Bull. Korean Math. Soc. **45** (2008), No. 4, pp. 729–737

ON MULTI-JENSEN FUNCTIONS AND JENSEN DIFFERENCE

Krzysztof Ciepliński

Abstract. In this paper we characterize multi-Jensen functions $f : V^n \to W$, where n is a positive integer, V, W are commutative groups and V is uniquely divisible by 2. Moreover, under the assumption that $f : \mathbb{R} \to \mathbb{R}$ is Borel measurable, we obtain representation of f (respectively, f, g, $h : \mathbb{R} \to \mathbb{R}$) such that the Jensen difference

$$2f\left(\frac{x+y}{2}\right) - f(x) - f(y)$$

(respectively, the Pexider difference

$$2f\left(\frac{x+y}{2}\right) - g(x) - h(y))$$

takes values in a countable subgroup of $\mathbb{R}$.

1. Introduction

In 2005 W. Prager and J. Schwaiger (see [23]) introduced the notion of multi-Jensen functions $f : V^n \longrightarrow W$ (V and W being vector spaces over the rationals) with the connection with generalized polynomials. On the other hand, the stability of the Jensen functional equation

$$2f\left(\frac{x+y}{2}\right) = f(x) + f(y)$$

(f satisfying this equation is called a *Jensen mapping*) was studied by a number of mathematicians (see for instance [20], [10], [17], [16] and [7]).

Speaking of the stability of a functional equation we follow the question of S. Ulam: "when is it true that the solution of an equation differing slightly from a given one, must of necessity be close to the solution of the given equation?" (see [25]). As the words "differing slightly" and "be close" may have various meanings, different kinds of stability can be dealt with (see for instance [15]).

Received February 22, 2008.

2000 *Mathematics Subject Classification.* Primary 39B72, 39B82, 39A70, 39B22, 39B52.

Key words and phrases. multi-Jensen function, multi-additive mapping, stability, Jensen difference, Pexider difference.

In 2006 J.-H. Bae and W.-G. Park (see [1]) gave the general solution the system of equations

$$\begin{cases} 2f(\frac{x+y}{2}, z) = f(x,z) + f(y,z), \\ 2f(x, \frac{y+z}{2}) = f(x,y) + f(x,z), \end{cases}$$

where $f : V^2 \longrightarrow W$ and V, W are vector spaces. Under some additional assumptions they also proved the stability of this system.

Our first goal is to generalize these results. More precisely, generalizing some outcomes from [23] and [1], we obtain the form of multi-Jensen functions in the case when V, W are commutative groups and V is uniquely divisible by 2. Moreover, we generalize some results from [16] and [1] dealing with the stability in the spirit of P. Gavruta (see [12] and also [21], [22]).

From an example of G. Godini (see [14]) it can be seen that it is not generally true that a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ for which the Cauchy difference

$$f(x+y) - f(x) - f(y)$$

belongs to $\mathbb{Z}$ for all $x, y \in \mathbb{R}$ has to be of the form $A + k$, where A is an additive mapping and k takes integer values only. However, such a representation is possible under some regularity condition imposed on f. It seems that J. G. van der Corput was the first who gave such a condition (see [9]). Further results and their generalizations (also concerning the Pexider difference

$$f(x+y) - g(x) - h(y))$$

were obtained for instance by K. Baron, PL. Kannappan, J. Brzdęk, N. Frantzikinakis, M. Bajger and the author (see [4], [5], [11], [2], and [8]).

Later the *Jensen difference*

$$2f\left(\frac{x+y}{2}\right) - f(x) - f(y)$$

(see [5], [13], and [19]) as well as the "quadratic" difference

$$f(x+y) + f(x-y) - 2f(x) - 2f(y)$$

(see [6] and [18]) were also investigated.

The second aim of this paper is to study the Jensen difference and the *Pexider difference*

$$2f\left(\frac{x+y}{2}\right) - g(x) - h(y),$$

where $f, g, h : \mathbb{R} \to \mathbb{R}$. Under the assumption that f is Borel measurable we obtain representation of f (respectively, f, g and h) with the Jensen (respectively, Pexider) difference taking values in a countable subgroup of $\mathbb{R}$.

2. Main results

2.1. Multi-Jensen functions

We follow the notation used in [23].

Denote by $|S|$ the cardinality of the finite set S and put

$$\mathbf{n} := \{1, \ldots, n\}, \qquad n \in \mathbb{N},$$

where $\mathbb{N}$ stands for the set of all positive integers.

Let V, W be commutative groups and V be uniquely divisible by 2. For sets S, T with $S \subseteq T$ and $x = (x_t)_{t \in T} \in V^T$, let $x_S := x\,|_S \in V^S$, i.e., $x\,|_S$ is the restriction of the mapping $x : T \longrightarrow V$ to S. For $y \in V^S$ let $y^T \in V^T$ be given by

$$(y^T)_S := y \quad \text{and} \quad (y^T)_{T \setminus S} := 0.$$

Let us also recall that a function $f : V^T \longrightarrow W$ is called *multi-additive* or *T-additive* if it is additive in each variable. Similarly, define f to be *multi-Jensen* or *T-Jensen* if it is a Jensen mapping in each variable.

Since (see Theorem 1.4 in [5]) every Jensen function $f : V \longrightarrow W$ is of the form

$$f(x) = a(x) + c, \qquad x \in V$$

with, uniquely determined, a $c \in W$ and an additive mapping $a : V \longrightarrow W$, we have the following

Lemma 2.1. *Let V, W be commutative groups and V be uniquely divisible by 2. Assume also that T is a set and $f : V^T \to W$. Then f is multi-additive if and only if it is a multi-Jensen mapping such that for any $i \in T, x \in V^T$ with $x(i) = 0$ we have $f(x) = 0$.*

Finally, for $T \subseteq S \subseteq \mathbf{n}$ and $y \in V^S$ we write $y_T^{\mathbf{n}}$ for $(y_T)^{\mathbf{n}}$, i.e., $y_T^{\mathbf{n}} = z \in V^{\mathbf{n}}$ with $z_T = y_T$ and $z_{\mathbf{n} \setminus T} = 0$.

Our first theorem characterizes multi-Jensen functions.

Theorem 2.2. *Let V, W be commutative groups and V be uniquely divisible by 2. Assume also that $n \in \mathbb{N}$ and $f : V^{\mathbf{n}} \longrightarrow W$. Then f is multi-Jensen if and only if there is a family $(M_S)_{S \subseteq \mathbf{n}}$ of S-additive functions such that*

$$f(x) = \sum_{S \subseteq \mathbf{n}} M_S(x_S), \qquad x \in V^{\mathbf{n}}. \tag{1}$$

Proof. Assume that $f : V^{\mathbf{n}} \to W$ is multi-Jensen and put

$$M_S(y) = \sum_{T \subseteq S} (-1)^{|S \setminus T|} f(x_T^{\mathbf{n}}), \qquad y \in V^S,\ S \subseteq \mathbf{n}, \tag{2}$$

where $x := y^{\mathbf{n}}$. We shall show that these mappings are multi-additive.

Fix an $S \subseteq \mathbf{n}$ and note that for every $T \subseteq S$ the mapping $V^S \ni y \longmapsto f(y_T^{\mathbf{n}}) = f(x_T^{\mathbf{n}}) \in W$ is multi-Jensen and, in consequence, so is M_S. Take

$i \in S, y \in V^S$ with $y(i) = 0$ and let $x := y^{\mathbf{n}}$. Clearly, if $T \subset S$ and $i \in S \setminus T$, then $x^{\mathbf{n}}_T = x^{\mathbf{n}}_{T\cup\{i\}}$. Therefore,

$$\begin{aligned} M_S(y) &= \sum_{T\subseteq S, i\in S\setminus T} (-1)^{|S\setminus T|} f(x^{\mathbf{n}}_T) + \sum_{T\subseteq S, i\in S\setminus T} (-1)^{|S\setminus (T\cup\{i\})|} f(x^{\mathbf{n}}_{T\cup\{i\}}) \\ &= \sum_{T\subseteq S, i\in S\setminus T} (-1)^{|S\setminus T|} (f(x^{\mathbf{n}}_T) - f(x^{\mathbf{n}}_T)) = 0, \end{aligned}$$

and the multi-additivity of M_S follows from Lemma 2.1.

Next, note that

$$\sum_{S\subseteq \mathbf{n}} M_S(x_S) = \sum_{S\subseteq \mathbf{n}} \sum_{T\subseteq S} (-1)^{|S\setminus T|} f(x^{\mathbf{n}}_T) = \sum_{T\subseteq \mathbf{n}} \alpha_T f(x^{\mathbf{n}}_T),$$

where

$$\alpha_T := \sum_{T\subseteq S\subseteq \mathbf{n}} (-1)^{|S\setminus T|} = \sum_{U\subseteq \mathbf{n}\setminus T} (-1)^{|U|}.$$

But

$$\alpha_T = \begin{cases} 1, & T = \mathbf{n}, \\ \sum_{l=0}^{|\mathbf{n}\setminus T|} \binom{|\mathbf{n}\setminus T|}{l} (-1)^l = 0, & T \neq \mathbf{n}, \end{cases}$$

and therefore

$$\sum_{S\subseteq \mathbf{n}} M_S(x_S) = f(x^{\mathbf{n}}_{\mathbf{n}}) = f(x), \qquad x \in V^{\mathbf{n}},$$

that is (1) holds.

On the other hand, if the function f is given by (1), where $(M_S)_{S\subseteq \mathbf{n}}$ is a family of S-additive mappings, then the fact that it is multi-Jensen follows from obvious fact that for every $S \subseteq \mathbf{n}$ so is the function $V^{\mathbf{n}} \ni x \longmapsto M_S(x_S) \in W$. □

2.2. Stability

Our next theorem generalizes Theorem 6 from [1] (see also Theorems 1 and 2 and Corollary 3 in [16]).

Theorem 2.3. *Let V be a commutative group uniquely divisible by 2 and W be a Banach space. Assume also that $n \in \mathbb{N}$ and for every $i \in \mathbf{n}$, $\varphi_i : V^{n+1} \to [0,\infty)$ is a mapping such that*

$$\begin{aligned} &\widetilde{\varphi}_i(x_1, \ldots, x_{n+1}) \\ &:= \sum_{j=0}^{\infty} \frac{1}{3^{j+1}} [\varphi_i(3^j x_1, x_2, \ldots, x_{n+1}) + \cdots \\ &\quad + \varphi_i(x_1, \ldots, x_{i-2}, 3^j x_{i-1}, x_i, \ldots, x_{n+1}) \\ &\quad + \varphi_i(x_1, \ldots, x_{i-1}, 3^j x_i, 3^j x_{i+1}, x_{i+2}, \ldots, x_{n+1}) \\ &\quad + \varphi_i(x_1, \ldots, x_{i+1}, 3^j x_{i+2}, x_{i+3}, \ldots, x_{n+1}) + \cdots \\ &\quad + \varphi_i(x_1, \ldots, x_n, 3^j x_{n+1})] < \infty, \;\; (x_1, \ldots, x_{n+1}) \in V^{n+1}. \end{aligned} \tag{3}$$

If $f : V^n \longrightarrow W$ is a function satisfying

$$
\begin{aligned}
&\Big\|2f\Big(x_1,\dots,x_{i-1},\frac{x_i+x_i'}{2},x_{i+1},\dots,x_n\Big)-f(x_1,\dots,x_n)\\
&\quad -f(x_1,\dots,x_{i-1},x_i',x_{i+1},\dots,x_n)\Big\|\\
&\le \varphi_i(x_1,\dots,x_i,x_i',x_{i+1},\dots,x_n),\\
&\qquad (x_1,\dots,x_i,x_i',x_{i+1},\dots,x_n)\in V^{n+1},\ i\in\mathbf{n},
\end{aligned}
\tag{4}
$$

then for every $i \in \mathbf{n}$ there exists a multi-Jensen mapping $F_i : V^n \to W$ for which

$$
\begin{aligned}
&\|f(x_1,\dots,x_n)-f(x_1,\dots,x_{i-1},0,x_{i+1},\dots,x_n)-F_i(x_1,\dots,x_n)\|\\
&\le \widetilde{\varphi}_i(x_1,\dots,x_i,-x_i,x_{i+1},\dots,x_n)\\
&\quad +\widetilde{\varphi}_i(x_1,\dots,x_{i-1},-x_i,3x_i,x_{i+1},\dots,x_n),\quad (x_1,\dots,x_n)\in V^n.
\end{aligned}
\tag{5}
$$

For every $i \in \mathbf{n}$ the function F_i is given by

$$
\begin{aligned}
&F_i(x_1,\dots,x_n)\\
&:=\lim_{j\to\infty}\frac{1}{3^j}f(x_1,\dots,x_{i-1},3^jx_i,x_{i+1},\dots,x_n),\ (x_1,\dots,x_n)\in V^n.
\end{aligned}
\tag{6}
$$

Proof. Fix $x_1, \dots, x_n \in V$ and $i \in \mathbf{n}$. By (4) we get

$$
\begin{aligned}
&\|2f(x_1,\dots,x_{i-1},0,x_{i+1},\dots,x_n)-f(x_1,\dots,x_n)\\
&\quad -f(x_1,\dots,x_{i-1},-x_i,x_{i+1},\dots,x_n)\|\\
&\le \varphi_i(x_1,\dots,x_i,-x_i,x_{i+1},\dots,x_n)
\end{aligned}
$$

and

$$
\begin{aligned}
&\|2f(x_1,\dots,x_n)-f(x_1,\dots,x_{i-1},-x_i,x_{i+1},\dots,x_n)\\
&\quad -f(x_1,\dots,x_{i-1},3x_i,x_{i+1},\dots,x_n)\|\\
&\le \varphi_i(x_1,\dots,x_{i-1},-x_i,3x_i,x_{i+1},\dots,x_n).
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\|3f(x_1,\dots,x_n)-f(x_1,\dots,x_{i-1},3x_i,x_{i+1},\dots,x_n)\\
&\quad -2f(x_1,\dots,x_{i-1},0,x_{i+1},\dots,x_n)\|\\
&\le \varphi_i(x_1,\dots,x_i,-x_i,x_{i+1},\dots,x_n)\\
&\quad +\varphi_i(x_1,\dots,x_{i-1},-x_i,3x_i,x_{i+1},\dots,x_n),
\end{aligned}
$$

and consequently for any non-negative integers l and m such that $l < m$ we obtain

$$\begin{aligned}
&\|\frac{1}{3^l}f(x_1,\dots,x_{i-1},3^l x_i,x_{i+1},\dots,x_n)\\
&\quad-\frac{1}{3^m}f(x_1,\dots,x_{i-1},3^m x_i,x_{i+1},\dots,x_n)\\
&\quad-\sum_{j=l}^{m-1}\frac{2}{3^{j+1}}f(x_1,\dots,x_{i-1},0,x_{i+1},\dots,x_n)\| \qquad (7)\\
&\le\sum_{j=l}^{m-1}\frac{1}{3^{j+1}}[\varphi_i(x_1,\dots,x_{i-1},3^j x_i,-3^j x_i,x_{i+1},\dots,x_n)\\
&\quad+\varphi_i(x_1,\dots,x_{i-1},-3^j x_i,3^{j+1}x_i,x_{i+1},\dots,x_n)].
\end{aligned}$$

Therefore from (3) it follows that $(\frac{1}{3^j}f(x_1,\dots,x_{i-1},3^j x_i,x_{i+1},\dots,x_n))_{j\in\mathbb{N}}$ is a Cauchy sequence. Since the space W is complete, this sequence is convergent and we define $F_i : V^n \longrightarrow W$ by (6). Putting $l = 0$, letting $m \longrightarrow \infty$ in (7) and using (3) we see that (5) holds.

Finally, fix $x_i' \in V$, $j \in \mathbb{N}$ and note that according to (4) we have

$$\begin{aligned}
&\|\frac{2}{3^j}f(x_1,\dots,x_{i-1},3^j\frac{x_i+x_i'}{2},x_{i+1},\dots,x_n)\\
&\quad-\frac{1}{3^j}f(x_1,\dots,x_{i-1},3^j x_i,x_{i+1},\dots,x_n)\\
&\quad-\frac{1}{3^j}f(x_1,\dots,x_{i-1},3^j x_i',x_{i+1},\dots,x_n)\|\\
&\le\frac{1}{3^j}\varphi_i(x_1,\dots,x_{i-1},3^j x_i,3^j x_i',x_{i+1},\dots,x_n).
\end{aligned}$$

Next, fix $k \in \mathbf{n}\setminus\{i\}$, $x_k' \in V$ and assume that $k < i$ (the same arguments apply to the case where $k > i$). From (4) it follows that

$$\begin{aligned}
&\|\frac{2}{3^j}f(x_1,\dots,x_{k-1},\frac{x_k+x_k'}{2},x_{k+1},\dots,x_{i-1},3^j x_i,x_{i+1},\dots,x_n)\\
&\quad-\frac{1}{3^j}f(x_1,\dots,x_{i-1},3^j x_i,x_{i+1},\dots,x_n)\\
&\quad-\frac{1}{3^j}f(x_1,\dots,x_{k-1},x_k',x_{k+1},\dots,x_{i-1},3^j x_i,x_{i+1},\dots,x_n)\|\\
&\le\frac{1}{3^j}\varphi_k(x_1,\dots,x_k,x_k',x_{k+1},\dots,x_{i-1},3^j x_i,x_{i+1},\dots,x_n).
\end{aligned}$$

Letting $j \to \infty$ in the above inequalities and using (3) we see that the mapping F_i is multi-Jensen. □

2.3. Jensen and Pexider differences

The results of this section correspond to some outcomes from [13], [19] and [5] (see also Theorem in [11] and Theorem 2 in [8]). The first one deals with the Jensen difference.

Theorem 2.4. *Let E be a countable subgroup of $\mathbb{R}$. If $f : \mathbb{R} \longrightarrow \mathbb{R}$ satisfies*

$$2f\left(\frac{x+y}{2}\right) - f(x) - f(y) \in E, \qquad x, y \in \mathbb{R} \tag{8}$$

and f is Borel measurable, then there exists a $c \in \mathbb{R}$ such that

$$f(x) - f(0) - cx \in E, \qquad x \in \mathbb{R}.$$

Proof. Put

$$\widehat{f}(x) := f(x) - f(0), \qquad x \in \mathbb{R}.$$

Then from (8) it follows that

$$2\widehat{f}\left(\frac{x+y}{2}\right) - \widehat{f}(x) - \widehat{f}(y) \in E, \qquad x, y \in \mathbb{R},$$

whence, setting $y := 0$, we get

$$2\widehat{f}\left(\frac{x}{2}\right) - \widehat{f}(x) \in E, \qquad x \in \mathbb{R}.$$

These two relations give

$$\begin{aligned} &\widehat{f}(x+y) - \widehat{f}(x) - \widehat{f}(y) \\ &= 2\widehat{f}\left(\frac{x+y}{2}\right) - \widehat{f}(x) - \widehat{f}(y) + \widehat{f}(x+y) - 2\widehat{f}\left(\frac{x+y}{2}\right) \in E, \qquad x, y \in \mathbb{R}. \end{aligned}$$

Since the mapping $\widehat{f}$ is Borel measurable, our assertion follows from Theorem in [11]. □

We finish with a theorem concerning the Pexider difference.

Theorem 2.5. *Let E be a countable subgroup of $\mathbb{R}$. If $f, g, h : \mathbb{R} \to \mathbb{R}$ satisfy*

$$2f\left(\frac{x+y}{2}\right) - g(x) - h(y) \in E, \qquad x, y \in \mathbb{R} \tag{9}$$

and f is Borel measurable, then there exists a $c \in \mathbb{R}$ such that

$$\left\{ \begin{array}{l} 2f(x) - 2f(0) - 2cx \in E, \\ g(x) - g(0) - cx \in E, \qquad x \in \mathbb{R}. \\ h(x) - h(0) - cx \in E, \end{array} \right. \tag{10}$$

Proof. Put

$$\widehat{f}(x) := f(x) - f(0), \quad \widehat{g}(x) := g(x) - g(0), \quad \widehat{h}(x) := h(x) - h(0), \qquad x \in \mathbb{R}.$$

Then from (9) it follows that

$$2\widehat{f}\left(\frac{x+y}{2}\right) - \widehat{g}(x) - \widehat{h}(y) \in E, \qquad x, y \in \mathbb{R}. \tag{11}$$

Set

$$\widetilde{f}(x) := 2\widehat{f}\left(\frac{x}{2}\right), \qquad x \in \mathbb{R}$$

and note that in view of (11) we have

$$\widetilde{f}(x+y) - \widehat{g}(x) - \widehat{h}(y) \in E, \qquad x, y \in \mathbb{R}. \tag{12}$$

Putting $y := 0$ and $x := 0$ in (12) separately we see that

$$\begin{cases} \widetilde{f}(x) - \widehat{g}(x) \in E, & x \in \mathbb{R}, \\ \widetilde{f}(y) - \widehat{h}(y) \in E, & y \in \mathbb{R}. \end{cases} \tag{13}$$

(12) and (13) give

$$\begin{aligned} &\widetilde{f}(x+y) - \widetilde{f}(x) - \widetilde{f}(y) \\ &= \widetilde{f}(x+y) - \widehat{g}(x) - \widehat{h}(y) + \widehat{g}(x) - \widetilde{f}(x) + \widehat{h}(y) - \widetilde{f}(y) \in E, \qquad x, y \in \mathbb{R}. \end{aligned}$$

Since the mapping $\widetilde{f}$ is Borel measurable, from Theorem in [11] it follows that there is a $c \in \mathbb{R}$ such that $\widetilde{f}(x) - cx \in E$ for $x \in \mathbb{R}$. This together with the definition of $\widetilde{f}$ and (13) finishes the proof. □

Note added in proof. Recently Hyers-Ulam stability of the multi-Jensen equation in the case when V, W are vector spaces over the rationals was investigated by W. Prager and J. Schwaiger (see [24]).

References

[1] J.-H. Bae and W.-G. Park, *On the solution of a bi-Jensen functional equation and its stability*, Bull. Korean Math. Soc. **43** (2006), no. 3, 499–507.

[2] M. Bajger, *On the composite Pexider equation modulo a subgroup*, Publ. Math. Debrecen **64** (2004), no. 1-2, 39–61.

[3] K. Baron, *Orthogonality and additivity modulo a discrete subgroup*, Aequationes Math. **70** (2005), no. 1-2, 189–190.

[4] K. Baron and Pl. Kannappan, *On the Pexider difference*, Fund. Math. **134** (1990), no. 3, 247–254.

[5] J. Brzdęk, *The Cauchy and Jensen diferences on semigroups*, Publ. Math. Debrecen **48** (1996), no. 1-2, 117–136.

[6] ——, *On orthogonally exponential functionals*, Pacific J. Math. **181** (1997), no. 2, 247–267.

[7] L. Cadariu and V. Radu, *Fixed points and the stability of Jensen's functional equation*, JIPAM. J. Inequal. Pure Appl. Math. **4** (2003), no. 1, Article 4.

[8] K. Ciepliński, *On a generalized Pexider equation and the Pexider difference*, Iteration theory (ECIT '06), 27–36, Grazer Math. Ber., **351**, Karl-Franzens-Univ. Graz, Graz, 2007.

[9] J. G. van der Corput, *Goniometrische functies gekarakteriseerd door een functionaal betrekking*, Euclides **17** (1940), 55–75.

[10] G. L. Forti, *Hyers-Ulam stability of functional equations in several variables*, Aequationes Math. **50** (1995), no. 1-2, 143–190.

[11] N. Frantzikinakis, *Additive functions modulo a countable subgroup of* $\mathbb{R}$, Colloq. Math. **95** (2003), no. 1, 117–122.

[12] P. Gavruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), no. 3, 431–436.

[13] P. Gavruta, S.-M. Jung, and K.-S. Lee, *Remarks on the Pexider equations modulo a subgroup*, Far East J. Math. Sci. (FJMS) **19** (2005), no. 2, 215–222.

[14] G. Godini, *Set-valued Cauchy functional equation*, Rev. Roumaine Math. Pures Appl. **20** (1975), no. 10, 1113–1121.

[15] D. H. Hyers, G. Isac, and Th. M. Rassias, *Stability of functional equations in several variables*, Progress in Nonlinear Differential Equations and their Applications, **34**. Birkhäser Boston, Inc., Boston, MA, 1998.

[16] K.-W. Jun and Y.-H. Lee, *A generalization of the Hyers-Ulam-Rassias stability of Jensen's equation*, J. Math. Anal. Appl. **238** (1999), no. 1, 305–315.

[17] S.-M. Jung, *Hyers-Ulam-Rassias stability of Jensen's equation and its application*, Proc. Amer. Math. Soc. **126** (1998), no. 11, 3137–3143.

[18] ———, *On the quadratic functional equation modulo a subgroup*, Indian J. Pure Appl. Math. **36** (2005), no. 8, 441–450.

[19] S.-M. Jung and K.-S. Lee, *On the Jensen functional equation modulo a subgroup*, J. Appl. Algebra Discrete Struct. **5** (2007), no. 1, 21–32.

[20] Z. Kominek, *On a local stability of the Jensen functional equation*, Demonstratio Math. **22** (1989), no. 2, 499–507.

[21] A. Najati, *Hyers-Ulam-Rassias stability of a cubic functional equation*, Bull. Korean Math. Soc. **44** (2007), no. 4, 825–840.

[22] C. Park and Th. M. Rassias, *d-isometric linear mappings in linear d-normed Banach modules*, J. Korean Math. Soc. **45** (2008), no. 1, 249–271.

[23] W. Prager and J. Schwaiger, *Multi-affine and multi-Jensen functions and their connection with generalized polynomials*, Aequationes Math. **69** (2005), no. 1-2, 41–57.

[24] ———, *Stability of the multi-Jensen equation*, Bull. Korean Math. Soc. **45** (2008), no. 1, 133–142.

[25] S. M. Ulam, *Problems in Modern Mathematics*, Science Editions John Wiley & Sons, Inc., New York, 1964.

Institute of Mathematics
Pedagogical University
Podchorążych 2
30-084 Kraków, Poland
E-mail address: kc@ap.krakow.pl