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NEW WEIGHTED OSTROWSKI-GRÜSS-ČEBYŠEV TYPE INEQUALITIES

WEN-JUN LIU, YU HUANG, AND XING-XIA PAN

ABSTRACT. In this paper, by introducing parameter $r > 1$, new weighted Ostrowski-Grüss-Čebyšev type inequalities for $1/p + 1/q = 1 - 1/r$ are established.

1. Introduction

In 1938, Ostrowski proved the following interesting integral inequality [7]:

Theorem 1.1. *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable in (a,b) and its derivative $f' : (a,b) \to \mathbb{R}$ is bounded in (a,b), that is, $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(x)| < \infty$. Then for any $x \in [a,b]$, we have the inequality:*

$$(1) \qquad \left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \le \left[\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right] (b-a)\|f'\|_\infty .$$

The inequality is sharp in the sense that the constant 1/4 cannot be replaced by a smaller one.

For two absolutely continuous functions $f, g : [a,b] \to \mathbb{R}$, consider the functional

$$(2) \quad T(f,g) = \frac{1}{b-a}\int_a^b f(x)g(x)dx - \left(\frac{1}{b-a}\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right)$$

provided the involved integrals exist. In 1882, Čebyšev [6] proved that, if f', $g' \in L^\infty[a,b]$, then

$$(3) \qquad |T(f,g)| \le \frac{1}{12}(b-a)^2\|f'\|_\infty\|g'\|_\infty .$$

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In 1934, Grüss [6] showed that

$$|T(f,g)| \le \frac{1}{4}(M-m)(N-n) \tag{4}$$

provided m, M, n, N are real numbers satisfying the condition $-\infty < m \le f(x) \le M < \infty$, $-\infty < n \le g(x) \le N < \infty$ for all $x \in [a,b]$.

During the past few years many researchers have given considerable attention to the above inequalities and various generalizations, extensions and variants of these inequalities have appeared in the literature, see [1, 2, 3, 4, 5, 9] and the references cited therein. In [10], Rafiq et al. gave a weighted Ostrowski type inequality for differentiable mappings whose first derivatives belong to $L^p[a,b]$, $p > 1$.

Theorem 1.2. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on $[a,b]$, whose first derivative, i.e., $f' : [a,b] \to \mathbb{R}$, belongs to $L^p[a,b]$, $p > 1$ for any $x \in [a,b]$. Then, we have the inequality:*

$$\left| f(x) - \frac{1}{m(a,b)} \int_a^b w(t) f(t) dt \right| \le \frac{(x-a)^{1+1/q} + (b-x)^{1+1/q}}{m(a,b)(q+1)^{1/q}} \|f'\|_{w,p}, \tag{5}$$

where $w(t)$ and $m(a,b)$ be given in Section 2, and the weighted norm of differentiable function whose derivatives belong to $L^p[a,b]$ is defined as

$$\|\phi\|_{w,p} = \left(\int_a^b |w(t)\phi(t)|^p dt \right)^{1/p}.$$

Recently, Pachpatte [8] established a new Grüss type inequality involving two functions and their derivatives.

Theorem 1.3. *Let $f, g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable in (a,b), whose derivatives $f', g' : (a,b) \to \mathbb{R}$ are bounded in (a,b), i.e., $\|f'\|_\infty < \infty$, $\|g'\|_\infty < \infty$. Then*

$$|T(f,g)| \le \frac{1}{2(b-a)^2} \int_a^b \{|g(x)| \|f'\|_\infty + |f(x)| \|g'\|_\infty\} \left(\int_a^b |x-y| dy \right) dx. \tag{6}$$

Motivated by the results of Pachpatte and Rafiq, in the present paper we establish some new weighted Ostrowski-Grüss-Čebyšev type inequalities for $1/p + 1/q = 1 - 1/r$ by introducing parameter $r > 1$. The analysis used in the proofs is elementary and based on the use of integral identities proved in [8].

2. Main results

Let the weight $w : [a,b] \to [0,\infty)$, be non-negative and integrable, i.e., $\int_a^b w(t)dt < \infty$. The domain of w is finite. We denote the zero moment as

$$m(a,b) = \int_a^b w(t) dt. \tag{7}$$

For suitable functions $f, g : [a,b] \to \mathbb{R}$ we set

$$
\begin{aligned}
T_w(f,g) &= \frac{1}{m(a,b)} \int_a^b w(x)f(x)g(x)dx \\
&\quad - \left(\frac{1}{m(a,b)} \int_a^b w(x)f(x)dx\right)\left(\frac{1}{m(a,b)} \int_a^b w(x)g(x)dx\right).
\end{aligned} \tag{8}
$$

Then the following theorem holds:

Theorem 2.1. *Let $f, g : [a,b] \to \mathbb{R}$ be absolutely continuous and have bounded first derivative, $w \in L^p[a,b]$, $p > 1, q > 0, r > 1$ and $1/p + 1/q = 1 - 1/r$. Then we have the inequalities*

$$|T_w(f,g)| \leq \frac{1}{2m^2(a,b)} \int_a^b w(x)M_w(x,r)\{\,|g(x)|\|f'\|_\infty + |f(x)|\|g'\|_\infty\}dx \tag{9}$$

and

$$|T_w(f,g)| \leq \frac{\|f'\|_\infty \|g'\|_\infty}{2m^2(a,b)} \int_a^b w(x)N_w(x,r)dx, \tag{10}$$

where

$$M_w(x,r) = \|w\|_p \left(\frac{q+r}{q+r+qr}\right)^{\frac{q+r}{qr}} \left[(x-a)^{\frac{q+r+qr}{qr}} + (b-x)^{\frac{q+r+qr}{qr}}\right]$$

and

$$N_w(x,r) = \|w\|_p \left(\frac{q+r}{q+r+2qr}\right)^{\frac{q+r}{qr}} \left[(x-a)^{\frac{q+r+2qr}{q+r}} + (b-x)^{\frac{q+r+2qr}{q+r}}\right]^{\frac{q+r}{qr}}.$$

Proof. For any $x, y \in [a,b]$ we have the following identities (see [8]):

$$f(x) - f(y) = \int_y^x f'(t)dt, \tag{11}$$

$$g(x) - g(y) = \int_y^x g'(t)dt. \tag{12}$$

Multiplying both sides of (11) and (12) by $g(x)$ and $f(x)$ respectively and adding the resulting identities, we have

$$2f(x)g(x) - [g(x)f(y) + f(x)g(y)] = g(x)\int_y^x f'(t)dt + f(x)\int_y^x g'(t)dt \tag{13}$$

for $x \in [a,b]$. Multiplying both sides of (13) by $w(y)$ and integrating with respect to y over $[a,b]$, we have

$$\begin{aligned}
&2f(x)g(x)\int_a^b w(y)dy \\
(14)\qquad &-\left[g(x)\int_a^b w(y)f(y)dy + f(x)\int_a^b w(y)g(y)dy\right] \\
=\ &\int_a^b w(y)\left[g(x)\int_y^x f'(t)dt + f(x)\int_y^x g'(t)dt\right]dy.
\end{aligned}$$

Multiplying both sides of (14) by $\frac{w(x)}{2m^2(a,b)}$, we obtain

$$\begin{aligned}
&\frac{w(x)f(x)g(x)}{m(a,b)} \\
(15)\qquad &-\left[\frac{w(x)g(x)}{2m^2(a,b)}\int_a^b w(y)f(y)dy + \frac{w(x)f(x)}{2m^2(a,b)}\int_a^b w(y)g(y)dy\right] \\
=\ &\frac{w(x)}{2m^2(a,b)}\int_a^b w(y)\left[g(x)\int_y^x f'(t)dt + f(x)\int_y^x g'(t)dt\right]dy.
\end{aligned}$$

Integrating both sides of (15) with respect to x over $[a,b]$, we have

$$\begin{aligned}
&T_w(f,g) \\
&= \frac{1}{2m^2(a,b)}\int_a^b w(x)\left\{\int_a^b w(y)\left[g(x)\int_y^x f'(t)dt + f(x)\int_y^x g'(t)dt\right]dy\right\}dx.
\end{aligned}$$

Using the properties of modulus, we get

(16)

$$\begin{aligned}
&|T_w(f,g)| \\
&\leq \frac{1}{2m^2(a,b)}\int_a^b w(x)\left(\int_a^b |x-y|w(y)dy\right)\{\,|g(x)|\|f'\|_\infty + |f(x)|\|g'\|_\infty\}dx.
\end{aligned}$$

By Hölder inequality, we have

$$\begin{aligned}
&\int_a^b |x-y|w(y)dy \\
&= \int_a^x (x-y)w(y)dy + \int_x^b (y-x)w(y)dy \\
(17)\qquad &\leq \|w\|_p\left[\left(\int_a^x (x-y)^{\frac{qr}{q+r}}dy\right)^{\frac{q+r}{qr}} + \left(\int_x^b (y-x)^{\frac{qr}{q+r}}dy\right)^{\frac{q+r}{qr}}\right] \\
&= \|w\|_p\left(\frac{q+r}{q+r+qr}\right)^{\frac{q+r}{qr}}\left[(x-a)^{\frac{q+rq+r}{qr}} + (b-x)^{\frac{q+rq+r}{qr}}\right] \\
&:= M_w(x,r).
\end{aligned}$$

From (16) and (17), we obtain (9).

Multiplying the left and right side of (11) and (12), we get

(18)

$$f(x)g(x) - [g(x)f(y) + f(x)g(y)] + f(y)g(y) = \left(\int_y^x f'(t)dt\right)\left(\int_y^x g'(t)dt\right)$$

for $x \in [a,b]$. Multiplying both sides of (18) by $w(y)$ and integrating with respect to y over $[a,b]$, we have

$$\begin{aligned} & f(x)g(x)\int_a^b w(y)dy - \left[g(x)\int_a^b w(y)f(y)dy + f(x)\int_a^b w(y)g(y)dy\right] \\ & + \int_a^b w(y)f(y)g(y)dy \\ = & \int_a^b w(y)\left(\int_y^x f'(t)dt\right)\left(\int_y^x g'(t)dt\right)dy. \end{aligned} \tag{19}$$

Multiplying both sides of (19) by $\frac{w(x)}{m^2(a,b)}$, we obtain

(20)

$$\begin{aligned} & \frac{w(x)f(x)g(x)}{m(a,b)} - \left[\frac{w(x)g(x)}{m^2(a,b)}\int_a^b w(y)f(y)dy + \frac{w(x)f(x)}{m^2(a,b)}\int_a^b w(y)g(y)dy\right] \\ & + \frac{w(x)}{m^2(a,b)}\int_a^b w(y)f(y)g(y)dy \\ = & \frac{w(x)}{m^2(a,b)}\int_a^b w(y)\left(\int_y^x f'(t)dt\right)\left(\int_y^x g'(t)dt\right)dy. \end{aligned}$$

Integrating both sides of (20) with respect to x over $[a,b]$, we have

$$T_w(f,g) = \frac{1}{2m^2(a,b)}\int_a^b w(x)\left\{\int_a^b w(y)\left(\int_y^x f'(t)dt\right)\left(\int_y^x g'(t)dt\right)dy\right\}dx.$$

Using the properties of modulus, we get

$$|T_w(f,g)| \le \frac{\|f'\|_\infty \|g'\|_\infty}{2m^2(a,b)}\int_a^b w(x)\left(\int_a^b (x-y)^2 w(y)dy\right)dx. \tag{21}$$

By Hölder inequality, we have

$$\begin{aligned} & \int_a^b (x-y)^2 w(y)dy \\ \le & \|w\|_p \left[\int_a^x (x-y)^{\frac{2qr}{q+r}}dy + \int_x^b (y-x)^{\frac{2qr}{q+r}}dy\right]^{\frac{q+r}{qr}} \\ = & \|w\|_p \left(\frac{q+r}{q+r+2qr}\right)^{\frac{q+r}{qr}}\left[(x-a)^{\frac{q+2rq+r}{q+r}} + (b-x)^{\frac{q+2rq+r}{q+r}}\right]^{\frac{q+r}{qr}} \\ := & N_w(x,r). \end{aligned} \tag{22}$$

From (21) and (22), we obtain (10). □

Remark 2.2. We note that in the special cases, if we set $w(x) = 1$, $r \to \infty$ and $q \to 1$ in (9), we obtain

(23)
$$|T(f,g)| \le \frac{1}{2(b-a)^2} \int_a^b \{|g(x)|\|f'\|_\infty + |f(x)|\|g'\|_\infty\} \left[\frac{(x-a)^2+(b-x)^2}{2}\right] dx,$$

which recaptures the inequality (6) since $\int_a^b |x-y|dy = \frac{(x-a)^2+(b-x)^2}{2}$. Taking $w(x) = 1$, $r \to \infty$ and $q \to 1$ in (10), we have the inequality

$$|T(f,g)| \le \frac{\|f'\|_\infty \|g'\|_\infty}{2(b-a)^2} \int_a^b \frac{(x-a)^3+(b-x)^3}{3} dx, \tag{24}$$

which recaptures the inequality (3) since $\int_a^b \frac{(x-a)^3+(b-x)^3}{3} dx = \frac{1}{6}(b-a)^4$.

Corollary 2.3. *Under the assumptions of Theorem 2.1 with* $r \to \infty$, *we have the inequalities*

$$|T_w(f,g)| \le \frac{1}{2m^2(a,b)} \int_a^b w(x) M_w(x)\{\,|g(x)|\|f'\|_\infty + |f(x)|\|g'\|_\infty\} dx, \tag{25}$$

and

$$|T_w(f,g)| \le \frac{\|f'\|_\infty \|g'\|_\infty}{2m^2(a,b)} \int_a^b w(x) N_w(x) dx, \tag{26}$$

where

$$M_w(x) = \|w\|_p \frac{\left[(x-a)^{1+1/q} + (b-x)^{1+1/q}\right]}{(q+1)^{1/q}}$$

and

$$N_w(x) = \|w\|_p \frac{\left[(x-a)^{2q+1} + (b-x)^{2q+1}\right]^{1/q}}{(2q+1)^{1/q}}.$$

Corollary 2.4. *Under the assumptions of Theorem 2.1 with* $w(x) = 1$, *we have the inequalities*

$$|T(f,g)| \le \frac{1}{2(b-a)^2} \int_a^b M(x,r)\{\,|g(x)|\|f'\|_\infty + |f(x)|\|g'\|_\infty\} dx, \tag{27}$$

and

$$|T_w(f,g)| \le \frac{\|f'\|_\infty \|g'\|_\infty}{2(b-a)^2} \int_a^b N(x,r) dx, \tag{28}$$

where

$$M(x,r) = \left(\frac{q+r}{q+r+qr}\right)^{\frac{q+r}{qr}} \left[(x-a)^{\frac{q+r+qr}{qr}} + (b-x)^{\frac{q+r+qr}{qr}}\right]$$

and

$$N(x,r) = \left(\frac{q+r}{q+r+2qr}\right)^{\frac{q+r}{qr}} \left[(x-a)^{\frac{q+r+2qr}{q+r}} + (b-x)^{\frac{q+r+2qr}{q+r}}\right]^{\frac{q+r}{qr}}.$$

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Wen-jun Liu
College of Mathematics and Physics
Nanjing University of Information Science and Technology
Nanjing 210044, China
E-mail address: wjliu@nuist.edu.cn

Yu Huang
College of Mathematics and Physics
Nanjing University of Information Science and Technology
Nanjing 210044, China
E-mail address: huangyu@nuist.edu.cn

Xing-xia Pan
Department of Calculation and Information Science
Nanchang Institute of Aeronautical Technology
Nanchang 330063, China
E-mail address: panpxx233@sina.com