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ON ϕ-RECURRENT (k, μ)-CONTACT METRIC MANIFOLDS

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Abstract. In this paper we prove that a ϕ-recurrent (k, μ)-contact metric manifold is an η-Einstein manifold with constant coefficients. Next, we prove that a three-dimensional locally ϕ-recurrent (k, μ)-contact metric manifold is the space of constant curvature. The existence of ϕ-recurrent (k, μ)-manifold is proved by a non-trivial example.

1. Introduction

The notion of local symmetry of a Riemannian manifold has been weakened by many authors in several ways to a different extent. As a weaker version of local symmetry, T. Takahashi [13] introduced the notion of locally ϕ-symmetry on a Sasakian manifold. Generalizing the notion of ϕ-symmetry, one of the authors, De [10] introduced the notion of ϕ-recurrent Sasakian manifold. In the context of contact geometry the notion of ϕ-symmetry is introduced and studied by Boeckx, Buecken, and Vanhecke [8] with several examples. In [E. Boeckx, *A class of locally ϕ-symmetric contact metric spaces*, Arch. Math. **72** (1999), 466–472], he proved that every non-Sasakian (k, μ)-manifold is locally ϕ-symmetric in the strong sense.

In the present paper we introduce a type of (k, μ)-contact metric manifolds called ϕ-recurrent (k, μ)-contact metric manifold which generalizes the notion of ϕ-symmetric (k, μ)-contact metric structure of Boeckx. The (k, μ)-contact metric manifold is one of special interest as it contains both the class of Sasakian and non-Sasakian cases. Hence, in our opinion, this is the first time that the notion of ϕ-recurrent manifold for the non-Sasakian case is appearing in the literature. After preliminaries in Section 3, it is proved that a ϕ-recurrent (k, μ)-contact metric manifold is an η-Einstein manifold with constant coefficients. Also it is shown that the characteristic vector field of the (k, μ)-contact metric

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manifold and the vector field associated to the 1-form of recurrence are co-directional. In Section 4, we study 3-dimensional locally ϕ-recurrent (k, μ)-contact metric manifold. The last section provides the existence of the locally ϕ-recurrent (k, μ)-contact metric manifold by an example which is neither locally symmetric nor locally ϕ-symmetric.

2. Contact metric manifolds

A $(2n+1)$-dimensional manifold M^{2n+1} is said to *admit an almost contact structure* if it admits a tensor field ϕ of type $(1,1)$, a vector field ξ and a 1-form η satisfying

$$\text{(2.1)} \qquad \text{(a) } \phi^2 = -I + \eta \otimes \xi, \quad \text{(b) } \eta(\xi) = 1, \quad \text{(c) } \phi\xi = 0, \quad \text{(d) } \eta \circ \phi = 0.$$

An almost contact metric structure is said to be *normal* if the induced almost complex structure J on the product manifold $M^{2n+1} \times \mathbb{R}$ defined by $J(X, f\frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt})$ is integrable, where X is tangent to M, t is the coordinate of $\mathbb{R}$ and f is a smooth function on $M \times \mathbb{R}$. Let g be a compatible Riemannian metric with almost contact structure (ϕ, ξ, η), that is,

$$\text{(2.2)} \qquad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y).$$

Then M becomes an almost contact metric manifold equipped with an almost contact metric structure (ϕ, ξ, η, g). From (2.2) it can be easily seen that

$$\text{(2.3)} \qquad \text{(a) } g(X, \phi Y) = -g(Y, \phi X), \quad \text{(b) } g(X, \xi) = \eta(X)$$

for all vector fields X and Y. An almost contact metric structure becomes a contact metric structure if

$$\text{(2.4)} \qquad g(X, \phi Y) = d\eta(X, Y)$$

for all vector fields X and Y. The 1-form η is then called a *contact form* and ξ is the *characteristic vector* field. We define a $(1,1)$-tensor field h by $h = \frac{1}{2}\pounds_\xi \phi$, where $\pounds$ denotes the Lie differentiation. Blair [3] proved that the tensor h is a symmetric operator. Then h satisfies $h\phi = -\phi h$. We have $\mathrm{Tr}(h) = \mathrm{Tr}(\phi h) = 0$ and $h\xi = 0$. Also,

$$\text{(2.5)} \qquad \nabla_X \xi = -\phi X - \phi h X$$

holds in a contact metric manifold. A normal contact metric manifold is a Sasakian manifold. An almost contact metric manifold is Sasakian if and only if

$$\text{(2.6)} \qquad (\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X, \quad X, Y \in TM,$$

where ∇ is Levi-Civita connection of the Riemannian metric g. A contact metric manifold $M^{2n+1}(\phi, \xi, \eta, g)$ for which ξ is a Killing vector field is said to be a *K-contact* manifold. A Sasakian manifold is K-contact but not conversely. However a 3-dimensional K-contact manifold is Sasakian [11]. It is well known that the tangent sphere bundle of a flat Riemannian manifold admits a contact metric structure satisfying $R(X,Y)\xi = 0$ [2]. On the other hand, on a Sasakian manifold the following holds:

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y. \tag{2.7}$$

It is well known that there exist contact metric manifolds for which the curvature tensor R and the direction of the characteristic vector field ξ satisfying $R(X,Y)\xi = 0$ for any vector fields X and Y. For example, the tangent sphere bundle of a flat Riemannian manifold admits such a structure.

As a generalization of both $R(X,Y)\xi = 0$ and the Sasakian case: D. E. Blair, T. Koufogiorgos, and B. J. Papantoniou [5] considered the (k,μ)-nullity condition on a contact metric manifold and gave several reasons for studying it. The (k,μ)-nullity distribution $N(k,\mu)$ ([5], [12]) of a contact metric manifold M is defined by

$$\begin{aligned} N(k,\mu) : p \longrightarrow N_p(k,\mu) = \ & \{W \in T_pM \mid R(X,Y)W \\ & = (kI + \mu h)\,(g(Y,W)X - g(X,W)Y)\} \end{aligned}$$

for all X and $Y \in TM$, where $(k,\mu) \in \mathbb{R}^2$. A contact metric manifold M^{2n+1} with $\xi \in N(k,\mu)$ is called a (k,μ)-contact metric manifold. We have

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY]. \tag{2.8}$$

Applying a D-homothetic deformation to a contact metric manifold with $R(X,Y)\xi = 0$, we obtain a contact metric manifold satisfying (2.8). In [5], it is proved that the standard contact metric structure on the tangent sphere bundle $T_1(M)$ satisfies the condition that ξ belongs to the (k,μ)-nullity distribution if and only if the base manifold is the space of constant curvature. There exist examples in all dimensions and the condition that ξ belongs to the (k,μ)-nullity distribution is invariant under D-homothetic deformations; in dimensions greater than 5, the condition determines the curvature completely; dimension 3 include the 3-dimensional unimodular Lie groups with a left invariant metric.

On a (k,μ)-contact metric manifold, $k \le 1$. If $k = 1$, the structure is Sasakian ($h = 0$ and μ is indeterminant) and if $k < 1$, the (k,μ)-nullity condition completely determines the curvature of M^{2n+1} [5]. In fact, for a (k,μ)-manifold, the condition of being a Sasakian manifold, a K-contact manifold, $k = 1$ and $h = 0$ are all equivalent.

In a (k,μ)-contact metric manifold, the following relations hold ([5], [7]):

$$h^2 = (k-1)\phi^2, \quad k \le 1, \tag{2.9}$$

$$(\nabla_X \phi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX), \tag{2.10}$$

$$R(\xi,X)Y = k[g(X,Y)\xi - \eta(Y)X] + \mu[g(hX,Y)\xi - \eta(Y)hX], \tag{2.11}$$

$$S(X,\xi) = 2nk\eta(X), \tag{2.12}$$

$$\begin{aligned} S(X,Y) &= [2(n-1)-n\mu]g(X,Y)+[2(n-1)+\mu]g(hX,Y) \\ &\quad +[2(1-n)+n(2k+\mu)]\eta(X)\eta(Y),\ n\geq 1, \end{aligned} \tag{2.13}$$

$$\tau = 2n(2n-2+k-n\mu), \tag{2.14}$$

$$S(\phi X,\phi Y) = S(X,Y)-2nk\eta(X)\eta(Y)-2(2n-2+\mu)g(hX,Y), \tag{2.15}$$

where S is the Ricci tensor of type $(0,2)$ and τ is the scalar curvature of the manifold. From (2.5), it follows that

$$(\nabla_X\eta)Y = g(X+hX,\phi Y). \tag{2.16}$$

Also in a (k,μ)-manifold, the following holds

$$\begin{aligned} &\eta(R(X,Y)Z) \\ &= k[g(Y,Z)\eta(X)-g(X,Z)\eta(Y)]+\mu[g(hY,Z)\eta(X)-g(hX,Z)\eta(Y)]. \end{aligned} \tag{2.17}$$

Especially for the case $\mu = 2(1-n)$, from (2.13) it follows that the manifold is η-Einstein.

The k-nullity distribution $N(k)$ of a Riemannian manifold M [10] is defined by

$$N(k): p \longrightarrow N_p(k) = \{Z \in T_pM \mid R(X,Y)Z = k(g(Y,Z)X-g(X,Z)Y\},$$

k being a constant. If the characteristic vector field $\xi \in N(k)$, then we call a contact metric manifold an $N(k)$-contact metric manifold [4]. The ϕ-recurrent $N(k)$-contact metric manifolds have been studied by De and Gazi [9].

If $k=1$, then $N(k)$-contact metric manifold is Sasakian and if $k=0$, then $N(k)$-contact metric manifold is locally isometric to the product $E^{n+1}\times S^n(4)$ for $n>1$ and flat for $n=1$. If $k<1$, the scalar curvature is $\tau = 2n(2n-2+k)$. If $\mu=0$, then a (k,μ)-contact metric manifold reduces to a $N(k)$-contact metric manifold.

3. ϕ-recurrent (k,μ)-contact metric manifolds

Definition 3.1 ([13])**.** A Sasakian manifold is said to be locally ϕ-symmetric if the relation

$$\phi^2((\nabla_W R)(X,Y,Z) = 0$$

holds for all vector fields X,Y,Z,W orthogonal to ξ.

Definition 3.2 ([10])**.** A (k,μ)-contact metric manifold is said to be ϕ-recurrent if and only if there exists a non-zero 1-form A such that

$$\phi^2((\nabla_W R)(X,Y,Z) = A(W)R(X,Y,Z) \tag{3.1}$$

for all vector fields X,Y,Z,W. Here X,Y,Z,W are arbitrary vector fields which are not necessarily orthogonal to ξ.

If X,Y,Z,W are orthogonal to ξ, then the manifold is called *locally ϕ-recurrent*. If the 1-form A vanishes identically, then the manifold is said to be a *locally ϕ-symmetric* manifold.

Definition 3.3 ([5])**.** A contact metric manifold is said to be η-Einstein if the Ricci tensor S of type $(0,2)$ satisfies the condition

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \tag{3.2}$$

where a and b are smooth functions on M^{2n+1}.

Now we prove the main theorem of the paper.

Theorem 3.1. *A ϕ-recurrent (k,μ)-contact metric manifold is an η-Einstein manifold with constant coefficients.*

Proof. By virtue of (2.1)(a) and (3.1) we have

$$-(\nabla_W R)(X,Y,Z) + \eta((\nabla_W R)(X,Y,Z)\xi = A(W)R(X,Y,Z), \tag{3.3}$$

from which it follows that

$$\begin{aligned} &- g((\nabla_W R)(X,Y,Z),U) + \eta((\nabla_W R)(X,Y,Z)\eta(U) \\ &= A(W)g(R(X,Y,Z),U). \end{aligned} \tag{3.4}$$

Let $\{e_i\}$, $i = 1,2,3,\ldots,2n+1$, be an orthonormal basis of the tangent space at any point of the manifold. Putting $X = U = \{e_i\}$ in (3.4) and taking summation over i, $1 \le i \le 2n+1$, we get

$$-(\nabla_W S)(Y,Z) + \sum_{i=1}^{2n+1} \eta((\nabla_W R)(e_i,Y)Z)\eta(e_i) = A(W)S(Y,Z). \tag{3.5}$$

The second term of (3.5) by putting $Z = \xi$ takes the form $g((\nabla_W R)(e_i,Y)\xi, \xi)g(e_i,\xi)$, which is denoted by E. In this case E vanishes. Since the following equation is well known,

$$\begin{aligned} g((\nabla_W R)(e_i,Y)\xi,\xi) = g(\nabla_W R(e_i,Y)\xi,\xi) &- g(R(\nabla_W e_i,Y)\xi,\xi) \\ &- g(R(e_i,\nabla_W Y)\xi,\xi) - g(R(e_i,Y)\nabla_W \xi,\xi) \end{aligned}$$

at $p \in M$. Using (2.3)(b) and (2.8), we obtain

$$\begin{aligned} & g(R(e_i,\nabla_W Y)\xi,\xi) \\ = \;& g(k[\eta(\nabla_W Y)e_i - \eta(e_i)\nabla_W Y] + \mu[\eta(\nabla_W Y)he_i - \eta(e_i)h\nabla_W Y],\xi) \\ = \;& k[\eta(\nabla_W Y)\eta(e_i) - \eta(e_i)\eta(\nabla_W Y)] = 0, \end{aligned}$$

since $g(hX,Y) = g(X,hY)$.

Thus, we obtain

$$g((\nabla_W R)(e_i,Y)\xi,\xi) = g(\nabla_W R(e_i,Y)\xi,\xi) - g(R(e_i,Y)\nabla_W \xi,\xi).$$

In virtue of $g(R(e_i,Y)\xi,\xi) = g(R(\xi,\xi)e_i,Y) = 0$, we have

$$g((\nabla_W R)(e_i,Y)\xi,\xi) + g(R(e_i,Y)\xi,\nabla_W \xi) = 0,$$

since $(\nabla_W g) = 0$, which implies

$$g((\nabla_W R)(e_i,Y)\xi,\xi) = -g(R(e_i,Y)\xi,\nabla_W \xi) - g(R(e_i,Y)\nabla_W \xi,\xi) = 0.$$

Using (2.5) and applying skew-symmetry of R, we get

$$\begin{aligned} & g((\nabla_W R)(e_i, Y)\xi, \xi) \\ = \; & g(R(e_i, Y)\xi, \phi W + \phi hW) + g(R(e_i, Y)(\phi W + \phi hW), \xi) \\ = \; & g(R(\phi W + \phi hW, \xi)Y, e_i) + g(R(\xi, \phi W + \phi hW)Y, e_i). \end{aligned}$$

Hence, we obtain

$$\begin{aligned} E = & \sum_{i=1}^{2n+1} [g(R(\phi W + \phi hW, \xi)Y, e_i)g(\xi, e_i) + g(R(\xi, \phi W + \phi hW)Y, e_i)g(\xi, e_i)] \\ = & \; g(R(\phi W + \phi hW, \xi)Y, \xi) + g(R(\xi, \phi W + \phi hW)Y, \xi) = 0. \end{aligned}$$

Replacing Z by ξ in (3.5) and using (2.12), we have

$$-(\nabla_W S)(Y, \xi) = 2nkA(W)\eta(Y). \tag{3.6}$$

Now, we have

$$(\nabla_W S)(Y, \xi) = \nabla_W S(Y, \xi) - S(\nabla_W Y, \xi) - S(Y, \nabla_W \xi).$$

Using (2.5) and (2.12) in the above relation, it follows that

$$(\nabla_W S)(Y, \xi) = 2nk(\nabla_W \eta)Y + S(Y, \phi W + \phi hW). \tag{3.7}$$

By virtue of (2.3)(a) and (2.16), we get from (3.7)

$$(\nabla_W S)(Y, \xi) = -2nkg(\phi W + \phi hW, Y) + S(Y, \phi W + \phi hW). \tag{3.8}$$

By virtue of (3.6) and (3.8), we have

$$2nkA(W)\eta(Y) = 2nkg(\phi W + \phi hW, Y) - S(Y, \phi W + \phi hW). \tag{3.9}$$

Replacing Y by ϕY in (3.9) and using (2.1)(d), (2.2) and (2.15), we get

$$2nkg(\phi W + \phi hW, \phi Y) - S(\phi Y, \phi W + \phi hW) = 0,$$

or,

$$\begin{aligned} & 2nkg[g(W + hW, Y) - \eta(W + hW)\eta(Y)] - S(Y, W + hW) \\ & +2nk\eta(Y)\eta(W + hW) + 2(2n - 2 + \mu)g(W + hW, hY) = 0, \end{aligned}$$

or,

$$\begin{aligned} & 2nkg(Y, W) + 2nkg(hW, Y) - S(Y, W) - S(Y, hW) \\ & +2(2n - 2 + \mu)g(hW, Y) + 2(2n - 2 + \mu)g(h^2 W, Y) = 0, \end{aligned}$$

since $g(X, hY) = g(hX, Y)$.

Now by (2.9), the above equation takes the form

$$\begin{aligned} S(Y, W) + S(Y, hW) = & \; 2nkg(Y, W) + [2nk + 2(2n - 2 + \mu)]g(Y, hW) \\ & + 2(2n - 2 + \mu)(k - 1)g(Y, -W + \eta(W)\xi). \end{aligned} \tag{3.10}$$

Now, by using (2.13), it follows that

$$\begin{aligned} S(Y, hW) = & \; (2n - 2 - n\mu)g(Y, hW) - (2n - 2 + \mu)(k - 1)g(Y, W) \\ & + (2n - 2 + \mu)(k - 1)\eta(W)\eta(Y). \end{aligned} \tag{3.11}$$

Hence from (3.10), we get

(3.12)

$$\begin{aligned}&S(Y,W)+(2n-2-n\mu)g(Y,hW)-(2n-2+\mu)(k-1)g(Y,W)\\&\quad+(2n-2+\mu)(k-1)\eta(Y)\eta(W)\\&=2nkg(Y,W)+[2nk+2(2n-2+\mu)]g(Y,hW)-2(2n-2+\mu)(k-1)g(Y,W)\\&\quad+2(2n-2+\mu)(k-1)\eta(Y)\eta(W),\end{aligned}$$

or,

$$\begin{aligned}S(Y,W)&=[\mu(1-k)+2(n-1)+2k]g(Y,W)\\&\quad+[2(nk+n-1)+\mu(n+2)]g(Y,hW)\\&\quad+(2n-2+\mu)(k-1)\eta(Y)\eta(W).\end{aligned}\tag{3.13}$$

Replacing W by hW and using (2.1)(a), we get from (3.13)

$$\begin{aligned}S(Y,hW)&=[\mu(1-k)+2(n-1)+2k]g(Y,hW)\\&\quad+[2(nk+n-1)+\mu(n+2)]g(Y,h^2W).\end{aligned}\tag{3.14}$$

From (3.11) and (3.14), using (2.9), it follows that

$$\begin{aligned}[\mu(k-1-n)-2k]g(Y,hW)&=(k-1)[-2nk-\mu(n+1)]g(Y,W)\\&\quad+(k-1)[2nk+\mu(n+1)]\eta(Y)\eta(W).\end{aligned}\tag{3.15}$$

From (3.13) and (3.15), we get

$$S(Y,W)=\alpha g(Y,W)+\beta\eta(Y)\eta(W),$$

where $\alpha=[[\mu(1-k)+2(n-1)+2k]+[\,2(nk+n-1)+\mu(n+2)]\frac{[-2nk-\mu(n+1)](k-1)]}{\mu(k-1-n)-2k}]$ and $\beta=[[2(n-1)+\mu](k-1)+[2(nk+n-1)+\mu(n+2)]\frac{[2nk+\mu(n+1)](k-1)]}{\mu(k-1-n)-2k}]$. So, the manifold is an η-Einstein manifold with constant coefficients. Hence the theorem is proved. □

Theorem 3.2. *In a ϕ-recurrent (k,μ)-contact metric manifold (M^{2n+1},g) $(n>1)$ the characteristic vector field ξ and the vector field ρ associated to the 1-form A are co-directional and the 1-form A is given by*

$$A(W)=\eta(W)\eta(\rho),$$

provided that $(2n-1)^2k^2+\mu^2(k-1)\neq 0$.

Proof. In a (k,μ)-contact metric manifold, the relation (3.3) holds. Changing W,X,Y cyclically in (3.3) and then adding the results we obtain

$$\begin{aligned}&-[(\nabla_W R)(X,Y)Z+(\nabla_X R)(Y,W)Z+(\nabla_Y R)(W,X)Z]\\&+[\eta((\nabla_W R)(X,Y)Z)+\eta((\nabla_X R)(Y,W)Z)+\eta((\nabla_Y R)(W,X)Z)]\xi\\&=A(W)R(X,Y)Z+A(X)R(Y,W)Z+A(Y)R(W,X)Z,\end{aligned}$$

which yields by virtue of Bianchi's identity that

$$A(W)\eta(R(X,Y)Z)+A(X)\eta(R(Y,W)Z)+A(Y)\eta(R(W,X)Z)=0.\tag{3.16}$$

With the help of (2.17), (3.16) reduces to

$$\begin{aligned} &A(W)[k\{g(Y,Z)\eta(X)-g(X,Z)\eta(Y)\} \\ &\qquad + \mu\{g(hY,Z)\eta(X)-g(hX,Z)\eta(Y)\}] \\ &+ A(X)[k\{g(W,Z)\eta(Y)-g(Y,Z)\eta(W)\} \\ &\qquad + \mu\{g(hW,Z)\eta(Y)-g(hY,Z)\eta(W)\}] \\ &+ A(Y)[k\{g(X,Z)\eta(W)-g(W,Z)\eta(X)\} \\ &\qquad + \mu\{g(hX,Z)\eta(W)-g(hW,Z)\eta(X)\}] \\ &= 0. \end{aligned} \tag{3.17}$$

Putting $Y = Z = e_i$ in (3.17) and taking summation over i, $1 \le i \le 2n+1$, we get

$$(2n-1)k[A(W)\eta(X)-A(X)\eta(W)]+\mu[A(hX)\eta(W)-A(hW)\eta(X)] = 0. \tag{3.18}$$

Substituting X by ξ in (3.18), we have

$$(2n-1)k[A(W) - A(\xi)\eta(W)] - \mu A(hW) = 0. \tag{3.19}$$

Replacing W by hW in (3.20) and using (2.9), we get

$$(2n-1)kA(hW) = \mu(k-1)[-A(W) + \eta(W)A(\xi)]. \tag{3.20}$$

From (3.19) and (3.20), we obtain

$$A(W) = A(\xi)\eta(W) = \eta(\rho)\eta(W),$$

provided that

$$(2n-1)^2k^2 + \mu^2(k-1) \neq 0,$$

where $A(\xi) = g(\xi, \rho)$. This proves the theorem. □

4. 3-dimensional locally ϕ-recurrent (k, μ)-contact metric manifolds

On any 3-dimensional Riemannian manifold we have

$$\begin{aligned} R(X,Y)Z = {} & g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X \\ & - S(X,Z)Y - \frac{\tau}{2}[g(Y,Z)X - g(X,Z)Y] \end{aligned} \tag{4.1}$$

for any vector fields X, Y, Z, where Q is the Ricci operator, that is, $g(QX,Y) = S(X,Y)$ and τ is the scalar curvature of the manifold. Moreover, using Remark 3.2 [5], we have

$$QX = \mu(\lambda - 1)X, \tag{4.2}$$

where $\lambda = \sqrt{1-k}$, $k < 1$. Therefore, it follows from (4.2) that

$$S(X,Y) = \mu(\lambda - 1)g(X,Y). \tag{4.3}$$

Thus from (4.1), (4.2), and (4.3), we get

$$R(X,Y)Z = 2\mu(\lambda-1)[g(Y,Z)X - g(X,Z)Y] - \frac{\tau}{2}[g(Y,Z)X - g(X,Z)Y]. \tag{4.4}$$

Taking the covariant differentiation to the both sides of the equation (4.4), we get

$$(\nabla_W R)(X,Y)Z = \frac{-d\tau(W)}{2}[g(Y,Z)X - g(X,Z)Y]. \tag{4.5}$$

Applying ϕ^2 to the both sides of (4.5) and using (2.1)(a) and (2.1)(c), we get

$$\phi^2(\nabla_W R)(X,Y)Z = \frac{d\tau(W)}{2}[g(X,Z)\phi^2 Y - g(Y,Z)\phi^2 X]. \tag{4.6}$$

By (3.1) the equation (4.6) reduces to

$$A(W)R(X,Y)Z = \frac{d\tau(W)}{2}[g(Y,Z)X - g(X,Z)Y + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi].$$

Noting that we may assume that all vector fields X, Y, Z, W are orthogonal to ξ, then we get

$$A(W)R(X,Y)Z = \frac{d\tau(W)}{2}[g(Y,Z)X - g(X,Z)Y].$$

Putting $W = \{e_i\}$, where $\{e_i\}$, $i = 1,2,3$, is an orthonormal basis of the tangent space at any point of the manifold and taking summation over i, $1 \le i \le 3$, we obtain

$$R(X,Y)Z = \lambda[g(Y,Z)X - g(X,Z)Y],$$

where $\lambda = \frac{d\tau(e_i)}{2A(e_i)}$ is a scalar, since A is a non-zero 1-form. Then by Schur's theorem λ will be a constant on the manifold. Therefore, M^3 is of constant curvature λ. Thus we get the following theorem:

Theorem 4.1. *A 3-dimensional connected locally ϕ-recurrent (k,μ)-contact metric manifold is the space of constant curvature.*

5. Existence of locally ϕ-recurrent (k,μ)-contact metric manifolds

In this section, we construct an example of a locally ϕ-recurrent (k,μ)-contact metric manifold to prove the existence. We consider the 3-dimensional manifold $M = \{(x,y,z) \in \mathbb{R}^3 \mid x \neq 0\}$, where (x,y,z) are the standard coordinates in $\mathbb{R}^3$. Let $\{e_1, e_2, e_3\}$ be linearly independent global frame on M given by

$$e_1 = \frac{2}{x}\frac{\partial}{\partial y}, \quad e_1 = 2\frac{\partial}{\partial x} - \frac{4z}{x}\frac{\partial}{\partial y} + xy\frac{\partial}{\partial z}, \quad e_3 = \frac{\partial}{\partial z}.$$

Let g be the Riemannian metric defined by

$$g(e_1,e_3) = g(e_2,e_3) = g(e_1,e_2) = 0, \quad g(e_1,e_1) = g(e_2,e_2) = g(e_3,e_3) = 1.$$

Let η be the 1-form defined by

$$\eta(U) = g(U, e_3)$$

for any $U \in \chi(M)$. Let ϕ be the $(1,1)$-tensor field defined by

$$\phi e_1 = e_2, \ \phi e_2 = -e_1, \ \phi e_3 = 0.$$

Then using the linearity of ϕ and g we have

$$\eta(e_3) = 1,$$

$$\phi^2(U) = -U + \eta(U)e_3$$

and

$$g(\phi U, \phi W) = g(U, W) - \eta(U)\eta(W)$$

for any $U, W \in \chi(M)$. Moreover

$$he_1 = -e_1, \quad he_2 = e_2, \text{ and } he_3 = 0.$$

Thus for $e_3 = \xi$, (ϕ, ξ, η, g) defines a contact metric structure on M.

Let ∇ be the Levi-Civita connection with respect to the Riemannian metric g and R be the curvature tensor of g. Then we have

$$[e_1, e_2] = 2e_3 + \frac{2}{x}e_1, \quad [e_1, e_3] = 0, \quad [e_2, e_3] = 2e_1.$$

The Riemannian connection ∇ of the metric tensor g is given by

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]).$$

Taking $e_3 = \xi$ and using the above formula for the Riemannian metric g, we can easily calculate that

$$\nabla_{e_1} e_3 = 0, \quad \nabla_{e_2} e_3 = 2e_1, \quad \nabla_{e_3} e_3 = 0, \quad \nabla_{e_3} e_1 = 0, \quad \nabla_{e_1} e_2 = \frac{2}{x}e_1,$$
$$\nabla_{e_1} e_1 = -2e_3, \quad \nabla_{e_2} e_2 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_2} e_1 = -\frac{2}{x}e_2.$$

From the above it can be easily seen that (ϕ, ξ, η, g) is a (k, μ)-contact metric structure on M. Consequently $M^3(\phi, \xi, \eta, g)$ is a (k, μ)-contact metric manifold with $k = -\frac{2}{x} \neq 0$ and $\mu = -\frac{2}{x} \neq 0$.

Using the above relations, we can easily calculate the non-vanishing components of the curvature tensor as follows:

$$R(e_2, e_3)e_2 = -\frac{4}{x}e_1, \quad R(e_2, e_3)e_1 = \frac{4}{x}e_2,$$

and components which can be obtained from these by the symmetry properties.

We shall now show that such a (k, μ)-contact metric manifold is ϕ-recurrent. Since $\{e_1, e_2, e_3\}$ form a basis of M^3, any vector field $X \in \chi(M)$ can be taken as

$$X = a_1 e_1 + a_2 e_2 + a_3 e_3,$$

where a_i are positive real numbers, $i = 1, 2, 3$. Thus the covariant derivatives of the curvature tensor are given by

$$(\nabla_X R)(e_2, e_3)e_1 = \frac{-8a_2}{x^2} e_2, \quad (\nabla_X R)(e_2, e_3)e_2 = \frac{8a_2}{x^2} e_1.$$

This implies that

$$\phi^2((\nabla_X R)(e_2, e_3)e_1) = \frac{8a_2}{x^2} e_2, \quad \phi^2((\nabla_X R)(e_2, e_3)e_2) = \frac{-8a_2}{x^2} e_1.$$

Let us consider the non-vanishing 1-form

$$A(X) = \frac{2a_2}{x}$$

at any point $p \in M^3$. Then we get

$$\phi^2((\nabla_X R)(e_2, e_3)e_1) = A(X)R(e_2, e_3)e_1,$$

and

$$\phi^2((\nabla_X R)(e_2, e_3)e_2) = A(X)R(e_2, e_3)e_2.$$

This implies that the manifold under consideration is a locally ϕ-recurrent (k,μ)-contact metric manifold which is neither locally symmetric nor locally ϕ-symmetric.

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