Bull. Korean Math. Soc. **45** (2008), No. 4, pp. 615–620

NEW CHARACTERIZATION OF THE CLASS OF STARLIKE FUNCTIONS

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Abstract. For analytic functions $f(z)$ in the open unit disc $\mathbb{D}$, an operator $\mathcal{N}_\alpha(f(z))$ relating with starlike functions is introduced. The object of the present paper is to discuss some properties of the operator $\mathcal{N}_\alpha(f(z))$.

1. Introduction

Let Ω be the class of functions $w(z)$ which are regular in the open unit disc $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ and satisfy $w(0) = 0$ and $|w(z)| < 1$ for all $z \in \mathbb{D}$.

For $w(z) \in \Omega$, let $\mathcal{P}$ denote the family of functions $p(z)$ of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$$

that are regular in $\mathbb{D}$ satisfying

$$p(z) = \frac{1 + w(z)}{1 - w(z)} \qquad (z \in \mathbb{D}).$$

A function $p(z) \in \mathcal{P}$ is called as Carathéodory function in $\mathbb{D}$ (*cf.* [1], [2]).

Further, let $\mathcal{A}$ be the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in $\mathbb{D}$. A function $f(z) \in \mathcal{A}$ is said to be starlike of order α in $\mathbb{D}$ if it satisfies

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \qquad (z \in \mathbb{D})$$

for some α $(0 \leqq \alpha < 1)$. We denote by $\mathcal{S}^*(\alpha)$ the subclass of $\mathcal{A}$ consisting of all starlike functions of order α in $\mathbb{D}$.

For $h(z)$ and $s(z)$ in the class $\mathcal{A}$, $h(z)$ is said to be subordinate to $s(z)$, written by $h(z) \prec s(z)$, if there exists $w(z) \in \Omega$ such that $h(z) = s(w(z))$

Received March 7, 2007; Revised October 15, 2007.
2000 *Mathematics Subject Classification.* Primary 30C45.
Key words and phrases. Carathéodory function, starlike function, operator.

for $z \in \mathbb{D}$. In particular, if $s(z)$ is univalent in $\mathbb{D}$, then the subordination $h(z) \prec s(z)$ is equivalent to $h(0) = s(0)$ and $h(\mathbb{D}) \subset s(\mathbb{D})$.

For $f(z) \in \mathcal{A}$, let us introduce the operator $\mathcal{N}_\alpha(f(z))$ defined by

$$\mathcal{N}_\alpha(f(z)) = z - 2\alpha \frac{f(z)}{f'(z)} \quad (z \in \mathbb{D})$$

for some α $(0 \leqq \alpha < 1)$. If $\alpha = \frac{1}{2}$, the operator

$$\mathcal{N}_{\frac{1}{2}}(f(z)) = z - \frac{f(z)}{f'(z)} \quad (z \in \mathbb{D})$$

is said to be Newtonian.

2. Properties of $\mathcal{N}_\alpha(f(z))$

To discuss some properties of the operator $\mathcal{N}_\alpha(f(z))$, we have to recall here the following lemma for Carathéodory functions (see [3]).

Lemma 2.1. *If $p(z) \in \mathcal{P}$, then*

$$\frac{1-|z|}{1+|z|} \leqq |p(z)| \leqq \frac{1+|z|}{1-|z|}$$

and

$$|p'(z)| \leqq \frac{2}{(1-|z|)^2}$$

for $z \in \mathbb{D}$.

Now we derive

Theorem 2.2. *If $f(z) \in \mathcal{A}$ satisfies*

$$\left| \left(\frac{\mathcal{N}_\alpha(f(z))}{z} \right)' \right| \leqq \beta |z|^\gamma \quad (z \in \mathbb{D})$$

for some β and γ such that $0 < \beta \leqq \gamma + 1$ and $\gamma \geqq 0$, then $f(z) \in \mathcal{S}^*(\alpha)$.

Proof. For $f(z) \in \mathcal{A}$, we define the function $w(z)$ by

$$w(z) = z\left(1 - 2\alpha \frac{f(z)}{zf'(z)}\right) = \mathcal{N}_\alpha(f(z)) \quad (z \in \mathbb{D}).$$

Then $w(z)$ is regular in $\mathbb{D}$ and $w(0) = 0$.

Since $\mathcal{N}_\alpha(f(z)) = w(z)$, the condition of the theorem gives us that

$$\left| \left(\frac{\mathcal{N}_\alpha(f(z))}{z} \right)' \right| = \left| \left(\frac{w(z)}{z} \right)' \right| \leqq \beta |z|^\gamma \quad (z \in \mathbb{D}).$$

It follows from the above that

$$\begin{aligned}\left|\frac{w(z)}{z}\right| &= \left|\int_0^z \left(\frac{w(t)}{t}\right)' dt\right| \\ &\leqq \int_0^{|z|} \beta|t|^\gamma d|t| = \frac{\beta}{\gamma+1}|z|^{\gamma+1}.\end{aligned}$$

This implies that

$$\left|\frac{w(z)}{z}\right| \leqq \frac{\beta}{\gamma+1}|z|^{\gamma+1} < 1 \quad (z \in \mathbb{D})$$

because $0 < \beta \leqq \gamma + 1$ and $\gamma \geqq 0$. Therefore, by the definition for $w(z)$, we conclude that

$$\left|2\alpha\frac{f(z)}{zf'(z)} - 1\right| < 1 \quad (z \in \mathbb{D}),$$

which is equivalent to

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in \mathbb{D}).$$

□

Next, we show

Theorem 2.3. *If $f(z) \in \mathcal{S}^*\left(\frac{1}{2}\right)$, then*

$$\left|\mathcal{N}_{\frac{1}{2}}(f(z)) - \frac{1+|z|^2}{2}\right| \geqq \frac{1-|z|^2}{2} \quad (z \in \mathbb{D})$$

and

$$\left|\mathcal{N}_{\frac{1}{2}}(f(z)) + \frac{1+|z|^2}{2}\right| \geqq \frac{1-|z|^2}{2} \quad (z \in \mathbb{D}).$$

Furthermore,

$$\left|(\mathcal{N}_{\frac{1}{2}}(f(z)))'\right| \leqq \frac{2|z|(1+|z|^2)}{(1-|z|)^2} \quad (z \in \mathbb{D}).$$

Proof. For $f(z) \in \mathcal{S}^*\left(\frac{1}{2}\right)$ let us define the function $w(z)$ by

$$w(z) = 1 - \frac{f(z)}{zf'(z)} \quad (z \in \mathbb{D}).$$

Then we see that $w(0) = 0$ and

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) = \operatorname{Re}\left(\frac{1}{1-w(z)}\right) > \frac{1}{2} \quad (z \in \mathbb{D}).$$

This implies that $|w(z)| < 1$ $(z \in \mathbb{D})$ so that $w(z) \in \Omega$. If we define the function $p(z)$ by

$$p(z) = \frac{1+w(z)}{1-w(z)},$$

then $p(z) \in \mathcal{P}$. Applying Lemma 2.1 for $p(z)$, we have that

$$\frac{1-|z|}{1+|z|} \leqq \left|\frac{1+w(z)}{1-w(z)}\right| \leqq \frac{1+|z|}{1-|z|} \quad (z \in \mathbb{D}).$$

From the right hand inequality, we see that

$$\left|\mathcal{N}_{\frac{1}{2}}(f(z)) - \frac{1+|z|^2}{2}\right| \geqq \frac{1-|z|^2}{2} \quad (z \in \mathbb{D}).$$

Also, from the left hand inequality, we have that

$$\left|\mathcal{N}_{\frac{1}{2}}(f(z)) + \frac{1+|z|^2}{2}\right| \geqq \frac{1-|z|^2}{2} \quad (z \in \mathbb{D}).$$

Further, Lemma 2.1 implies that

$$|p'(z)| = \left|\frac{2w'(z)}{(1-w(z))^2}\right| \leqq \frac{2}{(1-|z|)^2} \quad (z \in \mathbb{D}).$$

Since $|w(z)| \leqq |z| < 1$ $(z \in \mathbb{D})$, we obtain that

$$|w'(z)| \leqq \left(\frac{1+|z|}{1-|z|}\right)^2 \quad (z \in \mathbb{D}).$$

Thus we see that

$$\begin{aligned}\left|(\mathcal{N}_{\frac{1}{2}}(f(z)))'\right| &\leqq |w(z)| + |zw'(z)| \\ &\leqq |z|(1+|w'(z)|) \\ &\leqq \frac{2|z|(1+|z|^2)}{(1-|z|)^2},\end{aligned}$$

which completes the proof of the theorem. □

To discuss our next problem, we need the following lemma in [2].

Lemma 2.4. *If* $w(z) \in \Omega$, *then*

$$|w'(z)| \leqq \begin{cases} 1 & (0 \leqq |z| \leqq \sqrt{2}-1), \\ \dfrac{(1+|z|^2)^2}{4|z|(1-|z|)^2} & (\sqrt{2}-1 \leqq |z| < 1). \end{cases}$$

Next our result is contained in

Theorem 2.5. *A function* $f(z) \in \mathcal{A}$ *is in the class* $\mathcal{S}^*\left(\frac{1}{2}\right)$ *for* $0 \leqq |z| \leqq \sqrt{2}-1$ *if and only if*

$$\left|(\mathcal{N}_{\frac{1}{2}}(f(z)))'\right| \leqq 2|z| \quad (0 \leqq |z| \leqq \sqrt{2}-1).$$

Proof. Let us put $\alpha = \frac{1}{2}$, $\beta = 2$ and $\gamma = 1$ in Theorem 2.2. Then we see that $\left|(\mathcal{N}_{\frac{1}{2}}(f(z)))'\right| \leqq 2|z|$ implies $f(z) \in \mathcal{S}^*\left(\frac{1}{2}\right)$. Furthermore, from the proof of Theorem 2.3, we have that $f(z) \in \mathcal{S}^*\left(\frac{1}{2}\right)$ implies

$$|(\mathcal{N}_{\frac{1}{2}}(f(z)))'| \leqq |z|(1+|w'(z)|) \leqq 2|z|$$

for $0 \leqq |z| \leqq \sqrt{2}-1$. □

Finally, in view of Theorem 2.5, we derive the following corollaries.

Corollary 2.6. *If* $f \in \mathcal{S}^*\left(\frac{1}{2}\right)$ *for* $0 \leqq |z| \leqq \sqrt{2}-1$, *then*

$$\left|\frac{f''(z)}{f'(z)}\right| \leqq 2\left|\frac{zf'(z)}{f(z)}\right| \quad (0 \leqq |z| \leqq \sqrt{2}-1).$$

Proof. Note that

$$\mathcal{N}_{\frac{1}{2}}(f(z)) = z - \frac{f(z)}{f'(z)} \quad (z \in \mathbb{D}).$$

Therefore, Theorem 2.5 shows that

$$\left|(\mathcal{N}_{\frac{1}{2}}(f(z)))'\right| = \left|\frac{\frac{f''(z)}{f'(z)}}{\frac{f'(z)}{f(z)}}\right| \leqq 2|z|,$$

that is, that

$$\left|\frac{f''(z)}{f'(z)}\right| \leqq 2\left|\frac{zf'(z)}{f(z)}\right| (0 \leqq |z| \leqq \sqrt{2}-1).$$

□

Corollary 2.7. *If* $f(z) \in \mathcal{S}^*\left(\frac{1}{2}\right)$ *for* $0 \leqq |z| \leqq \sqrt{2}-1$, *then*

$$\left|\frac{f''(z)}{f'(z)}\right| < \frac{2}{1-|z|} \quad (0 \leqq |z| \leqq \sqrt{2}-1).$$

Proof. Since $f(z) \in \mathcal{S}^*\left(\frac{1}{2}\right)$ implies that

$$\frac{zf'(z)}{f(z)} \prec \frac{1}{1-z} \quad (0 \leqq |z| \leqq \sqrt{2}-1),$$

we obtain that

$$\left|\frac{zf'(z)}{f(z)} - \frac{1}{1-|z|^2}\right| < \frac{|z|}{1-|z|^2} \quad (0 \leqq |z| \leqq \sqrt{2}-1).$$

Therefore, applying Corollary 2.6, we complete the proof of the corollary. □

Remark 2.8. We note that the convex function is starlike of order $\frac{1}{2}$. Therefore, all convex functions $f(z) \in \mathcal{A}$ satisfy Corollary 2.6 and Corollary 2.7 for $0 \leqq |z| \leqq \sqrt{2}-1$.

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