

A CHARACTERIZATION OF PARABOLA

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ABSTRACT. In this paper, the parabola will be characterized as the plane curve whose curvature function and support function satisfy a condition, where we define the support function as the distance from origin to the tangent line.

1. Introduction

Prof. Dong-Soo Kim and Prof. Young Ho Kim studied the relationship between the support function and the curvature of the ellipse and the hyperbola in [2]. For the parabola, their method doesn't work. In this note, using our method, we give a characterization of the parabola with the support function.

2. Preliminaries

Let $r(s) : I \rightarrow E^2$ be an unit-speed curve of class C^2 in the Euclidean plane E^2 defined on an open interval I , where s denotes the arc length parameter of curve $r(s)$. The unit tangent vector field α of $r(s)$ is defined by $\alpha(s) = r'(s)$. The unit normal vector field β of $r(s)$ is the unique vector field such that $\{\alpha(s), \beta(s)\}$ gives a right handed orthonormal basis of E^2 for each s . The plane curvature κ or signed curvature of $r(s)$ is given by $\kappa(s) = \langle \alpha'(s), \beta(s) \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the natural inner product in E^2 .

As we know that the best known plane curves are the straight lines and the circles, which are characterized as the plane curves of constant curvature. In [2], the ellipse and the hyperbola have been characterized by the curvature and the support function. In the following, we also use the curvature function and the support function to characterize the parabola.

Let $h(s) = \langle r(s), \beta(s) \rangle$. This means that $h(s)$ is the signed distance from the origin to the tangent line at $r(s)$. The function $h(s)$ is called the support function of the curve $r(s)$. Without loss of generality, we may consider the parabola as $\Gamma : y^2 = 2px + p^2$, where p is a positive constant. In this case we know that $\kappa(s) \neq 0$ and $h(s) \neq 0$.

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3. Main results

Theorem 3.1. *For an unit-speed parametrization $r(s)$ of the parabola $\Gamma : y^2 = 2px + p^2$, the curvature function $\kappa(s)$ and the support function $h(s)$ are related by $\kappa = -\frac{p^2}{8}h^{-3}$.*

Proof. Let $r(s) : I \rightarrow E^2$ be an unit-speed curve. Denote the position vector field $r(s) = (x(s), y(s))$ and s as the arc length parameter of $r(s)$. We may assume that $r(s)$ is positive oriented. Denoting $\gamma(s) = (-p, y(s))$, so the inward unit normal vector $\beta(s)$ is given by

$$\beta(s) = -\frac{\gamma(s)}{|\gamma(s)|},$$

where $|\gamma(s)| = \sqrt{p^2 + y^2}$. Then we get

$$(1) \quad h(s) = \langle r(s), \beta(s) \rangle = -\frac{|\gamma(s)|}{2}.$$

Since $\{\alpha(s), \beta(s)\}$ is a right-handed orthonormal basis for E^2 , we have

$$(2) \quad \alpha(s) = (x'(s), y'(s)) = \frac{1}{|\gamma(s)|}(y, p).$$

Hence the curvature $\kappa(s)$ satisfies

$$(3) \quad \begin{aligned} \kappa(s) = \langle \alpha'(s), \beta(s) \rangle &= -\frac{1}{|\gamma(s)|} \langle \alpha'(s), \gamma(s) \rangle \\ &= \frac{1}{|\gamma(s)|} \langle \alpha(s), \gamma'(s) \rangle. \end{aligned}$$

From (2), (4) and the equation of Γ , we complete the proof of Theorem 3.1. \square

Given an unit-speed curve $r(s) : I \rightarrow E^2$, we denote $\theta = \theta(s)$ the angle between the positive x -axis and the unit normal $\beta(s)$, measured counterclockwise. Then, for $r(s)$ describing the parabola $\Gamma : y^2 = 2px + p^2$ we know that

$$(4) \quad \beta(s) = \frac{1}{2h(s)}(-p, y) = (\cos \theta(s), \sin \theta(s)).$$

Hence we get $2h(\theta) \cos \theta + p = 0$.

We say that an unit-speed curve $r(s)$ satisfies condition (*) if $\kappa = -\frac{p^2}{8}h^{-3}$ for some positive constant p .

Theorem 3.2. *Let $r(s) : I \rightarrow E^2$ be an unit-speed curve of class C^2 in E^2 whose curvature function $\kappa(s)$ dose not vanish identically. Then $r(s)$ satisfies the condition (*) if and only if $r(s)$ is a parabola whose focus is at the origin.*

Proof. We define $\theta(s)$ as in (4). Then θ is continuous on I by dividing it into small intervals over which β does not change much and adjusting θ on each interval and furthermore differentiable. Hence we assume that

$$(5) \quad \begin{cases} \alpha(s) = (\sin \theta(s), -\cos \theta(s)), \\ \beta(s) = (\cos \theta(s), \sin \theta(s)). \end{cases}$$

By differentiating, we obtain

$$(6) \quad \kappa(s) = \theta'(s).$$

Then for a fixed maximal interval I_0 on which $\kappa(s)$ does not vanish, $\theta : I_0 \rightarrow \theta(I_0)$ gives a well-defined reparametrization $r \circ \theta^{-1}$ of $r(s)$. Hereafter we use the parameter θ on $J = \theta(I_0)$ when dealing with the curve $r(s)$. By the definition of h , we have

$$(7) \quad r(\theta) = \lambda(\theta)\alpha(\theta) + h(\theta)\beta(\theta)$$

for a function $\lambda : J \rightarrow R$. If we denote by $\dot{r}(\theta)$ the derivation with respect to θ , then we get $\lambda(\theta) = -\dot{h}(\theta)$, since $\dot{r}(\theta)$ is parallel to $\alpha(\theta)$. Accordingly, we have

$$(8) \quad r(\theta) = -\dot{h}(\theta)\alpha(\theta) + h(\theta)\beta(\theta),$$

and

$$(9) \quad \dot{r}(\theta) = -(\ddot{h}(\theta) + h(\theta))\alpha(\theta).$$

This shows that the curvature is given by

$$(10) \quad \kappa(\theta) = -\frac{1}{\ddot{h}(\theta) + h(\theta)}.$$

Now suppose that $r(\theta)$ satisfies the condition $(*) : \kappa = -\frac{p^2}{8}h^{-3}$, where p is a positive constant. From (10) we see that the support function h satisfies

$$(11) \quad \ddot{h}(\theta) + h(\theta) = \frac{8}{p^2}h^3(\theta)$$

on the interval J . Therefore, we have

$$(12) \quad \ddot{h}(\theta)\dot{h}(\theta) + h(\theta)\dot{h}(\theta) = \frac{8h^3(\theta)\dot{h}(\theta)}{p^2}.$$

Then

$$(13) \quad \dot{h}(\theta) = \pm \sqrt{\frac{4}{p^2}h^4(\theta) - h^2(\theta) + \frac{p^2}{16} + c - \frac{p^2}{16}}.$$

If $c = \frac{p^2}{16}$, we have three particular solutions for the equation (13):

$$(14) \quad \begin{cases} h(\theta) = h_1(\theta) = -\frac{p}{2 \cos \theta}, \\ h(\theta) = h_2(\theta) = \frac{p}{2\sqrt{2}} \tanh \frac{\theta}{\sqrt{2}}, \\ h(\theta) = h_3(\theta) = \frac{p}{2\sqrt{2}} \coth \frac{\theta}{\sqrt{2}}. \end{cases}$$

And otherwise, the solutions of the equation (13) are elliptic functions. It is easy to check that (13) satisfies the condition of Lipschitz. We can also find that the solutions $h_2(\theta)$ and $h_3(\theta)$ are meaningless in our case.

Using (5) and (8), we can determine $r(\theta) = (x(\theta), y(\theta))$ as follows:

$$(15) \quad \begin{cases} x(\theta) = h(\theta) \cos \theta - \dot{h}(\theta) \sin \theta, \\ y(\theta) = h(\theta) \sin \theta + \dot{h}(\theta) \cos \theta. \end{cases}$$

Together with (14) and the Pythagorean identity $\cos^2 \theta + \sin^2 \theta = 1$, a straightforward calculation reveals that $r(\theta)$ satisfies

$$(16) \quad y^2(\theta) = 2px(\theta) + p^2.$$

This completes the proof of Theorem 3.2. \square

References

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