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ON SOME PROPERTIES OF MALCEV-NEUMANN MODULES

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ABSTRACT. Let M be a right R-module, G an ordered group and σ a map from G into the group of automorphisms of R. The conditions under which the Malcev-Neumann module $M*((G))$ is a PS module and a p.q.Baer module are investigated in this paper. It is shown that: (1) If M_R is a reduced σ-compatible module, then the Malcev-Neumann module $M*((G))$ over a PS-module is also a PS-module; (2) If M_R is a faithful σ-compatible module, then the Malcev-Neumann module $M*((G))$ is a p.q.Baer module if and only if the right annihilator of any G-indexed family of cyclic submodules of M in R is generated by an idempotent of R.

1. Introduction and preliminaries

The Malcev-Neumann construction appeared for the first time in the latter part of the 1940's (the Laurent series ring, a particular case of Malcev-Neumann ring, was used before by Hilbert). Using them, Malcev and Neumann independently showed (in 1948 and 1949 resp.) that the group ring of an ordered group over a division ring can be embedded in a division ring. Since then, the construction has appeared in many papers, mainly in the study of various properties of division rings and related topics. For instance, Makar-Limanov in [10] used a particular skew-Laurent series division ring to prove that the skew field of fractions of the first Weyl algebra contains a free noncommutative subalgebra. The study of Malcev-Neumann group ring over arbitrary rings was initiated in [9] by Lorenz while investigating properties of group algebras of nilpotent groups. Other results on Malcev-Neumann rings can be found in Musson and Stafford [11] and Sonin [14].

In [14], Sonin generalized the construction to obtain Malcev-Neumann modules over Malcev-Neumann rings. In this paper, the PS property and the p.q.Baerness of Malcev-Neumann modules will be investigated. These results

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generalize the corresponding results for polynomial rings and Laurent power series rings.

Throughout the paper all rings are associative with unity and all modules are right unitary.

We construct the Malcev-Neumann (group) ring in the following. Let R be a ring, G an ordered group, and suppose that σ is a map from G into the group of automorphisms of R, $x \mapsto \sigma_x$. Suppose also that we are given a map t from $G \times G$ to $U(R)$, the group of invertible elements of R. Now $R((G, \sigma, t))$ is the set of all formal sums $f = \sum_{x \in G} r_x x$ with $r_x \in R$ such that supp(f)= $\{x \in G \mid r_x \neq 0\}$ is well ordered. Addition is defined as usual, that is

$$\sum_{x \in G} a_x x + \sum_{y \in G} b_y y = \sum_{z \in G} (a_z + b_z) z,$$

and multiplication is defined by

$$\Big(\sum_{x \in G} a_x x\Big)\Big(\sum_{y \in G} b_y y\Big) = \sum_{z \in G}\Big(\sum_{\{x,y \mid xy = z\}} a_x \sigma_x(b_y) t(x, y)\Big) z.$$

It is necessary to impose two additional conditions on σ and t to insure associativity, namely that for all $x, y, z \in G$,

$$\text{(i) } t(xy, z)\sigma_z(t(x, y)) = t(x, yz)t(y, z), \quad \text{(ii) } \sigma_y \sigma_z = \sigma_{yz}\delta(y, z),$$

where $\delta(y, z)$ denotes the automorphism of R induced by the unit $t(y, z)$ (see, [13, Lemma 1.1]). It is now routine to check that $R((G, \sigma, t))$ is a ring which we call the Malcev-Neumann ring. We make no explicit use of conditions (i) and (ii), so we will denote the construction simply by $R * ((G))$. Basic properties of it (without twisting t), and the original Malcev-Neumann theorem can be found in [13].

If M is a module over R, then the Malcev-Neumann module $M * ((G))$ is the set of all formal sums $\sum_{x \in G} m_x x$ with coefficients in M and well-ordered supports. With operations defined as above, one can easily check that (i) and (ii) insure that $M * ((G))$ is a right unitary module over $R * ((G))$.

For example, if $G = \mathbb{Z}$, $\sigma_x = id$ for all $x \in G$, $t(x, y) = 1$ for all $x, y \in G$, then $M * ((G))_{R*((G))}$ is the Laurent series extension of M. If σ happens to be the trivial homomorphism and $t(x, y) = 1$ for all $x, y \in G$, the resulting untwisted module will denoted by $M((G))$.

As usual, we shall identify R with the subring $R \cdot 1 \subseteq R * ((G))$, and identify G with the subgroup $1 \cdot G$ of invertible elements in $R * ((G))$.

2. PS-modules

According to [12], a right R-module M is called PS-module if its socle $Soc(M_R)$ is projective, and a ring R is called a right PS-ring if R_R is a PS-module. In [12], it was proved that if R is a right PS-ring then so is $R[[x]]$. If R is a commutative ring and $(S, \leq)$ is a strictly totally ordered monoid which satisfied the condition that $0 \leq s$ for every $s \in S$, in [7], it was proved that if

M is a PS-module, then the module $[[M^{S,\leq}]]$ of generalized power series over M is a PS $[[R^{S,\leq}]]$-module. In this section, we will consider the PS property of Malcev-Neumann modules.

Let α be an endomorphism of ring R (with $\alpha(1) = 1$). Following from [6], a module M_R is called α-reduced if, for any $m \in M$ and any $a \in R$,

(1) $ma = 0$ implies $mR \bigcap Ma = 0$.

(2) $ma = 0$ if and only if $m\alpha(a) = 0$.

The module M_R is called reduced if M_R is 1-reduced.

The following result appeared in [6, Lemma 1.2].

Lemma 2.1. *The following conditions are equivalent:*

(1) *M_R is α-reduced.*

(2) *For any $m \in M$ and $a \in R$, the following conditions hold:*

  (a) *$ma = 0$ implies $mRa = mR\alpha(a) = 0$.*

  (b) *$ma\alpha(a) = 0$ implies $ma = 0$.*

  (c) *$ma^2 = 0$ implies $ma = 0$.*

Definition 2.2. Given M_R and σ as above, we say that M_R is σ-compatible if for each $m \in M$, $r \in R$ and $x \in G$, $mr = 0 \Leftrightarrow m\sigma_x(r) = 0$.

Clearly, if $\sigma_x = 1_R$, the identity map of R for any $x \in G$, then any module M_R is σ-compatible. If $G = \mathbb{Z}$, $\sigma_x = \alpha^x$ for all $x \in G$, then M_R is reduced σ-compatible if and only if M_R is α-reduced, where $\alpha \in \mathrm{Aut}(R)$.

Lemma 2.3. *Let M be a reduced σ-compatible right R-module and G an ordered group. If $\phi = \sum_{x \in G} m_x x \in M * ((G))$ and $f = \sum_{y \in G} a_y y \in R * ((G))$ are such that $\phi f = 0$, then $m_x a_y = 0$ for any $x, y \in G$.*

Proof. Let $0 \neq \phi \in M * ((G))$, $0 \neq f \in R * ((G))$ be such that $\phi f = 0$. Then

$$0 = \phi f = \sum_{z \in G} \sum_{\{x,y \mid xy = z\}} m_x \sigma_x(a_y) t(x,y) z. \tag{1}$$

We will use transfinite induction on the ordered group $(G, \leq)$ to show that $m_x a_y = 0$ for any $x \in \mathrm{supp}(\phi)$ and any $y \in \mathrm{supp}(f)$.

Let x_0 and y_0 be the minimal elements of $\mathrm{supp}(\phi)$ and $\mathrm{supp}(f)$ in the $\leq$ order, respectively. If $x \in \mathrm{supp}(\phi)$ and $y \in \mathrm{supp}(f)$ are such that $xy = x_0 y_0$, then $x_0 \leq x$ and $y_0 \leq y$. If $x_0 < x$, then $x_0 y_0 < x y_0 \leq xy = x_0 y_0$, a contradiction. Thus $x = x_0$. Similarly, $y = y_0$. Hence from (1) it follows that $m_{x_0} \sigma_{x_0}(a_{y_0}) t(x_0, y_0) = 0$. Thus $m_{x_0} \sigma_{x_0}(a_{y_0}) = 0$ since $t(x_0, y_0)$ is invertible. So $m_{x_0} a_{y_0} = 0$ since M is σ-compatible.

Now suppose that $w \in G$ is such that for any $x \in \mathrm{supp}(\phi)$ and $y \in \mathrm{supp}(f)$ with $xy < w$, $m_x a_y = 0$. We will show that $m_x a_y = 0$ for any $x \in \mathrm{supp}(\phi)$ and $y \in \mathrm{supp}(f)$ with $xy = w$. For convenience, we write $\{(x,y) \mid xy = w\}$ as $\{(x_i, y_i) \mid i = 1, 2, \ldots, n\}$ with $x_1 < x_2 < \cdots < x_n$ (Note that if $x_1 = x_2$, then from $x_1 y_1 = x_2 y_2$ it follows that $y_1 = y_2$, and thus $(x_1, y_1) = (x_2, y_2)$). Now, from (1), we have

$$(2)\qquad 0=\sum_{\{x,y|xy=w\}} m_x\sigma_x(a_y)t(x,y)=\sum_{i=1}^{n} m_{x_i}\sigma_{x_i}(a_{y_i})t(x_i,y_i).$$

For any $1\le i\le n-1$, $x_iy_n < x_ny_n = w$, and thus, by induction hypothesis, we have $m_{x_i}a_{y_n}=0$. Then $m_{x_i}\sigma_{x_i}(a_{y_i})t(x_i,y_i)a_{y_n}=0$ since M is reduced. Hence, multiplying (2) on the right hand side by a_{y_n}, we obtain

$$0=\sum_{i=1}^{n} m_{x_i}\sigma_{x_i}(a_{y_i})t(x_i,y_i)a_{y_n}=m_{x_n}\sigma_{x_n}(a_{y_n})t(x_n,y_n)a_{y_n}.$$

Then $m_{x_n}\sigma_{x_n}(a_{y_n})t(x_n,y_n)\sigma_{x_n}(a_{y_n})=0$ since M is σ-compatible. Thus

$$m_{x_n}\big(\sigma_{x_n}(a_{y_n})t(x_n,y_n)\big)^2=0.$$

Since M is reduced, we have $m_{x_n}\sigma_{x_n}(a_{y_n})t(x_n,y_n)=0$. Thus $m_{x_n}a_{y_n}=0$ since $t(x_n,y_n)$ is invertible and M is σ-compatible. Now (2) becomes

$$(3)\qquad \sum_{i=1}^{n-1} m_{x_i}\sigma_{x_i}(a_{y_i})t(x_i,y_i)=0.$$

Multiplying $a_{y_{n-1}}$ on (3) from the right-hand side we obtain $m_{x_{n-1}}a_{y_{n-1}}=0$ by the same way as above. Continuing this process, one can prove that $m_{x_i}a_{y_i}=0$ for $i=1,2,\dots,n$. Thus $m_xa_y=0$ for any $x\in\operatorname{supp}(\phi)$ and $y\in\operatorname{supp}(f)$ with $xy=w$.

Therefore, by transfinite induction, $m_xa_y=0$ for any $x\in\operatorname{supp}(\phi)$ and $y\in\operatorname{supp}(f)$. □

Let M be a right R-module. For any subset X of R, denote $l_M(X)=\{m\in M \mid mX=0\}$. The following result appeared in [15].

Lemma 2.4. *The following statements are equivalent for a module M_R:*

(1) *M_R is a* PS*-module.*

(2) *If L is a maximal right ideal of R, then either $l_M(L)=0$ or $L=eR$, where $e^2=e\in R$.*

Theorem 2.5. *Let M_R be a reduced σ-compatible module, G an ordered group. If M_R is a* PS*-module, then so is $M*((G))$.*

Proof. Let L be a maximal right ideal of $R*((G))$. By Lemma 2.4, it is enough to show that either $l_{M*((G))}(L)=0$ or $L=\alpha R*((G))$ for some $\alpha^2=\alpha\in R*((G))$. Let I be the set of all constant coefficients of elements in L. Let J be the right ideal of R generated by I. If $J=R$, then there exist $a_1^1,\ a_1^2,\dots,a_1^n\in I,\ f_1,\ f_2,\dots,f_n\in L$ and $r_1,\ r_2,\dots,r_n\in R$ such that $1=a_1^1r_1+a_1^2r_2+\cdots+a_1^nr_n$ with $f_i=\sum_{x_i\in G}a_{x_i}^ix_i,\ i=1,2,\dots,n$. Suppose that $\phi=\sum_{y\in G}m_yy\in l_{M*((G))}(L)$. Then $\phi f_i=0$. Thus $m_ya_{x_i}^i=0$ by Lemma 2.3. Particularly, $m_ya_1^i=0$ for any $y\in G$ and any $i=1,2,\dots,n$. Thus $m_y=m_y(a_1^1r_1+a_1^2r_2+\cdots+a_1^nr_n)=0$, and so $\phi=0$. Thus $l_{M*((G))}(L)=0$.

Now suppose that $J \neq R$. We show that J is a maximal right ideal of R. Let $r \in R - J$. Then $r \in R * ((G))$. If $r \in L$, then $r \in J$, a contradiction. Thus $r \notin L$. So $R * ((G)) = L + r \cdot R * ((G))$. It follows that there exist $f \in L$ and $g \in R * ((G))$ such that $1 = f + rg$. Suppose that $f = \sum_{x \in G} a_x x$ and $g = \sum_{y \in G} b_y y$. Then $1 = a_1 + r\sigma_1(b_1)t(1,1) \in J + rR$. Thus $R = J + rR$. Hence J is a maximal right ideal of R.

Since M_R is a PS-module, it follows that either $l_M(J) = 0$ or there exists an $e^2 = e \in R$ such that $J = eR$.

Case 1. Suppose that $l_M(J) = 0$. We will show that $l_{M*((G))}(L) = 0$. Let $\phi = \sum_{y \in G} m_y y \in l_{M*((G))}(L)$, $r \in J$. Then there exist $a_1^1, a_1^2, \ldots, a_1^n \in I$, $f_1, f_2, \ldots, f_n \in L$ and $r_1, r_2, \ldots, r_n \in R$ such that $r = a_1^1 r_1 + a_1^2 r_2 + \cdots + a_1^n r_n$, where a_1^i is the constant coefficient of f_i. Since $\phi \in l_{M*((G))}(L)$, $\phi f_i = 0$ for every $i = 1, 2, \ldots, n$. By Lemma 2.3, we have $m_y a_1^i = 0$ for any $y \in G$ and any $i = 1, 2, \ldots, n$. Thus $m_y r = m_y(a_1^1 r_1 + a_1^2 r_2 + \cdots + a_1^n r_n) = 0$ for any $y \in G$. This means that $m_y \in l_M(J) = 0$ for any $y \in G$. Thus $\phi = 0$, and so $l_{M*((G))}(L) = 0$.

Case 2. Suppose that $J = eR$ where $e^2 = e \in R$. We will show that $L = e \cdot R * ((G))$. If $e \notin L$, then $R * ((G)) = L + e \cdot R * ((G))$. Thus $1 = f + eg$, where $f = \sum_{x \in G} a_x x \in L$ and $g = \sum_{y \in G} b_y y \in R * ((G))$, and so $1 = a_1 + e\sigma_1(b_1)t(1,1) \in J + eR = J$, a contradiction. Therefore $e \in L$, and so $e \cdot R * ((G)) \subseteq L$. Conversely, suppose that $f = \sum_{x \in G} a_x x \in L$. For any $x \in G$, there exists $x^{-1} \in G$ such that $xx^{-1} = 1$ since G is a group, and $fx^{-1} \in L$ since L is a right ideal of $R * ((G))$. Thus $a_x \sigma_x(1) t(x, x^{-1}) \in J = eR$ for any $x \in G$. Thus $a_x \in J = eR$ since $t(x, x^{-1})$ is invertible and J is a right ideal of R, and so $a_x = ea_x$. Thus $f = e \sum_{x \in G} \sigma_1^{-1}(a_x t(1,x)^{-1})x \in e \cdot R * ((G))$. Thus $L \subseteq e \cdot R * ((G))$. Hence $L = e \cdot R * ((G))$ and the result follows. □

Corollary 2.6. *Let M be a reduced module and G an ordered group. If M is a* PS*-module, then $M((G))$ is a* PS*-module.*

Corollary 2.7. *Let $\alpha \in \mathrm{Aut}(R)$ and M be an α-reduced module. If M is a* PS*-module, then $M[[x, x^{-1}; \alpha]]_{R[[x,x^{-1};\alpha]]}$ is a* PS*-module.*

Proof. Take $G = \mathbb{Z}$ and $t(x, y) = 1$ for any $x, y \in \mathbb{Z}$. For any $x \in \mathbb{Z}$, let $\sigma_x = \alpha^x$. Then M is reduced and σ-compatible. Now the result follows from Theorem 2.5. □

3. p.q-Baer modules

In [5], Kaplansky introduced Baer rings as rings in which the right (left) annihilator of every nonempty subset is generated by an idempotent. According to Clark [4], a ring R is said to be quasi-Baer if the right annihilator of each right ideal of R is generated by an idempotent. These definitions are left-right symmetric. As a generalization of quasi-Baer rings, Birkenmeier, Kim and Park in [2] introduced the concept of principally quasi-Baer rings. A ring R is called right principally quasi-Baer (or simply right p.q.Baer) if the right annihilator of a principal right ideal of R is generated by an idempotent. Similarly, left p.q.Baer rings can be defined. A ring is called p.q.Baer if it is both right and left p.q.Baer ring. The Baerness, the quasi-Baerness and the p.q.Baerness of the (Laurent) polynomial extension and the (Laurent) power series extension of rings have been discussed by many authors, see for example [1, 2, 3, 8]. Recently, in [6], Lee-Zhou introduced Baer modules and quasi-Baer modules as follows:

(1) M_R is called Baer if, for any subset X of M, $r_R(X) = eR$ where $e^2 = e \in R$.

(2) M_R is called quasi-Baer if, for any submodule X of M, $r_R(X) = eR$ where $e^2 = e \in R$.

Also, the results on (Laurent) polynomial extension and (Laurent) power series extension of Baer rings and quasi-Baer rings were extended to the corresponding module extensions, for more details, see [6]. In this section, the concept of p.q.Baer modules will be introduced, and a necessary and sufficient condition for some modules under which the Malcev-Neumann module $M * ((G))$ is p.q.Baer will be given.

Definition 3.1. A module M_R is called principally quasi-Baer (p.q.Baer for short) if for any $m \in M$, $r_R(mR) = eR$ where $e^2 = e \in R$.

It is clear that R is a right p.q.Baer ring if and only if R_R is a p.q.Baer module. If R is a p.q.Baer ring, then for any right ideal I of R, I_R is a p.q.Baer module. Moreover, every quasi-Baer module is p.q.Baer.

Lemma 3.2. *Let G be an ordered group and M_R a σ-compatible p.q.Baer module. If $\phi = \sum_{x\in G} m_x x \in M * ((G))$ and $f = \sum_{y \in G} a_y y \in R * ((G))$ are such that $\phi R * ((G)) f = 0$, then $m_x R a_y = 0$ for any $x, y \in G$.*

Proof. Let $0 \neq \phi = \sum_{x\in G} m_x x \in M * ((G))$ and $0 \neq f = \sum_{y\in G} a_y y \in R*((G))$ be such that $\phi R * ((G)) f = 0$. Then for any $r \in R$, from

$$0 = \phi r f = \sum_{z\in G} \sum_{\{x,y|xy=z\}} m_x \sigma_x\big(r\sigma_1(a_y)t(1,y)\big)t(x,y)z$$

it follows that

$$\sum_{\{x,y|xy=z\}} m_x\sigma_x\big(r\sigma_1(a_y)t(1,y)\big)t(x,y) = 0, \quad \forall z \in G.$$

Let x_0 and y_0 denote the minimal elements of $\mathrm{supp}(\phi)$ and $\mathrm{supp}(f)$ in the $\leq$ order, respectively. If $x \in \mathrm{supp}(\phi)$ and $y \in \mathrm{supp}(f)$ are such that $xy = x_0y_0$, then $x_0 \leq x$ and $y_0 \leq y$. If $x_0 < x$, then $x_0y_0 < xy_0 \leq xy = x_0y_0$, a contradiction. Thus $x = x_0$. Similarly, $y = y_0$. Hence

$$\sum_{\{x,y\,|\,xy=x_0y_0\}} m_x\sigma_x(r\sigma_1(a_y)t(1,y))t(x,y) = m_{x_0}\sigma_{x_0}(r\sigma_1(a_{y_0})t(1,y_0))t(x_0,y_0) = 0.$$

Thus $m_{x_0}\sigma_{x_0}(r\sigma_1(a_{y_0})t(1,y_0)) = 0$ since $t(x_0,y_0)$ is invertible. Hence, by the σ-compatibility of M, we have $m_{x_0}r\sigma_1(a_{y_0})t(1,y_0) = 0$. By the way as above, we can get $m_{x_0}ra_{y_0} = 0$, which means that $m_{x_0}Ra_{y_0} = 0$.

Now suppose that $w \in G$ is such that for any $x \in \operatorname{supp}(\phi)$ and $y \in \operatorname{supp}(f)$ with $xy < w$, $m_xRa_y = 0$. We will show that $m_xRa_y = 0$ for any $x \in \operatorname{supp}(\phi)$ and $y \in \operatorname{supp}(f)$ with $xy = w$. If there are not $x \in \operatorname{supp}(\phi)$ and $y \in \operatorname{supp}(f)$ such that $xy = w$, then clearly the conclusion holds. Now suppose that $x \in \operatorname{supp}(\phi)$ and $y \in \operatorname{supp}(f)$ are such that $xy = w$. For convenience we write $\{(x,y) \mid xy = w\}$ as $\{(x_i,y_i) \mid i = 1,2,\ldots,n\}$ with $x_1 < x_2 < \cdots < x_n$. Then for any $r \in R$, from

$$\sum_{\{x,y|xy=w\}} m_x\sigma_x(r\sigma_1(a_y)t(1,y))t(x,y) = 0$$

it follows that

$$\sum_{i=1}^{n} m_{x_i}\sigma_{x_i}(r\sigma_1(a_{y_i})t(1,y_i))t(x_i,y_i) = 0. \tag{4}$$

For each $i = 1,2,\ldots,n$, since M_R is a p.q.Baer module, there exists $e_{x_i}^2 = e_{x_i} \in R$ such that $r_R(m_{x_i}R) = e_{x_i}R$. Let $r' \in R$ and take $r = r'e_{x_1}$ in (4). From $m_{x_1}r'e_{x_1} = 0$ it follows that $m_{x_1}r'e_{x_1}\sigma_1(a_{y_1})\ t(1,y_1) = 0$. Thus $m_{x_1}\sigma_{x_1}(r'e_{x_1}\sigma_1(a_{y_1})t(1,y_1)) = 0$ since M_R is σ-compatible. Hence

$$\sum_{i=2}^{n} m_{x_i}\sigma_{x_i}(r'e_{x_1}\sigma_1(a_{y_i})t(1,y_i))t(x_i,y_i) = 0. \tag{5}$$

Note that $x_1y_i < x_iy_i = w$ for any $i = 2,\ldots,n$. Thus by induction hypothesis, $m_{x_1}Ra_{y_i} = 0$. Thus $a_{y_i} \in r_R(m_{x_1}R) = e_{x_1}R$. So $a_{y_i} = e_{x_1}a_{y_i}$. Thus $m_{x_i}r'(1-e_{x_1})a_{y_i} = 0$, and so $m_{x_i}r'(1-e_{x_1})\sigma_1(a_{y_i}) = 0$ since M_R is σ-compatible. Thus $m_{x_i}r'(1-e_{x_1})\sigma_1(a_{y_i})t(1,y_i) = 0$. Thus $m_{x_i}\sigma_{x_i}\big(r'(1-e_{x_1})\sigma_1(a_{y_i})t(1,y_i)\big) = 0$ since M_R is σ-compatible. Hence

$$m_{x_i}\sigma_{x_i}(r'\sigma_1(a_{y_i})t(1,y_i)) = m_{x_i}\sigma_{x_i}(r'e_{x_1}\sigma_1(a_{y_i})t(1,y_i)).$$

Now from (5) it follows that

$$\sum_{i=2}^{n} m_{x_i}\sigma_{x_i}(r'\sigma_1(a_{y_i})t(1,y_i))t(x_i,y_i) = 0. \tag{6}$$

Let $p \in R$ and take $r' = pe_{x_2}$. Then, since $m_{x_2}pe_{x_2} = 0$, we have

$$m_{x_2}\sigma_{x_2}(pe_{x_2}\sigma_1(a_{y_2})t(1,y_2)) = 0.$$

Thus

$$\begin{aligned}&\sum_{i=3}^{n} m_{x_i}\sigma_{x_i}(pe_{x_2}\sigma_1(a_{y_i})t(1,y_i))t(x_i,y_i)\\ =&\sum_{i=3}^{n} m_{x_i}\sigma_{x_i}(p\sigma_1(a_{y_i})t(1,y_i))t(x_i,y_i) = 0.\end{aligned}$$

Continuing in this manner, we have $m_{x_n}\sigma_{x_n}(q\sigma_1(a_{y_n})t(1,y_n))t(x_n,y_n) = 0$, where q is an arbitrary element of R. Thus $m_{x_n}\sigma_{x_n}(q\sigma_1(a_{y_n})t(1,y_n)) = 0$ since $t(x_n,y_n)$ is invertible. This implies that $m_{x_n}qa_{y_n} = 0$ since M_R is σ-compatible. Hence

$$m_{x_{n-1}}qa_{y_{n-1}} = 0, \ldots, m_{x_1}qa_{y_1} = 0.$$

Therefore, by transfinite induction, we have shown that for any $x \in \text{supp}(\phi)$ and $y \in \text{supp}(f)$, $m_xRa_y = 0$. □

Lemma 3.3. *Let G be an ordered group and M_R a σ-compatible module. Then the following are equivalent:*

(1) *For any $\phi = \sum_{x\in G} m_x x \in M * ((G))$ and any $f = \sum_{y\in G} a_y y \in R * ((G))$, $\phi R * ((G)) f = 0$ implies $m_xRa_y = 0$ for all x and y.*

(2) *For any $\phi = \sum_{x\in G} m_x x \in M * ((G))$, $r_{R*((G))}(\phi R * ((G))) = r_R(X) * ((G))$, where $X = \{m_xR \mid x \in G\}$.*

Proof. (1) $\Longrightarrow$ (2) Assume that $f = \sum_{y\in G} a_y y \in r_{R*((G))}(\phi R * ((G)))$ with $\phi \in M * ((G))$. By (1), $m_xRa_y = 0$ for all x and y. Thus $a_y \in r_R(X)$, and so $f \in r_R(X) * ((G))$. Hence $r_{R*((G))}(\phi R * ((G))) \subseteq r_R(X) * ((G))$. Conversely, suppose that $f = \sum_{y\in G} a_y y \in r_R(X) * ((G))$. Then $a_y \in r_R(X)$ for all $y \in G$. Thus $m_xRa_y = 0$ for all x and y. Then for any $g = \sum_{z\in G} b_z z \in R * ((G))$, by the σ-compatibility of M_R, $m_x\sigma_x(b_z)\sigma_x\sigma_z(a_y) = 0$ for any $x, y, z \in G$. Thus $m_x\sigma_x(b_z)\sigma_x\sigma_z(a_y)\sigma_x(t(z,y))t(x,p) = 0$ for any $x, y, z, p \in G$. Hence

$$\begin{aligned}\phi g f &= \Big(\sum_{x\in G} m_x x\Big)\Big(\sum_{p\in G}\sum_{\{z,y|zy=p\}} b_z\sigma_z(a_y)t(z,y)p\Big)\\ &= \sum_{q\in G}\sum_{\{x,p|xp=q\}}\sum_{\{z,y|zy=p\}} m_x\sigma_x(b_z)\sigma_x\sigma_z(a_y)\sigma_x(t(z,y))t(x,p)q = 0.\end{aligned}$$

This means that $f \in r_{R*((G))}(\phi R * ((G)))$. So $r_{R*((G))}(\phi R * ((G))) = r_R(X) * ((G))$.

(2) $\Longrightarrow$ (1) Suppose that $\phi = \sum_{x\in G} m_x x \in M * ((G))$ and $f = \sum_{y\in G} a_y y \in R * ((G))$ are such that $\phi R * ((G)) f = 0$. Thus $f \in r_{R*((G))}(\phi R * ((G))) = r_R(X) * ((G))$. Hence $a_y \in r_R(X)$. So $m_xRa_y = 0$ for all x and y. □

Lemma 3.4. *Let G be an ordered group and M_R a σ-compatible module. Then for any $m \in M$, $r_{R*((G))}(m \cdot R * ((G))) = r_R(mR) * ((G))$.*

Proof. Let $f = \sum_{x\in G} a_x x \in r_{R*((G))}(m \cdot R * ((G)))$. Then for any $r \in R$, $mr\sigma_1(a_x)\ t(1,x) = 0$. Thus $mra_x = 0$ since $t(1,x)$ is invertible and M_R is σ-compatible. Hence $a_x \in r_R(mR)$. So $f \in r_R(mR) * ((G))$. Conversely, suppose that $f = \sum_{x\in G} a_x x \in r_R(mR) * ((G))$. Then $mRa_x = 0$. Hence for any $g = \sum_{y\in G} b_y y \in R * ((G))$, $m\sigma_1(b_y)t(1,y)\sigma_y(a_x)t(y,x) = 0$. Thus

$$mgf = \sum_{z\in G}\sum_{\{y,x|yx=z\}} m\sigma_1(b_y)t(1,y)\sigma_y(a_x)t(y,x)z = 0.$$

Hence $f \in r_{R*((G))}(m \cdot R*((G)))$. So, $r_{R*((G))}(m \cdot R*((G))) = r_R(mR)*((G))$. □

In order to prove the main result, we first give the necessity of the module $M*((G))$ to be a p.q.Baer module.

Proposition 3.5. *Let G be an ordered group and M_R a faithful σ-compatible module. If $M*((G))$ is a p.q.Baer module, then M_R is a p.q.Baer module.*

Proof. Let $m \in M$. By Lemma 3.4, $r_{R*((G))}(m \cdot R*((G))) = r_R(mR)*((G))$. Since $M*((G))$ is a p.q.Baer module, there exists $f^2 = f \in R*((G))$ such that $r_{R*((G))}(m \cdot R*((G))) = fR*((G))$. Suppose that $f = \sum_{x\in G} a_x x$. We will show that $r_R(mR) = a_1 R$ and $a_1^2 = a_1$, which will imply that M_R is a p.q.Baer module. From $f \in r_{R*((G))}(m \cdot R*((G)))$ it follows that $mrf = 0$ for any $r \in R$. Thus $mr\sigma_1(a_x)t(1,x) = 0$. Thus for any $x \in G$, $mra_x = 0$ since $t(1,x)$ is invertible and M_R is σ-compatible. Thus $a_1 \in r_R(mR)$. Conversely, let $r \in r_R(mR)$. Then $r \in r_R(mR)*((G))$. Thus $r = fr$. Then $r = a_1\sigma_1(r)t(1,1) \in a_1R$. Hence $r_R(mR) = a_1R$. Since $a_1 = fa_1$, $(1-a_1)\sigma_1(a_1)t(1,1) = 0$. Thus $a_1^2 = a_1$ since $t(1,1)$ is invertible and M_R is a faithful σ-compatible module. □

Let X be a non-empty set. We will say that X is G-indexed if there exists a well-ordered subset I of G such that X is indexed by I.

Theorem 3.6. *Let G be an ordered group and M_R a faithful σ-compatible module. Then the following are equivalent:*

(1) *$M*((G))$ is a p.q.Baer module.*

(2) *For any G-indexed set X consisting of cyclic submodules of M, there exists an idempotent $e \in R$ such that $r_R(X) = eR$.*

Proof. (1) $\Longrightarrow$ (2) Suppose that $X = \{m_xR \mid m_x \in M, x \in I\}$ is a G-indexed family of cyclic submodules of M, meaningly I is a well-ordered subset of G. Let $m_x = 0$ when $x \in G-I$, then $\phi = \sum_{x\in G} m_x x \in M*((G))$ since supp$(\phi) \subseteq I$ is a well-ordered subset of G. Since $M*((G))$ is a p.q.Baer module, there exists $f^2 = f \in R*((G))$ such that $r_{R*((G))}(\phi R*((G))) = f \cdot R*((G))$. On the other hand, since $M*((G))$ is a p.q.Baer module, M is p.q.Baer by Proposition 3.5. Thus $r_{R*((G))}(\phi R*((G))) = r_R(X)*((G))$ by Lemma 3.2 and Lemma 3.3. Hence $r_R(X)*((G)) = f \cdot R*((G))$. Let $f = \sum_{y\in G} a_y y$, then by analogy with the proof of Proposition 3.5, we can show that $r_R(X) = a_1R$ and $a_1^2 = a_1$.

(2) $\Longrightarrow$ (1) Let $\phi = \sum_{x\in G} m_x x \in M*((G))$. Then $X = \{m_xR \mid m_x \in M, x \in \text{supp}(\phi)\}$ is a G-indexed family of cyclic submodules of M. By (2), there exists an idempotent $e \in R$ such that $r_R(X) = eR$. It is easy to see that M is p.q.Baer by (2). Thus $r_{R*((G))}(\phi R*((G))) = r_R(X)*((G)) = (eR)*((G)) = e \cdot R*((G))$ by Lemma 3.2 and Lemma 3.3, and which implies that $M*((G))$ is a p.q.Baer module. □

In the rest of this section, we will work with the special module R_R, which will lead to more interesting results.

Recall from [2], an idempotent $e \in R$ is left (resp. right) semicentral in R if $ere = re$ (resp. $ere = er$) for all $r \in R$. Equivalently, $e^2 = e \in R$ is left (resp. right) semicentral if eR (resp. Re) is an ideal of R. Since the right annihilator of a right ideal is an ideal, we see that if the right annihilator of a G-indexed family of principal right ideals of R is generated by an idempotent e, then e is a left semicentral idempotent.

Let $I(R)$ be the set of all idempotents of R, $\mathcal{S}_l(R)$ the set of all left semicentral idempotents of R and $C(R)$ the set of all central idempotents of R. Let S be a G-indexed subset of $I(R)$. We say that S has a generalized join in $I(R)$ if there exists an idempotent $e \in I(R)$ such that

(1) $gR(1-e) = 0$ for any $g \in S$.

(2) If $f \in I(R)$ is such that $gR(1-f) = 0$ for any $g \in S$, then $eR(1-f) = 0$.

Corollary 3.7. *Let G be an ordered group and R a σ-compatible ring. Then the following conditions are equivalent:*

(1) *$R * ((G))$ is a right p.q.Baer ring.*

(2) *The right annihilator of any G-indexed family of principal right ideals of R is generated by an idempotent of R.*

If $\mathcal{S}_l(R) \subseteq C(R)$, then the following conditions are equivalent to the conditions above:

(3) *R is a right p.q.Baer ring and for any G-indexed subset $\{e_s \mid s \in I\}$ of $I(R)$, $\bigcap_{s \in I} r_R(e_sR) = eR$.*

(4) *R is a right p.q.Baer ring and for any G-indexed subset $\{e_s \mid s \in I\}$ of $C(R)$, $\bigcap_{s \in I} r_R(e_sR) = eR$.*

(5) *R is a right p.q.Baer ring and any G-indexed subset of $C(R)$ has a generalized join in $I(R)$.*

(6) *R is a right p.q.Baer ring and any G-indexed subset of $I(R)$ has a generalized join in $I(R)$.*

Proof. (1) $\Longleftrightarrow$ (2) follows from Theorem 3.6.

(2) $\Longrightarrow$ (3). Note that for any $a \in R$, $\{aR\}$ is G- indexed. Thus (2) $\Longrightarrow$ (3) is straightforward.

(3) $\Longrightarrow$ (4). It is directly verified.

(4) $\Longrightarrow$ (5). Let $\{e_s \mid s \in I\}$ be a G-indexed subset of $C(R)$. By (4), there exists an $e \in I(R)$ such that $\bigcap_{s \in I} r_R(e_sR) = eR$. We will show that $1-e$ is a generalized join of the set $\{e_s \mid s \in I\}$. It is clearly that $e_sR(1-(1-e)) = e_sRe = 0$ for any $s \in I$. Assume that $f^2 = f \in R$ is such that $e_sR(1-f) = 0$ for any $s \in I$. Then $1-f \in \bigcap_{s \in I} r_R(e_sR) = eR$. So $(1-f) = e(1-f)$. Since $e \in \mathcal{S}_l(R)$, $(1-e)R(1-f) = 0$. Hence $1-e$ is a generalized join of $\{e_s \mid s \in I\}$ in $I(R)$.

(5) $\Longrightarrow$ (6). Let $\{e_s \mid s \in I\}$ be an G-indexed subset of $I(R)$. Since R is a right p.q.Baer ring, there exist $f_s \in \mathcal{S}_l(R) \subseteq C(R)$ such that $r_R(e_sR) = f_sR$ for all $s \in I$. By (5), $\{1-f_s \mid s \in I\}$ has a generalized join in $I(R)$, say e. Then $(1-f_s)R(1-e) = 0$ for any $s \in I$. Thus, for any $r \in R$ and any $s \in I$, $r(1-e) = f_sr(1-e)$. Hence $e_sr(1-e) = e_sf_sr(1-e) = 0$ for any $s \in I$.

This means that $e_sR(1-e)=0$ for any $s \in I$. Suppose that $f \in I(R)$ is such that $e_sR(1-f)=0$ for each $s \in I$. Then $1-f \in r_R(e_sR)=f_sR$, and so $(1-f)=f_s(1-f)$. Thus $(1-f_s)(1-f)=0$. Hence $(1-f_s)R(1-f)=0$. Since e is a generalized join of $\{1-f_s \mid s \in I\}$, it follows that $eR(1-f)=0$. Hence e is a generalized join of $\{e_s \mid s \in I\}$.

(6) $\Longrightarrow$ (2). Suppose that $X=\{a_sR \mid a_s \in R, s \in I\}$ is a G-indexed family of principal right ideals of R. Then there exists a left semicentral idempotent $e_s^2=e_s \in R$ such that $r_R(a_sR)=e_sR$ for each $s \in I$. By the hypothesis, the set $\{1-e_s \mid s \in I\}$ has a generalized join f. Then $(1-e_s)R(1-f)=0$. We will show that $r_R(X)=(1-f)R$. Since $(1-e_s)R(1-f)=0$, $r(1-f)=e_sr(1-f)$ for any $r \in R$. Thus $a_sr(1-f)=a_se_sr(1-f)=0$. This means that $(1-f) \in r_R(X)$. Conversely, suppose that $p \in r_R(X)$. Then $a_sRp=0$ for any $s \in I$. Thus $p \in r_R(a_sR)=e_sR$, and so $p=e_sp$ for any $s \in I$. Suppose that $r_R(pR)=gR$, where g is a left semicentral idempotent. Since e_s is left semicentral, by the hypothesis, e_s is central. Thus we have $pr=e_spr=pre_s$, which means $1-e_s \in gR$. Thus $1-e_s=g(1-e_s)$ for any $s \in I$. So $(1-e_s)R(1-g)=0$. Since f is a generalized join of $\{1-e_s \mid s \in I\}$, it follows that $fR(1-g)=0$. Hence $p=p-pg=p(1-g)=(1-g)p=(1-f)(1-g)p \in (1-f)R$. Therefore, $r_R(X)=(1-f)R$. □

In [8], it was shown that if $\mathcal{S}_l(R) \subseteq C(R)$, then $R[[x]]$ is a right p.q.Baer ring if and only if R is a right p.q.Baer ring and any countable subset of $I(R)$ has a generalized join in $I(R)$. Here we have

Corollary 3.8. *Let $\alpha \in \mathrm{Aut}(R)$ and R an α-compatible ring. If $\mathcal{S}_l(R) \subseteq C(R)$, then $R[[x,x^{-1};\alpha]]$ is a right p.q.Baer ring if and only if R is a right p.q.Baer ring and any countable subset of $C(R)$ has a generalized join in $I(R)$.*

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