROSS} from Kim *et al.* (2006) is

$$\text{eff}(P_{ROSS}) = \sqrt{c(c-1)}\left[\int_{-\infty}^{\infty} \left\{f_{(h)}^{(h)}(t)\right\}^2 dt\right], \quad h = \frac{k+1}{2}. \qquad (3.8)$$

Then, the ARE of J_{MROSS} with respect to J_{RSS}, J_{SRS} and P_{ROSS} are

$$\text{ARE}(J_{MROSS}, J_{RSS}) = \left\{\frac{\text{eff}(J_{MROSS})}{\text{eff}(J_{RSS})}\right\}^2, \qquad (3.9)$$
$$\text{ARE}(J_{MROSS}, J_{SRS}) = \left\{\frac{\text{eff}(J_{MROSS})}{\text{eff}(J_{SRS})}\right\}^2, \qquad (3.10)$$
$$\text{ARE}(J_{MROSS}, P_{ROSS}) = \left\{\frac{\text{eff}(J_{MROSS})}{\text{eff}(P_{ROSS})}\right\}^2. \qquad (3.11)$$

With the aid of computer, (3.9), (3.10) and (3.11) can be evaluated. We provide the computed results in Table 3.2 when the underlying distribution is uniform. The values of ARE we consider here are very good, in addition the above three AREs are greater than 1 and increase as k, n do.

Table 3.2: Variance of J_{MROSS} for some specified c, k, n.

c	k	n	$\mathrm{Var}_0(J_{MROSS})$	c	k	n	$\mathrm{Var}_0(J_{MROSS})$
3	3	3	3.2852	5	3	3	17.6228
3	3	5	3.2507	5	3	5	17.5023
3	5	3	5.5609	5	5	3	29.6809
3	5	5	5.4353	5	5	5	29.2304

Table 4.1: Empirical powers of tests($c = 3$, $n = 3$, $\alpha = 0.05$)

a) Uniform Distribution

k	Statistic	θ					
		0.0	0.2	0.4	0.6	0.8	1.0
3	J_{SRS}	0.052	0.173	0.365	0.601	0.817	0.942
	J_{RSS}	0.050	0.267	0.628	0.888	0.983	0.999
	P_{ROSS}	0.030	0.134	0.358	0.620	0.822	0.934
	J_{MROSS}	0.049	0.485	0.872	0.986	0.999	1.000
5	J_{SRS}	0.050	0.146	0.332	0.581	0.807	0.942
	J_{RSS}	0.048	0.326	0.770	0.974	0.999	1.000
	P_{ROSS}	0.032	0.234	0.616	0.893	0.981	0.998
	J_{MROSS}	0.057	0.788	0.992	1.000	1.000	1.000

b) Normal Distribution

k	Statistic	θ					
		0.0	0.4	0.8	1.2	1.6	2.0
3	J_{SRS}	0.050	0.108	0.212	0.401	0.564	0.717
	J_{RSS}	0.052	0.173	0.358	0.591	0.786	0.912
	P_{ROSS}	0.033	0.102	0.233	0.412	0.617	0.768
	J_{MROSS}	0.049	0.162	0.364	0.605	0.812	0.935
5	J_{SRS}	0.038	0.101	0.211	0.364	0.539	0.705
	J_{RSS}	0.041	0.177	0.433	0.714	0.903	0.979
	P_{ROSS}	0.032	0.119	0.300	0.537	0.749	0.888
	J_{MROSS}	0.058	0.271	0.609	0.877	0.978	0.998

c) Logistic Distribution

k	Statistic	θ					
		0.0	0.4	0.8	1.2	1.6	2.0
3	J_{SRS}	0.050	0.098	0.150	0.229	0.316	0.412
	J_{RSS}	0.050	0.106	0.191	0.321	0.450	0.579
	P_{ROSS}	0.030	0.066	0.128	0.212	0.317	0.429
	J_{MROSS}	0.053	0.104	0.180	0.274	0.390	0.527
5	J_{SRS}	0.042	0.077	0.123	0.184	0.266	0.358
	J_{RSS}	0.047	0.112	0.216	0.369	0.545	0.702
	P_{ROSS}	0.032	0.099	0.220	0.403	0.591	0.763
	J_{MROSS}	0.057	0.152	0.297	0.485	0.672	0.816

d) Double Exponential Distribution

k	Statistic	θ					
		0.0	0.4	0.8	1.2	1.6	2.0
3	J_{SRS}	0.055	0.137	0.248	0.382	0.527	0.660
	J_{RSS}	0.051	0.156	0.310	0.505	0.682	0.811
	P_{ROSS}	0.030	0.105	0.252	0.432	0.613	0.760
	J_{MROSS}	0.047	0.122	0.237	0.391	0.543	0.685
5	J_{SRS}	0.044	0.103	0.208	0.340	0.498	0.646
	J_{RSS}	0.045	0.159	0.372	0.612	0.802	0.922
	P_{ROSS}	0.032	0.191	0.496	0.774	0.923	0.976
	J_{MROSS}	0.054	0.195	0.424	0.685	0.869	0.957

e) Cauchy Distribution

k	Statistic	θ					
		0.0	0.6	1.2	1.8	2.4	3.0
3	J_{SRS}	0.061	0.162	0.329	0.526	0.693	0.812
	J_{RSS}	0.037	0.129	0.297	0.508	0.687	0.815
	P_{ROSS}	0.040	0.178	0.447	0.706	0.871	0.949
	J_{MROSS}	0.050	0.108	0.180	0.249	0.322	0.403
5	J_{SRS}	0.046	0.109	0.208	0.345	0.493	0.629
	J_{RSS}	0.050	0.154	0.347	0.577	0.764	0.880
	P_{ROSS}	0.032	0.145	0.363	0.622	0.818	0.924
	J_{MROSS}	0.066	0.168	0.329	0.527	0.719	0.853

4. Power Comparison

Now we compare the empirical power of the proposed test statistic J_{MROSS} with P_{ROSS}, J_{RSS} and J_{SRS}. The powers are obtained from the five underlying distribution, such as uniform($U(0,\ 1)$), normal, double exponential, logistic and Cauchy distributions. Except for uniform, the others have scale parameter 1.

Through the simulation study we consider the sample size $k = 3, 5$, the cycle size $n = 3, 5$, the population size $c = 3, 5$. The location parameter has $\theta = 0.0$ (0.2) 1.0 or $\theta = 0.0$ (0.4) 2.0. For Cauchy distribution, $\theta = 0.0$ (0.6) 3.0. The j^{th} population has $\theta_j = \theta_0 + (j-1)\times\triangle$, $\triangle = \theta/\sqrt{kc}$, $j = 1, \ldots, c$. Simulation size is taken as 10,000. The critical value is computed from the asymptotic distribution of the proposed statistic in Theorem 3.1. The empirical power at $\theta = 0.0$ is the empirical Type I error of the test statistic.

From Table 4.1, J_{MROSS} has the best power among the test statistics when the underlying distribution is uniform or normal distribution. Except for the case $c = 3$, $n = 3$, $k = 3$, J_{MROSS} is also superior to J_{RSS} on the other cases under logistic distribution.

If the underlying distributions have heavy tails and the sample size k is small, the power of J_{MROSS} is strangely lower than that of P_{ROSS}, J_{RSS}. As sample size k of J_{MROSS} increases, the power of the proposed test statistic increases, and is similar or superior to those of J_{RSS}.

Appendix: The Variance of the Proposed Test Statistic

$$
\begin{aligned}
\mathrm{Var}_0(J_{MROSS}) &= \mathrm{Var}_0\left(\sum_{j<j'}^{c}\sum_{r=1}^{k}\sum_{r'=1}^{k}\Psi\left(\tilde{X}_{j'(r')}^{(r')}-\tilde{X}_{j(r)}^{(r)}\right)\right) \\
&= \sum_{j<j'}^{c}\sum_{r=1}^{k}\sum_{r'=1}^{k}\mathrm{Var}\left(\Psi\left(\tilde{X}_{j'(r')}^{(r')}-\tilde{X}_{j(r)}^{(r)}\right)\right) \\
&+ \sum_{j_1\neq j_2<j'}^{c}\sum_{r=1}^{k}\sum_{r'=1}^{k}\mathrm{Cov}\left(\Psi\left(\tilde{X}_{j'(r')}^{(r')}-\tilde{X}_{1(r)}^{(r)}\right),\ \Psi\left(\tilde{X}_{j'(r')}^{(r')}-\tilde{X}_{2(r)}^{(r)}\right)\right) \\
&+ \sum_{j<j_1'\neq j_2'}^{c}\sum_{r=1}^{k}\sum_{r'=1}^{k}\mathrm{Cov}\left(\Psi\left(\tilde{X}_{1(r')}^{(r')}-\tilde{X}_{j(r)}^{(r)}\right),\ \Psi\left(\tilde{X}_{2(r')}^{(r')}-\tilde{X}_{j(r)}^{(r)}\right)\right)
\end{aligned}
$$

$$+\sum_{j<j'}^{c}\sum_{r_1\neq r_2}^{k}\sum_{r'=1}^{k}\mathrm{Cov}\left(\Psi\left(\tilde{X}_{j'(r')}^{(r')}-\tilde{X}_{j(r_1)}^{(r_1)}\right),\ \Psi\left(\tilde{X}_{j'(r')}^{(r')}-\tilde{X}_{j(r_2)}^{(r_2)}\right)\right)$$

$$+\sum_{j<j'}^{c}\sum_{r=1}^{k}\sum_{r_1'\neq r_2'}^{k}\mathrm{Cov}\left(\Psi\left(\tilde{X}_{j'(r_1')}^{(r_1')}-\tilde{X}_{j(r)}^{(r)}\right),\ \Psi\left(\tilde{X}_{j'(r_2')}^{(r_2')}-\tilde{X}_{j(r)}^{(r)}\right)\right)$$

$$=\sum_{j<j'}^{c}\sum_{r=1}^{k}\sum_{r'=1}^{k}P\left(\tilde{X}_{j'(r')}^{(r')}>\tilde{X}_{j(r)}^{(r)}\right)\left\{1-P\left(\tilde{X}_{j'(r')}^{(r')}>\tilde{X}_{j(r)}^{(r)}\right)\right\}$$

$$+\sum_{j_1\neq j_2<j'}^{c}\sum_{r=1}^{k}\sum_{r'=1}^{k}\left\{P\left(\max\left(\tilde{X}_{1(r)}^{(r)},\ \tilde{X}_{2(r)}^{(r)}\right)<\tilde{X}_{j'(r')}^{(r')}\right)-P\left(\tilde{X}_{j'(r')}^{(r')}>\tilde{X}_{1(r)}^{(r)}\right)^2\right\}$$

$$+\sum_{j<j_1'\neq j_2'}^{c}\sum_{r=1}^{k}\sum_{r'=1}^{k}\left\{P\left(\min\left(\tilde{X}_{1(r')}^{(r')},\ \tilde{X}_{2(r')}^{(r')}\right)>\tilde{X}_{j(r)}^{(r)}\right)-P\left(\tilde{X}_{1(r')}^{(r')}>\tilde{X}_{j(r)}^{(r)}\right)^2\right\}$$

$$+\sum_{j<j'}^{c}\sum_{r_1\neq r_2}^{k}\sum_{r'=1}^{k}\left\{P\left(\max\left(\tilde{X}_{j(r_1)}^{(r_1)},\ \tilde{X}_{j(r_2)}^{(r_2)}\right)<\tilde{X}_{j'(r')}^{(r')}\right)\right.$$

$$\left.-P\left(\tilde{X}_{j(r_1)}^{(r_1)}<\tilde{X}_{j'(r')}^{(r')}\right)P\left(\tilde{X}_{j(r_2)}^{(r_2)}<\tilde{X}_{j'(r')}^{(r')}\right)\right\}$$

$$+\sum_{j<j'}^{c}\sum_{r=1}^{k}\sum_{r_1'\neq r_2'}^{k}\left\{P\left(\min\left(\tilde{X}_{j'(r_1')}^{(r_1')},\ \tilde{X}_{j'(r_2')}^{(r_2')}\right)>\tilde{X}_{j(r)}^{(r)}\right)\right.$$

$$\left.-P\left(\tilde{X}_{j(r)}^{(r)}<\tilde{X}_{j'(r_1')}^{(r_1')}\right)P\left(\tilde{X}_{j(r)}^{(r)}<\tilde{X}_{j'(r_2')}^{(r_2')}\right)\right\}$$

$$=\sum_{j<j'}^{c}\left[\sum_{r=1}^{k}\sum_{r'=1}^{k}\int_{-\infty}^{\infty}F_{\tilde{X}_{(r)}^{(r)}}(x)dF_{\tilde{X}_{(r')}^{(r')}}(x)-\sum_{r=1}^{k}\sum_{r'=1}^{k}\left\{\int_{-\infty}^{\infty}F_{\tilde{X}_{(r)}^{(r)}}(x)dF_{\tilde{X}_{(r')}^{(r')}}(x)\right\}^2\right]$$

$$+\sum_{j_1\neq j_2<j'}^{c}\left[\sum_{r=1}^{k}\sum_{r'=1}^{k}\int_{-\infty}^{\infty}\left\{F_{\tilde{X}_{(r)}^{(r)}}(x)\right\}^2dF_{\tilde{X}_{(r')}^{(r')}}(x)\right.$$

$$\left.-\sum_{r=1}^{k}\sum_{r'=1}^{k}\left\{\int_{-\infty}^{\infty}F_{\tilde{X}_{(r)}^{(r)}}(x)dF_{\tilde{X}_{(r')}^{(r')}}(x)\right\}^2\right]$$

$$+\sum_{j<j_1'\neq j_2'}^{c}\left[\sum_{r=1}^{k}\sum_{r'=1}^{k}\int_{-\infty}^{\infty}\left\{1-F_{\tilde{X}_{(r')}^{(r')}}(x)\right\}^2dF_{\tilde{X}_{(r)}^{(r)}}(x)\right.$$

$$\left.-\sum_{r=1}^{k}\sum_{r'=1}^{k}\left\{\int_{-\infty}^{\infty}F_{\tilde{X}_{(r)}^{(r)}}(x)dF_{\tilde{X}_{(r')}^{(r')}}(x)\right\}^2\right]$$

$$+\sum_{j<j'}^{c}\left\{\sum_{r_1\neq r_2}^{k}\sum_{r'=1}^{k}\int_{-\infty}^{\infty}F_{\tilde{X}_{(r_1)}^{(r_1)}}(x)F_{\tilde{X}_{(r_2)}^{(r_2)}}(x)dF_{\tilde{X}_{(r_1)}^{(r_1)}}(x)\right.$$

$$\left.-\sum_{r_1\neq r_2}^{k}\sum_{r'=1}^{k}\int_{-\infty}^{\infty}F_{\tilde{X}_{(r_1)}^{(r_1)}}(x)dF_{\tilde{X}_{(r')}^{(r')}}(x)\int_{-\infty}^{\infty}F_{\tilde{X}_{(r_2)}^{(r_2)}}(x)dF_{\tilde{X}_{(r')}^{(r')}}(x)\right\}$$

$$
\begin{aligned}
&+\sum_{j<j'}^{c}\Bigg[\sum_{r=1}^{k}\sum_{r_1'\neq r_2'}^{k}\int_{-\infty}^{\infty}\Big\{1-F_{\tilde{X}^{(r_1')}_{(r_1')}}(x)\Big\}\Big\{1-F_{\tilde{X}^{(r_2')}_{(r_2')}}(x)\Big\}dF_{\tilde{X}^{(r)}_{(r)}}(x)\\
&\quad-\sum_{r=1}^{k}\sum_{r_1'\neq r_2'}^{k}\int_{-\infty}^{\infty}\Big\{1-F_{\tilde{X}^{(r_1')}_{(r_1')}}(x)\Big\}dF_{\tilde{X}^{(r)}_{(r)}}(x)\times\int_{-\infty}^{\infty}\Big\{1-F_{\tilde{X}^{(r_2')}_{(r_2')}}(x)\Big\}dF_{\tilde{X}^{(r)}_{(r)}}(x)\Bigg]\\
&=\sum_{j<j'}^{c}\left[\frac{k^2}{2}-\sum_{r=1}^{k}\sum_{r'=1}^{k}\left\{\int_{-\infty}^{\infty}F_{\tilde{X}^{(r)}_{(r)}}(x)dF_{\tilde{X}^{(r')}_{(r')}}(x)\right\}^2\right]\\
&\quad+2\sum_{j_1\neq j_2<j'}^{c}\Bigg[\sum_{r=1}^{k}\sum_{r'=1}^{k}\int_{-\infty}^{\infty}\Big\{F_{\tilde{X}^{(r)}_{(r)}}(x)\Big\}^2dF_{\tilde{X}^{(r')}_{(r')}}(x)\\
&\quad-\sum_{r=1}^{k}\sum_{r'=1}^{k}\left\{\int_{-\infty}^{\infty}F_{\tilde{X}^{(r)}_{(r)}}(x)dF_{\tilde{X}^{(r')}_{(r')}}(x)\right\}^2\Bigg]\\
&\quad+2\sum_{j<j'}^{c}\Bigg[\sum_{r_1=1}^{k}\sum_{r_2=1}^{k}\sum_{r'=1}^{k}\int_{-\infty}^{\infty}F_{\tilde{X}^{(r_1)}_{(r_1)}}(x)F_{\tilde{X}^{(r_2)}_{(r_2)}}(x)dF_{\tilde{X}^{(r')}_{(r')}}(x)\\
&\quad-\sum_{r=1}^{k}\sum_{r'=1}^{k}\int_{-\infty}^{\infty}\Big\{F_{\tilde{X}^{(r)}_{(r)}}(x)\Big\}^2dF_{\tilde{X}^{(r')}_{(r')}}(x)\\
&\quad-\sum_{r_1=1}^{k}\sum_{r_2=1}^{k}\sum_{r'=1}^{k}\int_{-\infty}^{\infty}F_{\tilde{X}^{(r_1)}_{(r_1)}}(x)dF_{\tilde{X}^{(r')}_{(r')}}(x)\int_{-\infty}^{\infty}F_{\tilde{X}^{(r_2)}_{(r_2)}}(x)dF_{\tilde{X}^{(r')}_{(r')}}(x)\\
&\quad+\sum_{r=1}^{k}\sum_{r'=1}^{k}\left\{\int_{-\infty}^{\infty}F_{\tilde{X}^{(r)}_{(r)}}(x)dF_{\tilde{X}^{(r')}_{(r')}}(x)\right\}^2\Bigg]\\
&=\frac{c(c-1)}{12}\Bigg[3k^2-2(4c-11)\sum_{r=1}^{k}\sum_{r'=1}^{k}\left\{\int_{-\infty}^{\infty}F_{\tilde{X}^{(r)}_{(r)}}(x)dF_{\tilde{X}^{(r')}_{(r')}}(x)\right\}^2\\
&\quad+4(2c-7)\sum_{r=1}^{k}\sum_{r'=1}^{k}\int_{-\infty}^{\infty}F_{\tilde{X}^{(r)}_{(r)}}(x)^2dF_{\tilde{X}^{(r')}_{(r')}}(x)\\
&\quad+12\sum_{r_1=1}^{k}\sum_{r_2=1}^{k}\sum_{r'=1}^{k}\Bigg\{\int_{-\infty}^{\infty}F_{\tilde{X}^{(r_1)}_{(r_1)}}(x)F_{\tilde{X}^{(r_2)}_{(r_2)}}(x)dF_{\tilde{X}^{(r')}_{(r')}}(x)\\
&\quad-\int_{-\infty}^{\infty}F_{\tilde{X}^{(r_1)}_{(r_1)}}(x)dF_{\tilde{X}^{(r')}_{(r')}}(x)\times\int_{-\infty}^{\infty}F_{\tilde{X}^{(r_2)}_{(r_2)}}(x)dF_{\tilde{X}^{(r')}_{(r')}}(x)\Bigg\}\Bigg]
\end{aligned}
$$

References

Bohn, L. L. and Wolfe, D. A. (1992). Nonparametric two-sample procedures for ranked-set samples data, *Journal of the American Statistical Association,* **87**, 552–561.

Bohn, L. L. and Wolfe, D. A. (1994). The effect of imperfect judgement rankings on properties of procedures based on the ranked-set samples analog of the Mann-Whitney-Wilcoxon statistic, *Journal of the American Statistical Association,* **89**, 168–176.

Dell, T. R. and Clutter, J. L. (1972). Ranked-set sampling theory with order statistics background, *Biometrics,* **28**, 545–553.

Hettmansperger, T. P. (1995). The ranked-set sample sign test, *Journal of Nonparametric Statistics,* **4**, 263–270.

Hoeffding, W. (1948). A class of statistics with asymptotically normal distribution, *The Annals of Mathematical Statistics,* **19**, 293–325.

Jonckheere, A. R. (1954). A distribution-free k-sample test against ordered alternatives, *Biometrika,* **41**, 133–145.

Kim, D. H. and Kim, H. G. (2003). Sign test using ranked ordering-set sampling, *Journal of Nonparametric Statistics,* **15**, 303–309.

Kim, D. H., Kim, H. G. and Park, H. K. (2000). Nonparametric test for ordered alternatives on multiple ranked-set samples, *The Korean Communications in Statistics,* **7**, 563–573.

Kim, D. H., Kim, H. G. and You, S. H. (2006). Nonparametric test for ordered alternatives on ranked ordering-set samples, *Journal of the Korean Data Analysis Society,* **8**, 459–467.

Koti, K. M. and Babu, G. J. (1996). Sign test for ranked-set sampling, *Communications in Statistics - Theory and Methods,* **25**, 1617–1630.

McIntyre, G. A. (1952). A method for unbiased selective sampling using ranked sets, *Australian Journal of Agricultural Research,* **3**, 385–390.

Öztürk, Ö. (1999). Two-sample inference based on one-sample ranked set sample sign statistics, *Journal of Nonparametric Statistics,* **10**, 197–212.

Randles, R. H. and Wolfe, D. A. (1979). *Introduction to the theory of nonparametric statistics*, John Wiley & Sons, New York.

Stokes, S. L. (1977). Ranked set sampling with concomitant variables, *Communications in Statistics - Theory and Methods,* **6**, 1207–1211.

Stokes, S. L. and Sager, T. W. (1988). Characterization of a ranked-set sample with application to estimating distribution functions, *Journal of the American Statistical Association,* **83**, 374–381.

Takahasi, K. and Wakimoto, K. (1968). On unbiased estimates of the population mean based on the sample stratified by means of ordering, *Annals of the Institute of Statistical Mathematics,* **20**, 1–31.

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