Bull. Korean Math. Soc. **45** (2008), No. 4, pp. 749–755

FUZZY ω^O-OPEN SETS

Talal Al-Hawary

Abstract. In this paper, we introduce the relatively new notion of fuzzy ω^O-open set. We prove that the collection of all fuzzy ω^O-open subsets of a fuzzy topological space forms a fuzzy topology that is finer than the original one. Several characterizations and properties of this class are also given as well as connections to other well-known "fuzzy generalized open" subsets.

1. Introduction

Fuzzy topological spaces were first introduced by [1, 2]. Let $(X, \mathfrak{T})$ be a fuzzy topological space (simply, Fts). If λ is a fuzzy set (simply, F-set), then the closure of λ, the interior of λ and the derived set of λ will be denoted by $Cl_{\mathfrak{T}}(\lambda)$, $Int_{\mathfrak{T}}(\lambda)$ and $d_{\mathfrak{T}}(\lambda)$, respectively. If no ambiguity appears, we use $\overline{\lambda}$, $\overset{o}{\lambda}$ and λ' instead, respectively. A F-set λ is called *F-semi-open* (simply, *FSO*) [6] if there exists a fuzzy open (simply, F-open) set μ such that $\mu \leq \lambda \leq Cl_{\mathfrak{T}}(\mu)$. Clearly λ is a FSO-set if and only if $\lambda \leq Cl_{\mathfrak{T}}(Int_{\mathfrak{T}}(\lambda))$. A complement of a FSO-set is called *F-semi-closed* (simply, *FSC*). λ is called *fuzzy preopen* (simply, *FPO*) if $\lambda \leq Int_{\mathfrak{T}}(Cl_{\mathfrak{T}}(\lambda))$. Finally, λ is called *fuzzy regular-open* (simply, *FRO*) if $\lambda = Int_{\mathfrak{T}}(Cl_{\mathfrak{T}}(\lambda))$. Complements of FRO-sets are called *fuzzy regular-closed* (simply, *FRC*). The collection of all FSO (resp., FPO, FRO and FRC) subsets of X will be denoted by $FSO(X, \mathfrak{T})$ (resp., $FPO(X, \mathfrak{T})$, $FRO(X, \mathfrak{T})$ and $FRC(X, \mathfrak{T})$). For more on the preceding notions, the reader is referred to [1, 2, 3, 6, 8].

Our goal in this paper is to introduce the new concept of fuzzy ω^o-open set, discuss its connection to other well-known sets and study several interesting properties and constructions of these sets in case of anti locally countable Fts.

2. Fuzzy ω^o-open set

We begin this section by introducing the notion of ω^O-open and ω^O-closed subsets.

Received March 9, 2008; Revised July 27, 2008.
2000 *Mathematics Subject Classification.* 54A40.
Key words and phrases. fuzzy topological space, fuzzy open set, fuzzy ω^O-open.

Definition 1. A F-set λ of a Fts $(X, \mathfrak{T})$ is called *fuzzy ω^o-open* (simply, *$F\omega^O$-open*) if for every fuzzy set $\mu \leq \lambda$, there exists an F-open set η such that $\mu \leq \eta$ and $\eta - \lambda^o$ is countable. Complements of Fω^O-open sets are called *fuzzy ω^O-closed* (simply, *$F\omega^O$-closed*).

Clearly every F-open set is Fω^o-open, but the converse needs not be true.

Example 1. Let $X = \{a, b\}$ and $\mathfrak{T} = \{0, 1, \chi_{\{a\}}\}$. Then $\chi_{\{b\}}$ is Fω^o-open but not F-open.

Definition 2. A F-set λ in $(X, \mathfrak{T})$ is countable if $\{\lambda(x) : x \in X\}$ is countable.

Next, we show that the collection of all Fω^o-open subsets of a Fts $(X, \mathfrak{T})$ forms a fuzzy topology $\mathfrak{T}_{F\omega^o}$ that is strictly finer than $\mathfrak{T}$.

Theorem 1. *Let $(X, \mathfrak{T})$ be a Fts. Then $(X, \mathfrak{T}_{F\omega^o})$ is a Fts such that $\mathfrak{T} \subseteq \mathfrak{T}_{F\omega^o}$.*

Proof. Clearly $\mathfrak{T} \subseteq \mathfrak{T}_{F\omega^o}$ and $0, 1 \in \mathfrak{T}_{F\omega^o}$. If $\lambda_1, \lambda_2 \in \mathfrak{T}_{F\omega^o}$ and $\mu \leq \lambda_1 \wedge \lambda_2$, then there exist F-open sets η_1, η_2 such that $\mu \leq \eta_1$, $\mu \leq \eta_2$ and both $\eta_1 - \overset{\circ}{\lambda}_1$ and $\eta_2 - \overset{o}{\lambda}_2$ are countable. Hence $\mu \leq \eta_1 \wedge \eta_2$ and for every $\lambda \leq \eta_1 \wedge \eta_2 - (\lambda_1 \wedge \lambda_2)^o = (\eta_1 \wedge \eta_2) - (\overset{o}{\lambda}_1 \wedge \overset{o}{\lambda}_2)$, we have $\lambda \leq \eta_1 - \overset{o}{\lambda}_1$ and $\lambda \leq \eta_2 - \overset{o}{\lambda}_2$. Thus $\eta_1 \wedge \eta_2 - (\lambda_1 \wedge \lambda_2)^o \leq \eta_1 - \overset{o}{\lambda}_1$ and $\eta_1 \wedge \eta_2 - (\lambda_1 \wedge \lambda_2)^o \leq \eta_2 - \overset{o}{\lambda}_2$ and so $\eta_1 \wedge \eta_2 - (\lambda_1 \wedge \lambda_2)^o$ is countable. Therefore, $\lambda_1 \wedge \lambda_2 \in \mathfrak{T}_{F\omega^o}$.

If $\{\lambda_\alpha : \alpha \in \Delta\}$ is a collection of Fω^o-open subsets of X, then for every $\lambda \leq \bigvee_{\alpha \in \Delta} \lambda_\alpha$, $\lambda \leq \lambda_\beta$ for some $\beta \in \Delta$. Hence there exists an F-open subset μ of X such that $\lambda \leq \mu$ and $\mu - \overset{\circ}{\lambda}_\beta$ is countable. But $\mu - (\bigvee_{\alpha \in \Delta} \lambda_\alpha)^o \leq \mu - \bigvee_{\alpha \in \Delta} \overset{o}{\lambda}_\alpha \leq \mu - \overset{o}{\lambda}_\beta$. Thus $\mu - (\bigvee_{\alpha \in \Delta} \lambda_\alpha)^o$ is countable and hence $\bigvee_{\alpha \in \Delta} \lambda_\alpha \in \mathfrak{T}_{F\omega^o}$. □

Definition 3. A Fts $(X, \mathfrak{T})$ is called *a PFts* if for every countable collection $\{\lambda_n : n \in \mathbb{N}\}$of F-open sets, $\bigwedge_{n \in \mathbb{N}} \lambda_n$ is a F-open set.

Corollary 1. *If $(X, \mathfrak{T})$ is a PFts, then $\mathfrak{T} = \mathfrak{T}_{F\omega^o}$.*

Proof. By Theorem 1, $\mathfrak{T} \subseteq \mathfrak{T}_{F\omega^o}$. On the other hand, if $\lambda \in \mathfrak{T}_{F\omega^o}$, then for every fuzzy set $\mu \leq \lambda$, there exists an F-open set η such that $\mu \leq \eta$ and $\eta - \lambda^o$ is countable. Thus λ can be written as $\bigwedge_{n \in \mathbb{N}} \lambda_n$, where each λ_n is a F-open set. As $(X, \mathfrak{T})$ is a PFts, λ is a F-open set. Therefor, $\mathfrak{T}_{F\omega^o} \subseteq \mathfrak{T}_F$ and hence $\mathfrak{T} = \mathfrak{T}_{F\omega^o}$. □

Next we show that Fω^o-open notion is independent of both FPO and FSO ones.

Example 2. Consider the fuzzy real line $\mathbb{R}(\mathfrak{L})$ [5]. Then $\mathbb{Q}(\mathfrak{L})$ is FPO but not Fω^o-open. Also $[0, 1](\mathfrak{L})$ is FSO but not Fω^o-open.

Example 3. In Example 1, $\lambda_{\{b\}}$ is Fω^o-open but neither FPO nor F-open.

Definition 4. A Fts $(X, \mathfrak{T})$ is *locally countable* if each $\mu \in X$ has a countable neighborhood.

Next we characterize $\mathfrak{T}_{F\omega^o}$ when X is a locally countable Fts.

Theorem 2. *If $(X, \mathfrak{T})$ is a locally countable Fts, then $\mathfrak{T}_{F\omega^o}$ is the discrete fuzzy topology.*

Proof. Let $\lambda \in X$ and $\mu \leq \lambda$. Since X is locally countable, there exists a countable neighborhood η of μ. Hence there exists a F-open set μ_1 such that $\mu \leq \mu_1 \leq \eta$. Since $\mu_1 - \mathring{\lambda} \leq \eta - \mathring{\lambda}$, $\mu_1 - \mathring{\lambda}$ is countable. Therefore λ is Fω^o-open and so $\mathfrak{T}_{F\omega^o}$ is the discrete fuzzy topology. □

Corollary 2. *If $(X, \mathfrak{T})$ is a countable Fts, then $\mathfrak{T}_{F\omega^o}$ is the discrete fuzzy topology.*

Next, a new characterization of Fω^o-open subsets is given. It will be used most often throughout the rest of this paper.

Lemma 1. *A F-set λ of a Fts $(X, \mathfrak{T})$ is Fω^o-open if and only if for every $\mu \leq \lambda$, there exists a F-open set $\eta \geq \mu$ and a countable F-set C_η such that $\eta - C_\eta \leq \overset{o}{\lambda}$.*

Proof. Let $\lambda \in \mathfrak{T}_{F\omega^o}$ and $\mu \leq \lambda$. Then there exists a F-open subset $\eta \geq \lambda$ such that $\eta - \overset{o}{\lambda}$ is countable. Let $C_\eta = \eta - \overset{o}{\lambda} = \eta \wedge (\overset{o}{\lambda})'$. Then $\eta - C_\eta \leq \overset{o}{\lambda}$.

Conversely, let $\mu \leq \lambda$. Then there exists a F-open subset $\eta \geq \mu$ and a countable subset C_μ such that $\eta - C_\mu \leq \overset{o}{\lambda}$. Thus $\eta - \overset{o}{\lambda} = C_\mu$ is countable. □

The following is an immediate result that follows from Theorem 1:

Lemma 2. *A F-set λ of a Fts $(X, \mathfrak{T})$ is Fω^o-closed if and only if $Cl_{F\omega^o}(\lambda) = \lambda$.*

Theorem 3. *If λ is Fω^o-open subset of a Fts $(X, \mathfrak{T})$, then $\mathfrak{T}_{F\omega^o}|_\lambda \subseteq (\mathfrak{T}|_\lambda)_{F\omega^o}$.*

Proof. Let $\mu \in \mathfrak{T}_{F\omega^o}|_\lambda$. Then $\mu = \lambda \wedge \eta$ for some Fω^o-open subset η. For every $\lambda_1 \leq \mu$, $\lambda_1 \leq \lambda \in \mathfrak{T}_{F\omega^o}$ and so there exist F-open $\lambda_2 \geq \lambda_1$ and countable set C_{λ_2} such that $\lambda_2 - C_{\lambda_2} \leq \mathring{\eta} = \eta$. Now $\lambda_2 \wedge \lambda \in \mathfrak{T}|_\lambda$ and

$$\begin{aligned} \lambda_2 \wedge \lambda - C_{\lambda_2} &= (\lambda_2 - C_{\lambda_2}) \wedge \lambda \\ &\leq \mathring{\eta} \wedge \lambda \\ &= \mathring{\eta} \wedge \lambda^o \\ &\leq \mathring{\mu}. \end{aligned}$$

Therefore, $\mu \in (\mathfrak{T}|_\lambda)_{F\omega^o}$. □

Corollary 3. *If λ is F-open subset of X, then $\mathfrak{T}_{F\omega^o}|_\lambda \subseteq (\mathfrak{T}|_\lambda)_{F\omega^o}$.*

In the next example, we show that if λ in the preceding theorem is not Fω^o-open, then the result needs not be true.

Example 4. Consider the fuzzy real line $\mathbb{R}(\mathfrak{L})$ and let $\lambda = (\mathbb{R}\backslash\mathbb{Q})(\mathfrak{L})$. Then λ is not Fω^o-open and so not F-open. As $(0,1)(\mathfrak{L})$ is Fω^o-open, then $d = ((0,1)\wedge\lambda)(\mathfrak{L}) \in \mathfrak{T}_{F\omega^o}|_\lambda$ while if $d \in (\mathfrak{T}|_\lambda)_{F\omega^o}$ then for every $\mu \leq d$, there exists $\eta \in \mathfrak{T}|_\lambda$ and a countable F-set C_μ such that $\mu - C_\mu \leq \overset{o}{C}_\mu = 0$. Thus $\eta \leq C_\mu$ and hence η is countable which is a contradiction.

Theorem 4. *If* $(X, \mathfrak{T})$ *is a Fts in which every countable set of F-points is F-closed, then* $(\mathfrak{T}|_\lambda)_{F\omega^o} \subseteq \mathfrak{T}_{F\omega^o}|_\lambda$.

Proof. Let $\mu \in (\mathfrak{T}|_\lambda)_{F\omega^o}$ and $\eta \leq \mu$. Then there exist $\lambda_* \geq \eta$ and a countable set C_{λ_*} of F-points such that $C_{\lambda_*} \leq \lambda$ and $\lambda - C_{\lambda_*} \leq \mathring{\mu}$. But $\lambda_* = \eta_1 \wedge \lambda$ for some F-open set η_1. Hence $\eta_1 \in \mathfrak{T}_{F\omega^o}$. Now

$$(\eta_1 - C_{\lambda_*}) \wedge \lambda = (\eta_1 \wedge \lambda) - C_{\lambda_*} = \lambda_* - C_{\lambda_*} \leq \mathring{\lambda}_1.$$

Moreover, $\eta_1 - \lambda_* \in \mathfrak{T}_{F\omega^o}$ as for every $\delta \leq \eta_1 - C_{\lambda_*}$, η_1 is an F-open set, $\delta \leq \eta_1$ and C_{λ_*} is a countable subset such that

$$\lambda_1 - C_{\lambda_*} \leq (\eta_1 - C_{\lambda_*})^o = \mathring{\eta}_1 \wedge (\mathring{C}_{\lambda_*})' = \eta_1 \wedge (\mathring{C}_{\lambda_*})' = \eta_1 \wedge \acute{C}_{\lambda_*} = \eta_1 - C_{\lambda_*}.$$

Therefore, $\mu \in \mathfrak{T}_{F\omega^o}|_\lambda$. □

Next, we show that if $(X, \mathfrak{T})$ is a Fts having a countable set A of F-points that is not F-closed, then $(\mathfrak{T}|_A)_{F\omega^o} \subsetneq \mathfrak{T}_{F\omega^o}|_A$.

Example 5. Consider the fuzzy real line $\mathbb{R}(\mathfrak{L})$ and let $\lambda_1 = \mathbb{Q}(\mathfrak{L})$ and $\lambda_2 = (0,1)(\mathfrak{L})$. If $\lambda_2 \in \mathfrak{T}_{F\omega^o}|_{\lambda_1}$, then $\lambda_2 = \mu \wedge \lambda_1$ for some $\mu \in \mathfrak{T}_{F\omega^o}$, which is impossible since $\sqrt{2} \leq \lambda_2 - \lambda_1$. On the other hand, let $\eta \leq \lambda_2$. If $\eta \leq \lambda_1$, pick $q_1, q_2 \leq \lambda$ such that $0 < q_1 < \eta < q_2 < 2$ and let $\delta = (q_1, q_2)(\mathfrak{L}) \wedge \lambda_1$. Then $\eta \leq \delta - 0 \leq \lambda_2 - \mathring{\lambda}_2$. If $\eta \nleq \lambda_1$, then pick $q_1, q_2 \nleq \lambda_1$ such that $0 < q_1 < \eta < q_2 < 2$ and let $\delta = (q_1, q_2)(\mathfrak{L}) \wedge \lambda_1$. Then $\eta \leq \delta - 0 \leq \lambda_2 = \mathring{\lambda}_2$. Thus in both cases $\lambda_2 \in (\mathfrak{T}|_{\lambda_1)F\omega^o}$.

Definition 5 ([2, 4])**.** A family $\{\lambda_\alpha : \alpha \in \Delta\}$ of F-open subsets of a Fts $(X, \mathfrak{T})$ is called a *fuzzy cover* (simply, *F-cover*) of X if $\bigvee_{\alpha\in\Delta} \lambda_\alpha = X$.

Definition 6 ([7])**.** A Fts $(X, \mathfrak{T})$ is called *fuzzy Lindelöf* (simply, *F-Lindelöf*) if every F-cover of X has a countable subcover.

Lemma 3. *If* X *is a Lindelöf Fts, then* $\lambda - \overset{o}{\lambda}$ *is countable for every F-closed subset* $\lambda \in \mathfrak{T}_{F\omega^o}$.

Proof. Let λ be an F-closed set such that $\lambda \in \mathfrak{T}_{F\omega^o}$. If λ is F-open, then $\lambda - \overset{o}{\lambda} = 0$ is countable. Otherwise, as $\lambda \in \mathfrak{T}_{F\omega^o}$, then for every $\mu \leq \lambda$, there exists a F-open set $\eta_\mu \geq \mu$ such that $\eta_\mu - \overset{o}{\lambda}$ is countable. Thus $\{\eta_\mu : \mu \leq \lambda\}$ is a F-cover for λ and as λ is Lindelöf, it has a countable subcover $\{\eta_n : n \in \mathbb{N}\}$. Hence $\lambda - \overset{o}{\lambda} = \bigvee_{n\in\mathbb{N}} (\eta_n - \overset{o}{\lambda})$ is countable. □

Definition 7 ([8])**.** A Fts $(X, \mathfrak{T})$ is called *second countable* if it has a countable base of F-open sets.

Corollary 4. *If* $(X, \mathfrak{T})$ *is a second countable Fts, then* $\lambda - \overset{o}{\lambda}$ *is countable for every closed subset* $\lambda \in \mathfrak{T}_{F\omega^o}$*.*

We remark that the preceding result needs not hold for ω^o-closed sets as shown next.

Example 6. Consider $\mathbb{R}(\mathfrak{L})$ with the standard topology and let $\lambda = \mathbb{Q}$. Then $\lambda \leq \mathbb{R}(\mathfrak{L}) \vee \mathbb{Q}$ but λ is not ω^o-closed.

Theorem 5. *Let* $(X, \mathfrak{T})$ *be a Fts and* λ *is a F*ω^o*-closed set. Then* $Cl_{\mathfrak{T}}(\lambda) \leq \mu \vee \eta$ *for some F-closed subset* μ *and a countable F-set* η*.*

Proof. Since λ is Fω^o-closed, $\acute{\lambda}$ is Fω^o-open and hence for every $\delta \leq \acute{\lambda}$, there exist a F-open set $\sigma \geq \delta$ and a countable set η such that $\sigma - \eta \leq (\acute{\lambda})^o = (Cl_{\mathfrak{T}}(\lambda))\acute{}$. Thus

$$Cl_{\mathfrak{T}}(C) \leq (\sigma - \eta)' \leq (\sigma \wedge \acute{\eta})' \leq 1 \wedge (\acute{\sigma} \vee \eta) \leq \acute{\sigma} \vee \eta.$$

Letting $\mu = \acute{\sigma}$. Then μ is F-closed such that $Cl_{\mathfrak{T}}(\lambda) \leq \mu \vee \eta$. □

3. Anti locally countable Fts

In this section, several interesting properties and constructions of Fω^o-open subsets are discussed in case of anti locally countable Fts.

Definition 8. A Fts $(X, \mathfrak{T})$ is called *anti locally countable* if every non-zero F-open set is uncountable.

Theorem 6. *A Fts* $(X, \mathfrak{T})$ *is anti locally countable if and only if* $(X, \mathfrak{T}_{F\omega^o})$ *is anti locally countable.*

Proof. Let $\lambda \in \mathfrak{T}_{F\omega^o}$ and $\mu \leq \lambda$. By Lemma 1, there exist a F-open subset $\eta \geq \mu$ and a countable F-set C_μ such that $\eta - C_\mu \leq \overset{o}{\lambda}$. Hence $\overset{o}{\lambda}$ is uncountable and so is λ. The converse follows from the fact that every F-open subset is Fω^o-open. □

Corollary 5. *If* $(X, \mathfrak{T})$ *is anti locally countable Fts and* λ *is F*ω^o*-open, then* $Cl_{\mathfrak{T}}(\lambda) = Cl_{\mathfrak{T}_{F\omega^o}}(\lambda)$*, where* $Cl_{\mathfrak{T}_{F\omega^o}}(\lambda)$ *is the closure of* λ *in* $(X, \mathfrak{T}_{F\omega^o})$*.*

Proof. Clearly $Cl_{\mathfrak{T}_{F\omega^o}}(\lambda) \leq Cl_{\mathfrak{T}}(\lambda)$. On the other hand, let $\mu \leq Cl_{\mathfrak{T}}(\lambda)$ and η be an Fω^o-open subset such that $\eta \geq \mu$. Then by Lemma 1, there exist a F-open subset $\delta \geq \mu$ and a countable F-set σ such that $\delta - \sigma \leq \overset{o}{\eta}$. Thus $(\delta - \sigma) \wedge \lambda \leq \overset{o}{\eta} \wedge \lambda$ and so $(\delta \wedge \lambda) - \sigma \leq \overset{o}{\eta} \wedge \lambda$. As $\mu \leq Cl_{\mathfrak{T}}(\lambda)$, $\delta \wedge \lambda \neq 0$ and as δ and λ are Fω^o-open, $\delta \wedge \lambda$ is Fω^o-open and as X is ani locally countable, $\delta \wedge \lambda$ is uncountable and so is $(\delta \wedge \lambda) - \sigma$. Thus $\eta \wedge \lambda$ is uncountable as it contains the uncountable set $\overset{o}{\eta} \wedge \lambda$. Therefore, $\eta \wedge \lambda \neq 0$ which means $\mu \leq Cl_{\mathfrak{T}_{F\omega^o}}(\lambda)$. □

By a similar argument, we can easily prove the following result.

Corollary 6. *If* $(X, \mathfrak{T})$ *is anti locally countable Fts and* λ *is* $F\omega^o$*-closed, then* $Int_{\mathfrak{T}}(\lambda) = Int_{\mathfrak{T}_{F\omega^o}}(\lambda)$*, where* $Int_{\mathfrak{T}_{F\omega^o}}(\lambda)$ *is the interior of* λ *in* $(X, \mathfrak{T}_{F\omega^o})$.

Definition 9. A F-set λ in a Fts $(X, \mathfrak{T})$ is called an Fα-(resp., Fβ-) set if $\lambda \leq Int_{\mathfrak{T}}(Cl_{\mathfrak{T}}(Int_{\mathfrak{T}}(\lambda)))$ (resp., $\lambda \leq Cl_{\mathfrak{T}}(Int_{\mathfrak{T}}(Cl_{\mathfrak{T}}(\lambda))))$.

The set of all Fα-(resp., Fβ-) sets of a Fts $(X, \mathfrak{T})$ will be denoted by $F\alpha(X, \mathfrak{T})$ (resp., $F\beta(X, \mathfrak{T})$). Clearly every F-open set is both $F\alpha$-open and $F\beta$-open and every $F\beta$-open is $F\alpha$-open and $F\alpha(X, \mathfrak{T}) = FSO(X, \mathfrak{T}) \cap FPO(X, \mathfrak{T})$.

Theorem 7. *Let* $(X, \mathfrak{T})$ *be an anti locally countable Fts. Then* $F\alpha(X, \mathfrak{T}) \subseteq F\alpha(X, \mathfrak{T}_{F\omega^o})$.

Proof. If $\lambda \in F\alpha(X, \mathfrak{T})$, then $\lambda \leq Int_{\mathfrak{T}}(Cl_{\mathfrak{T}}(Int_{\mathfrak{T}}(\lambda)))$ and by Corollary 5, $\lambda \leq Int_{\mathfrak{T}}(Cl_{\mathfrak{T}_{F\omega^o}}(Int_{\mathfrak{T}}(\lambda)))$. Now by Corollary 6 and as $Cl_{\mathfrak{T}_{F\omega^o}}(Int_{\mathfrak{T}}(\lambda))$ is Fω^o-closed, $\lambda \leq Int_{\mathfrak{T}_{F\omega^o}}(Cl_{\mathfrak{T}_{F\omega^o}}(Int_{\mathfrak{T}}(\lambda)))$ and by Corollary 6 again, $\lambda \leq Int_{\mathfrak{T}_{F\omega^o}}(Cl_{\mathfrak{T}_{F\omega^o}}(Int_{\mathfrak{T}_{F\omega^o}}(\lambda)))$ which means $\lambda \in F\alpha(X, \mathfrak{T}_{F\omega^o})$. □

The converses of the preceding result needs not be true.

Example 7. Consider the Fts from Example 1. Then $\lambda = \chi_{\{b\}} \in F\alpha(X, \mathfrak{T}_{F\omega^o})$ but $\lambda \notin F\alpha(X, \mathfrak{T})$.

Similarly, one can show that in an anti locally countable Fts, $F\beta(X, \mathfrak{T}_{F\omega^o}) \subseteq F\beta(X, \mathfrak{T})$. Recall that a F-point μ in a Fts $(X, \mathfrak{T})$ is a *fuzzy cluster point* of a F-set λ if for every F-open set $\eta \geq \mu$, $\eta \wedge \lambda - \mu \neq 0$. The set of all cluster points of λ will be denoted by $d_{\mathfrak{T}}(\lambda)$.

Theorem 8. *Let* $(X, \mathfrak{T})$ *be an anti locally countable Fts. Then* $d_{\mathfrak{T}}(\lambda) = d_{\mathfrak{T}_{F\omega^o}}(\lambda)$ *for every F-set* λ.

Proof. If $\mu \in d_{\mathfrak{T}}(\lambda)$ and η is any Fω^o-open subset such that $\eta \geq \mu$, then there exist a F-open subset $\delta \geq \mu$ and a countable σ such that $\delta - \sigma \leq \overset{o}{\eta} \leq \eta$. Since $\delta \wedge \lambda - \mu \neq 0$ and as $(\sigma \wedge \lambda - \mu) - \sigma = (\delta - \sigma) \wedge (\lambda - \mu)$ is uncountable subset of $\eta \wedge \lambda - \mu$, $\eta \wedge \lambda - \mu \neq 0$ and hence $\mu \in d_{\mathfrak{T}_{F\omega^o}}(\lambda)$.

The converse is obvious as every F-open subset is Fω^o-open. □

Theorem 9. *Let* $(X, \mathfrak{T})$ *be an anti locally countable Fts. Then*

$$FRO(X, \mathfrak{T}) = FRO(X, \mathfrak{T}_{F\omega^o}).$$

Proof. If $\lambda \in FRO(X, \mathfrak{T})$, then $\lambda = Int_{\mathfrak{T}}(Cl_{\mathfrak{T}}(\lambda))$. By Corollary 5, $\lambda = Int_{\mathfrak{T}}(Cl_{\mathfrak{T}_{F\omega^o}}(\lambda))$ and by Corollary 6 and as $Cl_{\mathfrak{T}_{F\omega^o}}(\lambda)$ is Fω^o-closed, $\lambda = Int_{\mathfrak{T}_{F\omega^o}}(Cl_{\mathfrak{T}_{F\omega^o}}(\lambda))$ which means $\lambda \in FRO(X, \mathfrak{T}_{F\omega^o})$.

Conversely, if $\lambda \in FRO(X, \mathfrak{T}_{F\omega^o})$, then $\lambda = Int_{\mathfrak{T}_{F\omega^o}}(Cl_{\mathfrak{T}_{F\omega^o}}(\lambda))$. As λ is Fω^o-open, by Corollary 5, $\lambda = Int_{\mathfrak{T}_{F\omega^o}}(Cl_{\mathfrak{T}}(\lambda))$ and as $Cl_{\mathfrak{T}}(\lambda)$ is Fω^o-closed being a F-closed set, then $\lambda = Int_{\mathfrak{T}}(Cl_{\mathfrak{T}}(\lambda))$ which means $\lambda \in FRO(X, \mathfrak{T})$. □

The converse of the preceding result needs not be true.

Example 8. Let $X = \{a,b,c,d,e\}$ and $\mathfrak{T} = \{0,\ 1,\ \chi_{\{a\}},\ \chi_{\{a,b\}},\ \chi_{\{a,b,c\}},\ \chi_{\{a,b,c,d\}}\}$. Then $(X,\mathfrak{T})$ is an anti locally countable Fts where $FRO(X,\mathfrak{T}) = \{0,1\}$ while $FRO(X,\mathfrak{T}_{F\omega^o}) = \mathfrak{T}$.

We end this section by showing that $(X,\mathfrak{T}_{F\omega^o})$ is Urysohn when $(X,\mathfrak{T})$ is an anti locally countable Fts.

Definition 10. A Fts $(X,\mathfrak{T})$ is *Urysohn* if for every two distinct F-sets δ and σ, there exist two F-open sets λ and μ such that $\delta \le \lambda$, $\sigma \le \mu$ and $Cl_{\mathfrak{T}}(\lambda) \wedge Cl_{\mathfrak{T}}(\mu) = 0$.

Corollary 7. *Let $(X,\mathfrak{T})$ be an anti locally countable Fts that is Urysohn. Then $(X,\mathfrak{T}_{F\omega^o})$ is Urysohn.*

Proof. If $\delta \ne \sigma$ in X, then there exist F-open sets λ and μ such that $\delta \le \lambda$, $\sigma \le \mu$ and $Cl_{\mathfrak{T}}(\lambda) \wedge Cl_{\mathfrak{T}}(\mu) = 0$. By Corollary 5, $Cl_{\mathfrak{T}_{F\omega^o}}(\lambda) \wedge Cl_{\mathfrak{T}_{F\omega^o}}(\mu) = Cl_{\mathfrak{T}}(\lambda) \wedge Cl_{\mathfrak{T}}(\mu) = 0$. □

Acknowledgment. The author would like to thank the referees for useful comments and suggestions.

References

[1] M. K. Chakrabarty and T. M. G. Ahsanullah, *Fuzzy topology on fuzzy sets and tolerance topology*, Fuzzy Sets and Systems **45** (1992), no. 1, 103–108.
[2] C. L. Chang, *Fuzzy topological spaces*, J. Math. Anal. Appl. **24** (1968), 182–190.
[3] A. K. Chaudhuri and P. Das, *Some results on fuzzy topology on fuzzy sets*, Fuzzy Sets and Systems **56** (1993), no. 3, 331–336.
[4] A. H. Eş, *Almost compactness and near compactness in fuzzy topological spaces*, Fuzzy Sets and Systems **22** (1987), no. 3, 289–295.
[5] T. E. Gantner and R. C. Steinlange, *Compactness in fuzzy topological spaces*, J. Math. Anal. Appl. **62** (1978), no. 3, 547–562.
[6] F. S. Mahmoud, M. A. Fath Alla, and S. M. Abd Ellah, *Fuzzy topology on fuzzy sets: fuzzy semicontinuity and fuzzy semiseparation axioms*, Appl. Math. Comput. **153** (2004), no. 1, 127–140.
[7] C. K. Wong, *Covering properties of fuzzy topological spaces*, J. Math. Anal. Appl. **43** (1973), 697–704.
[8] ———, *Fuzzy points and local properties of fuzzy topology*, J. Math. Anal. Appl. **46** (1974), 316–328.

Department of Mathematics & Statistics
Mu'tah University
P. O. Box 6, Karak–Jordan
E-mail address: talalhawary@yahoo.com