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TRANS-SEPARABILITY IN THE STRICT AND COMPACT-OPEN TOPOLOGIES

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Abstract. We give a characterization of trans-separability for the function spaces $(C_b(X,E),\ \beta)$, $(C(X,E),k)$ and $(C_b(X,E),u)$ in the case of E any general topological vector space.

1. Introduction

The fundamental result on the characterization of separability of $(C_b(X),u)$ was obtained by M. Krein and S. Krein [12] in 1940. Later, similar results were obtained by Gulick and Schmets [5] and, independently, by Summers [15] for $(C_b(X),k)$ and $(C_b(X),\beta)$. On the other hand, Gulick and Schmets [5] also gave a characterization of seminorm-separability for $(C_b(X),u)$, $(C_b(X),k)$ and $(C_b(X),\beta)$. Characterization of separability for vector-valued function spaces have been considered in [16, 2, 8]. In [9], the author generalised these results by giving a characterization of neighbourhood-separability for the spaces $(C_b(X,E),\beta)$, $(C(X,E),k)$ and $(C_b(X,E),u)$ in the case of E a 'semi-convex' topological vector space (TVS) having non-trivial topological dual $E^{'}$. The purpose of this note is to extend these results further to the case of E any general TVS, using the terminology of trans-separability as in [10].

2. Preliminaries

For the convenience of the reader, we recall some terminology so that this note can be read independently of [9, 10]. Let X denote a completely regular Hausdorff space and E a non-trivial Hausdorff TVS with $\mathcal{W}$ a base of neighbourhoods of 0. A neighbourhood G of 0 in E is called *shrinkable* [11] if $r\overline{G} \subseteq \text{int}\, G$ for $0 \leq r < 1$. By ([11], Theorems 4 and 5), every Hausdorff TVS has a base of shrinkable neighbourhoods of 0 and also the Minkowski functional ρ_G of any such neighbourhood G is continuous and positively homogeneous.

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Definition 1. The *strict topology* β [1, 7] on $C_b(X,E)$ is defined as the linear topology which has a base of neighbourhoods of 0 consisting of all sets of the form

$$U(\varphi, W) = \{f \in C_b(X,E) : (\varphi f)(X) \subseteq W\},$$

where $\varphi \in B_0(X)$, the set of all bounded scalar-valued functions on X which vanish at infinity, and $W \in \mathcal{W}$.

Let u (resp. k) denotes the uniform (resp. compact-open topology) on $C_b(X,E)$ (resp. $C(X,E)$). Then $k \leq \beta \leq u$ on $C_b(X,E)$. For any $\varphi \in B_0(X)$, let $\| \,.\, \|_\varphi$ denote the seminorm on $C_b(X)$ given by $\| f \|_\varphi = \sup\{|\varphi(x)f(x)| : x \in X\}$, $f \in C_b(X)$. We shall denote by $C(X) \otimes E$ the vector subspace of $C(X,E)$ spanned by the set of all functions of the form $\varphi \otimes a$, where $\varphi \in C(X)$, $a \in E$, and $(\varphi \otimes a) = \varphi(x)a$, $x \in X$.

Recall that a locally convex space L is called *seminorm-separable* [5] if, for each continuous seminorm p on L, (L,p) is separable. The following classical result is stated for reference purpose.

Theorem 1 ([5])**.** *The following statements are equivalent*:

(a) $(C_b(X), \beta)$ *is seminorm-separable.*

(b) $(C(X), \kappa)$ *is seminorm-separable.*

(c) *Every compact subset of* X *is metrizable.*

Definition 2. A uniform space L is called *trans-separable* if every uniform cover of L admits a countable subcover [6]. In particular, a TVS L is *trans-separable* if, for each neighbourhood W of 0 in L, the open cover $\{a+W : a \in L\}$ of L admits a countable subcover.

Drewnowski [3] had actually coined the word "trans-separable" and it has been further used by Robertson [13] and Ferrando-Kakol-Pellicer [4]. Khan [9] introduced a generalized notion of separability, namely, the neighbourhood-separability in the TVS setting, as follows.

Definition 3. Let L be a TVS, and let V be a neighbourhood of 0 in L. A subset H of L is said to be V*-dense* in L if, for any $z \in L$ and $\delta > 0$, there exists an element $y \in H$ such that $y - z \in \delta V$. L is called *neighbourhood-separable* if, for each neighbourhood V of 0, there exists a countable V-dense subset of L.

Another notion of generalized separability may also be considered, as follows.

Definition 4. Let (L,τ) be a TVS whose topology is generated by a family $Q(\tau)$ of continuous F-seminorms [17]. Then (L,τ) is called *F-seminorm-separable* if (L,q) is separable for each $q \in Q(\tau)$.

Clearly, separability implies F-seminorm-separability; the converse holds in metrizable spaces.

The following result establishes the equivalence of all the above notions of generalized separabilities.

Lemma 1 (cf. [10])**.** *Let (L,τ) be a TVS. The following are equivalent*:
(1) (L,τ) *is trans-separable.*
(2) (L,τ) *is neighbourhood-separable.*
(3) (L,τ) *is F-seminorm-separable.*

Proof. (1) $\Rightarrow$ (2) Suppose L is trans-separable, and let V be a neighbourhood of 0. For each $n \geq 1$, $\mathbb{U}_n = \{x + n^{-1}V : x \in L\}$ is a uniform cover of L, and so it has a countable subcover $\mathbb{U}_n^* = \{x_k^{(n)} + n^{-1}V : k \in \mathbb{N}\}$. Let $D = \cup_{n=1}^{\infty}\{x_k^{(n)} : k \in \mathbb{N}\}$. To show that D is V-dense in L, let $y \in L$ and $\delta > 0$. Choose $N \geq 1$ such that $N^{-1} < \delta$. Since $\mathbb{U}_N^*$ is a cover of L, $y \in x_K^{(N)} + N^{-1}V$ for some $K \in \mathbb{N}$. Then $y - x_K^{(N)} \in \delta V$. Hence L is neighbourhood-separable.

(2) $\Leftrightarrow$ (3) This is trivial.

(3) $\Rightarrow$ (1) Suppose (L,q) is separable for each $q \in Q(\tau)$. Let $\{x+U : x \in L\}$ be any uniform cover of L, where U is neighbourhood of 0 in L. Choose a balanced neighbourhood V of 0 in L with $V + V \subseteq U$. Choose $q \in Q(\tau)$ such that $W = \{x \in L : q(x) < 1\} \subseteq V$. Let $\{z_n\}$ be a countable dense subset in (L,q). Since $L = \cup_{x\in L}(x+W)$, to each $z_n \in L$, there exists some $x_n \in L$ such that $z_n - x_n \in W$. Let $y \in L$. Choose z_k such that $q(y - z_k) < 1$. Then

$$y - x_k = (y - z_k) + (z_k - x_k) \in W + W \subseteq U,$$

and so $L = \cup_{n\geq 1}(x_n + U)$. □

3. Main results

Theorem 2. *Let E be any non-trivial TVS. Then the following statements are equivalent*:
(a) $(C_b(X) \otimes E, \beta)$ *is trans-separable.*
(b) $(C(X) \otimes E, k)$ *is trans-separable.*
(c) *Every compact subset of X is metrizable and E is trans-separable.*

Proof. (a) $\Rightarrow$ (b) This follows from the fact that $k \leq \beta$ on $C_b(X) \otimes E$ and that $C_b(X) \otimes E$ is k-dense in $C(X) \otimes E$.

(b) $\Rightarrow$ (c) This follows from Theorem 1 and the fact that both $(C(X), k)$ and E are isomorphic to subspaces of $(C(X) \otimes E, k)$ via the maps $g \to g \otimes a$ $(0 \neq a \in E$ fixed) and $a \to 1_X \otimes a$, respectively.

(c) $\Rightarrow$ (a) By Theorem 1, $(C_b(X), \beta)$ is trans-separable. Fix a $\varphi \in B_o(X)$, $0 \leq \varphi \leq 1$ and a balanced $W \in \mathcal{W}$. We need to show that there is a countable set $H \subseteq C_b(X) \otimes E$ such that $C_b(X) \otimes E = H + U(\varphi, W)$.

For every pair $m, n \in \mathbb{N}$ choose a balanced $U_{m,n} \in \mathcal{W}$ so that, denoting $V_{m,n} = U_{m,n} + mU_{m,n} + U_{m,n}$, one has

$$V_{m,n} + \cdots + V_{m,n} \ (n\text{-summands}) \ \subset W.$$

Also, choose a countable set $D_{m,n}$ in E so that $E = D_{m,n} + U_{m,n}$. Let D be the union of all these sets $D_{m,n}$ $(m, n \in \mathbb{N})$.

Next, for each $k \in \mathbb{N}$ denote $B_k = \{f \in C_b(X) : \|f\|_\varphi \leq 1/k\}$ and choose a countable set G_k in $C_b(X)$ so that $C_b(X) = G_k + B_k$. Let G be the union of all these sets G_k ($k \in \mathbb{N}$).

We are going to show that the countable set $H = H_{\varphi,W}$ of all functions in $C_b(X)\otimes E$ of the form $h = \sum_{i=1}^r g_i \otimes d_i$, where $g_i \in G$ and $d_i \in D$ ($i = 1, \dots, r$, $r \in \mathbb{N}$), is as required.

Take any $f \in C_b(X)\otimes E$. Then $f = \sum_{i=1}^n f_i \otimes a_i$ for some $f_1, \dots, f_n \in C_b(X)$ and $a_1, \dots, a_n \in E$. Let $m \in \mathbb{N}$ be such that $\|f_i\|_\varphi \leq m$ for $i = 1, \dots, n$, and next choose $k \in \mathbb{N}$ so that $k^{-1}a_i \in U_{m,n}$ for $i = 1, \dots, n$. By the definitions of $D_{m,n}$ and G_k, there are $d_1, \dots, d_n \in D_{m,n}$ and $g_1, \dots, g_k \in G_k$ such that

$$a_i - d_i \in U_{m,n} \quad \text{and} \quad \|f_i - g_i\|_\varphi \leq 1/k \qquad \text{for } i = 1, \dots, n.$$

Now, for $i = 1, \dots, n$ and $x \in A$,

$$\begin{aligned}\varphi(x)[f_i(x)a_i - g_i(x)d_i] &= \varphi(x)[f_i(x) - g_i(x)]a_i + \varphi(x)f_i(x)(a_i - d_i) \\ &\quad + \varphi(x)[g_i(x) - f_i(x)](a_i - d_i),\end{aligned} \tag{$*$}$$

hence (using the fact that $U_{m,n}$ is balanced)

$$\varphi(x)[f_i(x)a_i - g_i(x)d_i] \in U_{m,n} + mU_{m,n} + U_{m,n} = V_{m,n}.$$

In consequence, setting $h = \sum_{i=1}^n g_i \otimes d_i$ we have $h \in H$ and for every $x \in A$,

$$\varphi(x)[f(x) - h(x)] = \sum_{i=1}^n \varphi(x)[f_i(x)a_i - g_i(x)d_i] \in W$$

so that $f - h \in U(\varphi, W)$. □

Remark 1. A somewhat more transparent variant of the above proof that (c) implies (a) can be based on Lemma 1. We need to show that for any $\varphi \in B_o(X), 0 \leq \varphi \leq 1$, and any continuous F-seminorm q on E, the space $(C_b(X)\otimes E, p_\varphi)$ is separable, where $p_\varphi(f) = \sup_{x\in X} q(\varphi(x)f(x))$. Now, let G be a countable subset dense in $(C_b(X), \|\cdot\|_\varphi)$, and D a countable set dense in (E, q). Take any $f = \sum_{i=1}^n f_i \otimes a_i$ in $C_b(X) \otimes E$, and choose $m \in \mathbb{N}$ so that $\|f_i\|_\varphi \leq m$ for each i. Given $\varepsilon > 0$, let $g = \sum_{i=1}^n g_i \otimes d_i$, where $g_i \in G$ and $d_i \in D$. Assume that $\|f_i - g_i\|_\varphi \leq \delta$ for all i and some as yet unspecified $0 < \delta < 1$. Then, making use of $(*)$, it is easily seen that

$$\begin{aligned}p_\varphi(f - g) &\leq \sum_{i=1}^n (q(\|f_i - g_i\|_\varphi a_i) + q(\|f_i\|_\varphi (a_i - d_i)) + q(\|f_i - g_i\|_\varphi (a_i - d_i))) \\ &\leq \sum_{i=1}^n (q(\delta a_i) + (m+1)q((a_i - d_i)\end{aligned}$$

and this can be made smaller than ε by taking δ sufficiently small and choosing the d_i's in D sufficiently close to the a_i's. It follows that the countable set of all g's of the above form is dense in $(C_b(X) \otimes E, p_\varphi)$.

Remark 2. If X has finite covering dimension or E is locally convex, or E has the approximation property or E is complete metrizable with a basis, then $C_b(X) \otimes E$ is β-dense in $C_b(X, E)$ and that $C(X) \otimes E$ is k-dense in $C(X, E)$ (see [14, 7]). Hence, under these assumptions, the above theorem holds with $C_b(X) \otimes E$, $C(X) \otimes E$ and $C_o(X) \otimes E$ replaced by $C_b(X, E)$, $C(X, E)$ and $(C_o(X, E)$, respectively. It is not known whether or not these 'density' results hold for E a locally bounded space. However, we include the following analogue of ([8]; [9], Theorem 3.4) for the reader's interest.

Theorem 3. *Let X be any Hausdorff space and E any locally bounded space. Then $(C_b(X, E), \beta)$ is trans-separable $\Leftrightarrow$ $(C(X, E), k)$ is so.*

Proof. Suppose $(C(X, E), k)$ is trans-separable. Let $\varphi \in B_o(X)$ with $0 \leq \varphi \leq 1$, and let $W \in \mathcal{W}$. Let V be a balanced bounded neighbourhood of 0 in E, and let S be a closed shrinkable neighbourhood of 0 with $S \subseteq V$. The Minkowski functional $\rho = \rho_S$ of S is continuous and positive homogeneous and, consequently, for each $r > 0$, the function $h_r : E \to E$ defined by

$$h_r(a) = \begin{cases} a & \text{if } a \in rS \\ \frac{r}{\rho(a)} a & \text{if } a \in E \setminus rS \end{cases}$$

is continuous. Further, $h_r(E) \subseteq rS \subseteq rV$, which shows that, for each $f \in C(X, E)$, the function $h_r \circ f \in C_b(X, E)$. Choose $t \geq 1$ such that $V + V \subseteq tS$ and $V + V \subseteq tW$. For each $m = 1, 2, \ldots$, there exists a compact set $K_m \subseteq X$ such that $\varphi(x) < 1/tm^2$ for $x \in X \backslash K_m$. Corresponding to each K_m, choose $\{f_{mn} : n \in \mathbb{N}\}$ as a $N(K_m, V)$-dense of $C(X, E)$, where

$$U(K_m, W) = \{f \in C_b(X, E) : f(K_m) \subseteq W\}.$$

We show that $\{h_m \circ f_{mn} : m, n = 1, 2, \ldots\}$ is β-dense in $C_b(X, E)$. Let $f \in C_b(X, E)$ and $0 \leq \delta \leq 1$. Choose integers $M \geq 1/\delta$ and $N \geq 1$ such that $f(X) \subseteq (M\delta/t)V$ and $(f_{MN} - f)(K_M) \subseteq (\delta/t)V$. Let $y \in X$. If $y \in K_M$, then $f_{MN}(y) \in f(y) + (\delta/t)V \subseteq (M\delta/t)V + (M\delta/t)V \subseteq MS$ and so

$$\varphi(y)[h_M \circ f_{MN}(y) - f(y)] = \varphi(y)[f_{MN}(y) - f(y)] \in \delta W.$$

If $y \in X \backslash K_M$, then, since $h_M(f_{MN}(y)) \in h_M(E) \subseteq MS$,

$$\begin{aligned} \varphi(y)[h_M \circ f_{MN}(y) - f(y)] &\in \varphi(y)[MS - \frac{M\delta}{t}V] \\ &\subseteq \frac{1}{tM}[V + \frac{\delta}{t}V] \subseteq \frac{\delta}{t}[V + V] \subseteq \delta W. \end{aligned}$$

Thus $h_M \circ f_{MN} - f \in \delta U(\varphi, W)$. Consequently, $(C_b(X, E), \beta)$ is neighbourhood-separable and hence trans-separable by Lemma 1.

The converse follows from the fact that $C_b(X, E)$ is dense in $(C(X, E), k)$, using again the local boundedness of E. Indeed, let $f \in C(X, E)$, K a compact subset of X and $W \in \mathcal{W}$. Let V and S be as above with $S \subseteq V$. Choose

$r \geq 1$ with $f(K) \subseteq rS$. Then, as in the above part, we have a function $h_r \circ f \in C_b(X, E)$ such that

$$h_r \circ f(x) - f(x) = f(x) - f(x) = 0 \in W \text{ for all } x \in K.$$

□

Next, we obtain:

Theorem 4. *Let E be a non-trivial TVS. Then*

(a) $(C_b(X) \otimes E, u)$ *is trans-separable* $\Leftrightarrow$ X *is a compact metric space and* E *is trans-separable.*

(b) *Suppose X is locally compact. Then* $(C_o(X) \otimes E, u)$ *is trans-separable* $\Leftrightarrow$ X *is a* σ*-compact metric space and* E *is trans-separable.*

Proof. (a) In this case, $(C_b(X), u)$ is trans-separable $\Leftrightarrow$ it is separable $\Leftrightarrow$ X is a compact metric space [12]. The proof now follows just as in Theorem 2.

(b) If X is locally compact, then $(C_o(X), u)$ is trans-separable $\Leftrightarrow$ it is separable $\Leftrightarrow$ X is a σ-compact metric space [5, 15]. Again the proof follows just as in Theorem 2. □

Again we remark that, if $C_b(X) \otimes E$ (*resp.* $C_o(X) \otimes E$) *is* u-dense in $C_b(X, E)$ $(C_o(X, E))$, the above theorem remains valid with $C_b(X) \otimes E$ $(C_o(X) \otimes E)$ replaced by $C_b(X, E)$ $(C_o(X, E))$.

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References

[1] R. C. Buck, *Bounded continuous functions on a locally compact space*, Michigan Math. J. **5** (1958), 95–104.

[2] S. A. Choo, *Separability in the strict topology*, J. Math. Anal. Appl. **75** (1980), no. 1, 219–222.

[3] L. Drewnowski, *Another note on Kalton's theorems*, Studia Math. **52** (1974/75), 233–237.

[4] J. C. Ferrando, J. Kakol, and M. López Pellicer, *A characterization of trans-separable spaces*, Bull. Belg. Math. Soc. Simon Stevin **14** (2007), no. 3, 493–498.

[5] D. Gulick and J. Schmets, *Separability and semi-norm separability for spaces of bounded continuous functions*, Bull. Soc. Roy. Sci. Liege **41** (1972), 254–260.

[6] J. R. Isabell, *Uniform Spaces*, Mathematical Surveys, No. 12 American Mathematical Society, Providence, R.I. 1964.

[7] L. A. Khan, *The strict topology on a space of vector-valued functions*, Proc. Edinburgh Math. Soc. (2) **22** (1979), no. 1, 35–41.

[8] ———, *Separability in function spaces*, J. Math. Anal. Appl. **113** (1986), no. 1, 88–92.

[9] ———, *Generalized separability in vector-valued function spaces*, Ann. Univ. Sci. Budapest. Eötvös Sect. Math. **42** (1999), 3–8.

[10] ———, *Trans-separability in spaces of continuous vector-valued functions*, Demonstratio Math. **37** (2004), no. 3, 611–617.

[11] V. Klee, *Shrinkable neighborhoods in Hausdorff linear spaces*, Math. Ann. **141** (1960), 281–285.

[12] M. Krein and S. Krein, *On an inner characteristic of the set of all continuous functions defined on a bicompact Hausdorff space*, C. R. (Doklady) Acad. Sci. URSS (N.S.) **27** (1940), 427–430.

[13] N. Robertson, *The metrizability of precompact sets*, Bull. Austral. Math. Soc. **43** (1991), 131–135.

[14] A. H. Shuchat, *Approximation of vector-valued continuous functions*, Proc. Amer. Math. Soc. **31** (1972), 97–103.

[15] W. H. Summers, *Separability in the strict and substrict topologies*, Proc. Amer. Math. Soc. **35** (1972), 507–514.

[16] C. Todd, *Stone-Weierstrass theorems for the strict topology*, Proc. Amer. Math. Soc. **16** (1965), 654–659.

[17] L. Waelbroeck, *Topological Vector Spaces and Algebras*, Lecture Notes in Mathematics, Vol. 230. Springer-Verlag, Berlin-New York, 1971.

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