

ON THE NUMERICAL SOLUTION OF NEUTRAL DELAY  
DIFFERENTIAL EQUATIONS USING MULTIQUADRIC  
APPROXIMATION SCHEME

SOLAT KARIMI VANANI AND AZIM AMINATAEI

ABSTRACT. In this paper, the aim is to solve the neutral delay differential equations in the following form using multiquadric approximation scheme,

$$(1) \quad \begin{cases} y'(t) = f(t, y(t), y(t - \tau(t, y(t))), y'(t - \sigma(t, y(t))), & t_1 \leq t \leq t_f, \\ y(t) = \phi(t), & t \leq t_1, \end{cases}$$

where  $f : [t_1, t_f] \times R \times R \times R \rightarrow R$  is a smooth function,  $\tau(t, y(t))$  and  $\sigma(t, y(t))$  are continuous functions on  $[t_1, t_f] \times R$  such that  $t - \tau(t, y(t)) < t_f$  and  $t - \sigma(t, y(t)) < t_f$ . Also  $\phi(t)$  represents the initial function or the initial data. Hence, we present the advantage of using the multiquadric approximation scheme. In the sequel, presented numerical solutions of some experiments, illustrate the high accuracy and the efficiency of the proposed method even where the data points are scattered.

## 1. Introduction

Neutral delay differential equations (NDDEs) are considered as a branch of delay differential equations (DDEs). DDEs arise in many areas of various mathematical modeling. For instance; infectious diseases, population dynamics, physiological and pharmaceutical kinetics and chemical kinetics, the navigational control of ships and aircrafts and control problems. There are many books to the application of DDEs which we can point out to the books of Driver [6], Gopalsamy [9], Halanay [11], Kolmanovskii and Myshkis [16], Kolmanovskii and Nosov [17] and Kuang [18]. Some modelers ignore the ‘lag’ effect and use an ODE model as a substitute for a DDE model. Kuang ([18], p.11) comments under the heading “*Small Delay Can Have Large Effects*”, on the dangers that researchers risk if they ignore lags which they think are small; see also El’sgol’ts and Norkin ([7], p. 243 et seq.). Other modelers replace a scalar DDE by a system of ODE in an attempt to simulate phenomena more appropriately modeled by DDEs. There are inherit qualitative differences between DDEs and finite

---

Received July 25, 2007.

2000 *Mathematics Subject Classification.* 65N, 65L10, 65N55.

*Key words and phrases.* multiquadric approximation scheme, delay differential equations, neutral delay differential equations.

systems of ODEs that make such a strategy risky. Hence, it is bettered to discuss about the DDEs independently and try not to enter the issue of ODEs in the problem which it is a complete DDE problem. Many different methods have been presented for numerical solution of DDEs such that we can point out to the Radau IIA method [10], Bellman's method of steps [3], waveform relaxation method [19], Runge-Kutta method and continuous Runge-Kutta method ([4] and [5]).

In the following of these methods, we are interested to solve NDDEs by the multiquadric (MQ) approximation scheme, because this method of solution works excellently, in particular when the data points are scattered.

The MQ approximation scheme is an useful method for the numerical solution of ordinary and partial differential equations (ODEs and PDEs). It is a grid-free spatial approximation scheme which converges exponentially for the spatial terms of ODEs and PDEs. The MQ approximation scheme was first introduced by Hardy [12] who successfully applied this method for approximating surface and bodies from field data. Hardy [13] has written a detailed review article summarizing its explosive growth in use since it was first introduced. In 1972, Franke [8] published a detailed comparison of 29 different scattered data schemes against analytic problems. Of all the techniques tested, he concluded that MQ performed the best in accuracy, visual appeal, and ease of implementation, even against various finite element schemes.

The organization of this paper is as follows: Section 2 is devoted to introduce the MQ approximation scheme and its preliminary concepts. In Section 3, we apply the MQ approximation scheme to equation (1). In Section 4, we present some experiments in-which their numerical results illustrate the accuracy and efficiency of the proposed method.

## 2. MQ approximation scheme

The basic MQ approximation scheme assumes that any function can be expanded as a finite series of upper hyperboloids,

$$(2) \quad y(t) = \sum_{j=1}^N a_j h(t - t_j), \quad t \in R^d,$$

where  $N$  is the total number of data centers under consideration, and

$$h(t - t_j) = ((t - t_j)^2 + R^2)^{\frac{1}{2}}, \quad j = 1, 2, \dots, N.$$

$(t - t_j)^2$  is the square of Euclidean distance in  $R^d$  and  $R^2 > 0$  is an input shape parameter. Note that, the basis function  $h$  is continuously differentiable, and is a type of spline approximation.

The expansion coefficients  $a_j$  are found by solving a set of full linear equations,

$$(3) \quad y(t_i) = \sum_{j=1}^N a_j h(t_i - t_j), \quad i = 1, 2, \dots, N.$$

Zerroukut et al [24] found that a constant shape parameter ( $R^2$ ) has achieved a better accuracy. Mai-Duy and Tran-Cong [22] have developed new methods based on radial basis function networks (RBFN) for the approximation of both functions and their first and higher derivatives. The so called direct RBFN (DRBFN) and indirect RBFN (IRBFN) methods were studied and it was found that the IRBFN methods yields consistently better results for both functions and derivatives. Recently, Aminataei and Mazarei [1] stated that, in the numerical solution of elliptic PDEs using direct and indirect RBFN methods, the IRBFN method is very accurate than other methods and the error is very small. They have shown that, especially, on one dimensional equations, IRBFN method is more accurate than DRBFN method.

Micchelli [23] proved that MQ belongs to a class of conditionally positive definite RBFN. He showed that the equation (2) is always solvable for distinct points. Madych and Nelson [21] proved that the MQ interpolation always produces a minimal semi-norm error, and that the MQ interpolant and derivative estimates converge exponentially as the density of data centers increases.

In contrast, the MQ interpolant is continuously differentiable over the entire domain of data centers, and the spatial derivative approximations were found to be excellent, most especially in very steep gradient regions where traditional methods fail. This excellent ability to approximate spatial derivatives is due in large part by a slight modification of the original MQ scheme by permitting the shape parameter to vary with the basis function.

Instead of using the expansion in equation (2), we used from ([14, 15, 2]) the following:

$$(4) \quad y(t) = \sum_{j=1}^N a_j h(t - t_j), \quad t \in R^d,$$

where

$$h(t - t_j) = ((t - t_j)^2 + R_j^2)^{\frac{1}{2}}, \quad j = 1, 2, \dots, N,$$

$$R_j^2 = R_{\min}^2 \left( \frac{R_{\max}^2}{R_{\min}^2} \right)^{\left( \frac{j-1}{N-1} \right)}, \quad j = 1, 2, \dots, N,$$

and

$$R_{\min}^2 > 0.$$

$R_{\max}^2$  and  $R_{\min}^2$  are two input parameters chosen so that the ratio

$$\frac{R_{\max}^2}{R_{\min}^2} \cong 10 \text{ to } 10^6.$$

Madych [20] proved that under circumstances very large values of a shape parameter are desirable. The adhoc formula in equation (4) is a way to have at least one very large value of a shape parameter without incurring the onset of severe ill-conditioning problems.

Spatial partial derivatives of any function are formed by differentiating the spatial basis functions. Consider a one dimensional problem. The first derivative is given by simple differentiation:

$$y'(t_i) = \sum_{j=1}^N \frac{a_j(t_i - t_j)}{h_{ij}}, \quad h_{ij} = ((t_i - t_j)^2 + R_j^2)^{\frac{1}{2}}, \quad i = 1, 2, \dots, N.$$

### 3. Numerical solution of NDDEs

In this section, we are interested to solve equation (1), i.e.,

$$(5) \quad \begin{cases} y'(t) = f(t, y(t), y(t - \tau(t, y(t))), y'(t - \sigma(t, y(t)))), & t_1 \leq t \leq t_f, \\ y(t) = \phi(t), & t \leq t_1, \end{cases}$$

by the MQ approximation scheme mentioned in Section 2. For the solution of equation (5), it is sufficient to suppose that approximate solution is

$$(6) \quad y(t) = \sum_{j=1}^N a_j h(t - t_j), \quad t_1 \leq t \leq t_f.$$

Choosing  $t_i$ ,  $i = 1, 2, \dots, N$ , as collocating points, we have

$$(7) \quad y(t_i) = \sum_{j=1}^N a_j h(t_i - t_j),$$

$$(8) \quad y'(t_i) = \sum_{j=1}^N \frac{a_j(t_i - t_j)}{h_{ij}}, \quad i = 1, 2, \dots, N,$$

also, for  $i = 1, 2, \dots, N$ ,

$$(9) \quad y(t_i - \tau(t_i, y(t_i))) = \sum_{j=1}^N a_j h(t_i - \tau(t_i, \sum_{j=1}^N a_j h(t_i - t_j)) - t_j),$$

$$(10) \quad y'(t_i - \sigma(t_i, y(t_i))) = \sum_{j=1}^N \frac{a_j(t_i - \sigma(t_i, \sum_{j=1}^N a_j h(t_i - t_j)) - t_j)}{h_{ij}^{(1)}},$$

where

$$(11) \quad h_{ij}^{(1)} = h(t_i - \sigma(t_i, y(t_i)) - t_j) = ((t_i - \sigma(t_i, \sum_{j=1}^N a_j h(t_i - t_j)) - t_j)^2 + R_j^2)^{\frac{1}{2}}.$$

Substituting (7), (8), (9) and (10) in (5), and imposing the supplementary condition ( $y(t_1) = \phi(t_1)$ ) to the problem, we gain  $N - 1$  equations of differential forms and initial condition produce one equation. Hence, the system of  $N$

equations with  $N$  unknowns is available. Then we must solve this system to distinct the unknown coefficients. Hence, we have used the Gauss elimination method with total pivoting to solve such a system.

*Remark.* It is noticeable that collocating points can be scattered. This is the main difference between this method of solution and the other methods. In next section, the numerical results demonstrate this issue, easily and the efficiency of MQ approximation scheme in this sense, is observable.

### 4. Numerical experiments

In this part, we present some experiments in-which their numerical solutions illustrate the high accuracy and efficiency of MQ approximation scheme.

**Problem 1.** Consider the following NDDE,

$$\begin{cases} y'(t) + \sqrt{t}(y'(e^{-\frac{t}{2}}) - y(\sqrt{t}e^{-t}) + y(t)) \\ = \cos(t) + \sqrt{t}(\cos(e^{-\frac{t}{2}}) - \sin(\sqrt{t}e^{-t}) + \sin(t)), & 0 \leq t \leq 1, \\ y(t) = \sin(t), & t \leq 0. \end{cases}$$

The exact solution is  $y(t) = \sin(t)$ . The MQ approximate solution is obtained with  $R_{\max} = 150, R_{\min} = .99$  and  $N = 7$ , and the results are given in the following Table.

TABLE I

$t_i$	MQ approximate solution	Exact solution	Error
0	0.0000000	0	0.0000000
$\frac{1}{6}$	0.1658960	0.1658961	0.0000001
$\frac{1}{3}$	0.3271945	0.3271946	0.0000001
$\frac{1}{2}$	0.4794253	0.4794255	0.0000002
$\frac{2}{3}$	0.6183694	0.6183698	0.0000004
$\frac{5}{6}$	0.7401766	0.7401768	0.0000002
1	0.8414705	0.8414709	0.0000004

Numerical results show that, the MQ approximation scheme in the required domain needs the minimum number of data points for the solution. This demonstrates one of the good advantages of MQ approximation scheme in spite of its simplicity.

**Problem 2.** Consider the following NDDE,

$$\begin{cases} y'(t) + e^t y'(t - \sin(t^2)) + \cos(t)y(t - \sin(t)) \\ = -e^{-t} - e^{\sin(t^2)} + \cos(t)e^{\sin(t)-t}, & 0 \leq t \leq \frac{\pi}{e}, \\ y(t) = e^{-t}, & t \leq 0, \end{cases}$$

TABLE II

$t_i$	MQ approximate solution	Exact solution	Error
0	1.00000002	1	0.00000002
$\frac{\pi}{10e}$	0.89085576	0.89085577	0.00000001
$\frac{\pi}{5e}$	0.79362401	0.79362400	0.00000001
$\frac{3\pi}{10e}$	0.70700452	0.70700453	0.00000001
$\frac{2\pi}{5e}$	0.62983906	0.62983906	0.00000000
$\frac{\pi}{2e}$	0.56109578	0.56109577	0.00000001
$\frac{3\pi}{5e}$	0.49985540	0.49985540	0.00000000
$\frac{7\pi}{10e}$	0.44529909	0.44529907	0.00000002
$\frac{4\pi}{5e}$	0.39669726	0.39669725	0.00000001
$\frac{9\pi}{10e}$	0.35340005	0.35340003	0.00000002
$\frac{\pi}{e}$	0.31482846	0.31482846	0.00000000

the exact solution is  $e^{-t}$ .

By choosing  $R_{\max} = 40$ ,  $R_{\min} = 1.1$  and  $N = 11$ , we have the following Table for MQ approximate solution.

In the following, we present an experiment which its numerical results shows that, in spite of the complexity of problem, the MQ approximation scheme works excellently for scattered data points, too. Hence, this method isn't depend on the selection of points. Here, also the high accuracy and efficiency of this method is observable.

**Problem 3.** Consider the following NDDE,

$$\begin{cases} y'(t) + \sqrt{\cos(t)}y'(\sqrt{t}) + (\sin(\sqrt{t}) + e^t)y(\sin(t)) \\ = e^t + \sqrt{\cos(t)}e^{\sqrt{t}} + (\sin(\sqrt{t}) + e^t)e^{\sin(t)}, & 0 \leq t \leq 1, \\ y(t) = e^t, & t \leq 0, \end{cases}$$

the exact solution is  $e^t$ .

By choosing  $R_{\max} = 50$ ,  $R_{\min} = .499$  and  $N = 15$ , we have the following Table for MQ approximate solution.

We observed that, this method (MQ approximation scheme) isn't depend on collocating points in large scales. But, when we apply other methods which need collocating points, if scattered data points are used, the round off error may occurs, soon. This is an other excellent advantage on the application of MQ approximation scheme, too.

## 5. Conclusion

In this paper, the MQ approximation scheme is proposed for solving NDDEs. This method of solution is easy to implement and yields desired accuracy only in a few terms. As we have observed, the method works excellently for NDDEs of scattered data points, too. This exhibits one of the other good advantages

TABLE III

$t_i$	MQ approximate solution	Exact solution	Error
0.00	1.000000000	1	0.00000000
0.06	1.061836544	1.061836546	0.00000002
0.15	1.161834240	1.161834242	0.00000002
0.20	1.221402756	1.221402758	0.00000002
0.28	1.323129811	1.323129812	0.00000001
0.34	1.404947588	1.404947590	0.00000002
0.41	1.506817782	1.506817785	0.00000003
0.47	1.599994190	1.599994193	0.00000003
0.55	1.733253016	1.733253017	0.00000001
0.64	1.896480877	1.896480879	0.00000002
0.69	1.993715531	1.993715532	0.00000001
0.76	2.138276220	2.138276220	0.00000000
0.85	2.339646850	2.339646851	0.00000001
0.91	2.484322532	2.484322533	0.00000001
1.00	2.718281828	2.718281828	0.00000000

of MQ approximation scheme in spite of its simplicity. The computations associated with the experiments discussed above, were performed by using Maple 10.

### References

- [1] A. Aminataei and M. M. Mazarei, *Numerical solution of elliptic partial differential equations using direct and indirect radial basis function networks*, Euro. J. Scien. Res. **2** (2005), no. 2, p. 5.
- [2] A. Aminataei and M. Sharan, *Using multiquadric method in the numerical solution of ODEs with a singularity point and PDEs in one and two dimensions*, Euro. J. Scien. Res. **10** (2005), no. 2, p. 19.
- [3] R. Bellman, *On the computational solution of differential-difference equations*, J. Math. Anal. Appl. **2** (1961), 108–110.
- [4] A. Bellen and M. Zennaro, *Adaptive integration of delay differential equations*, Advances in time-delay systems, 155–165, Lect. Notes Comput. Sci. Eng., 38, Springer, Berlin, 2004.
- [5] ———, *Numerical Methods for Delay Differential Equations*, Numerical Mathematics and Scientific Computation, The Clarendon Press, Oxford University Press, New York, 2003.
- [6] R. D. Driver, *Ordinary and Delay Differential Equations*, Applied Mathematical Sciences, Vol. 20. Springer-Verlag, New York-Heidelberg, 1977.
- [7] L. E. El'sgol'ts and S. B. Norkin, *Introduction to the Theory and Applications of Differential Equations with Deviating Arguments*, Mathematics in Science and Engineering, Vol. 105. Academic Press [A Subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1973.
- [8] R. Franke, *Scattered data interpolation: Tests of some methods*, Math. Comput. **38** (1971), 181–199.

- [9] K. Gopalsamy, *Stability and Oscillations in Delay Differential Equations of Population Dynamics*, Mathematics and its Applications, 74. Kluwer Academic Publishers Group, Dordrecht, 1992.
- [10] N. Guglielmi and E. Hairer, *Implementing Radau IIA methods for stiff delay differential equations*, Computing **67** (2001), no. 1, 1–12.
- [11] A. Halanay, *Differential Equations: Stability, Scillations, Time Lags*, Academic Press, New York-London, 1966.
- [12] R. L. Hardy, *Multiquadric equations of topography and other irregular surfaces*, J. Geophys. Res. **76** (1971), no. 8, 1705–1915.
- [13] ———, *Theory and applications of the multiquadric-biharmonic method. 20 years of discovery 1968–1988*, Comput. Math. Appl. **19** (1990), no. 8-9, 163–208.
- [14] E. J. Kansa, *Multiquadrics—a scattered data approximation scheme with applications to computational fluid-dynamics. I. Surface approximations and partial derivative estimates*, Comput. Math. Appl. **19** (1990), no. 8-9, 127–145.
- [15] ———, *Multiquadrics—a scattered data approximation scheme with applications to computational fluid-dynamics. II. Solutions to parabolic, hyperbolic and elliptic partial differential equations*, Comput. Math. Appl. **19** (1990), no. 8-9, 147–161.
- [16] V. B. Kolmanovskii and A. Myshkis, *Applied Theory of Functional-Differential Equations*, Mathematics and its Applications (Soviet Series), 85. Kluwer Academic Publishers Group, Dordrecht, 1992.
- [17] V. B. Kolmanovskii and V. R. Nosov, *Stability of Functional-Differential Equations*, Mathematics in Science and Engineering, 180. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], London, 1986.
- [18] Y. Kuang, *Delay Differential Equations with Applications in Population Dynamics*, Mathematics in Science and Engineering, 191. Academic Press, Inc., Boston, MA, 1993.
- [19] E. Lelarsmee, A. Ruehli, and A. Sangiovanni-Vincentelli, *The waveform relaxation method for time domain analysis of large scale integrated circuits*, IEEE Transactions on CAD **1** (1982), no. 3, 131–145.
- [20] W. R. Madych, *Miscellaneous error bounds for multiquadric and related interpolators*, Comput. Math. Appl. **24** (1992), no. 12, 121–138.
- [21] W. R. Madych and S. A. Nelson, *Multivariate interpolation and conditionally positive definite functions. II*, Math. Comp. **54** (1990), no. 189, 211–230.
- [22] N. Mai-Duy and T. Tran-Cong, *Numerical solution of differential equations using multiquadric radial basis function networks*, Neural Networks **14** (2001), no.2 , 185–199.
- [23] C. A. Micchelli, *Interpolation of scattered data: distance matrices and conditionally positive definite functions*, Constr. Approx. **2** (1986), no. 1, 11–22.
- [24] M. Zerroukut, H. Power, and C. S. Chen, *A numerical method for heat transfer problems using collocation and radial basis functions*, Internat. J. Numer. Methods Engrg. **42** (1998), no. 7, 1263–1278.

SOLAT KARIMI VANANI  
DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE  
K. N. TOOSI UNIVERSITY OF TECHNOLOGY  
P.O. BOX: 16315-1618, TEHRAN, IRAN  
E-mail address: solatkarimi@yahoo.com

AZIM AMINATAEI  
DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE  
K. N. TOOSI UNIVERSITY OF TECHNOLOGY  
P.O. BOX: 16315-1618, TEHRAN, IRAN  
E-mail address: ataei@kntu.ac.ir