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# REAL HYPERSURFACES IN COMPLEX TWO-PLANE GRASSMANNIANS WITH COMMUTING STRUCTURE JACOBI OPERATOR

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Abstract. In this paper we give a complete classification of real hypersurfaces in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ with commuting structure Jacobi operator $R_\xi$ and another geometric condition.

## 0. Introduction

In the geometry of real hypersurfaces in complex space forms $M_n(c)$ Kimura [6] has proved that Hopf real hypersurfaces $M$ in a complex projective space $P_n(\mathbb{C})$ with commuting Ricci tensor are locally congruent to of type $(A)$, a tube over a totally geodesic $P_k(\mathbb{C})$, of type $(B)$, a tube over a complex quadric $Q_{n-1}$, $\cot^2 2r = n-2$, of type $(C)$, a tube over $P_1(\mathbb{C})\times P_{(n-1)/2}(\mathbb{C})$, $\cot^2 2r = \frac{1}{n-2}$ and $n$ is odd, of type $(D)$, a tube over a complex two-plane Grassmannian $G_2(\mathbb{C}^5)$, $\cot^2 2r = \frac{3}{5}$ and $n = 9$, of type $(E)$, a tube over a Hermitian symmetric space $SO(10)/U(5)$, $\cot^2 2r = \frac{5}{9}$ and $n = 15$.

The notion of Hopf real hypersurfaces means that the structure vector $\xi$ defined by $\xi = -JN$ satisfies $A\xi = \alpha\xi$, where $J$ denotes a Kaehler structure of $P_n(\mathbb{C})$, $N$ and $A$ a unit normal and the shape operator of $M$ in $P_n(\mathbb{C})$. We say such a structure vector $\xi$ on $M$ the *Reeb* vector field, and its flow the *Reeb* flow on M.

In a quaternionic projective space $\mathbb{Q}P^m$ Pérez [7] has classified real hypersurfaces in $\mathbb{Q}P^m$ with commuting Ricci tensor $S\phi_i = \phi_i S$, $i = 1, 2, 3$, where $S$ (resp. $\phi_i$) denotes the Ricci tensor (resp. the structure tensor) of $M$ in $\mathbb{Q}P^m$, is locally congruent to of $A_1, A_2$-type, that is, a tube over $\mathbb{Q}P^k$ with radius $0 < r < \frac{\pi}{2}$, $k \in \{0, \ldots, m-1\}$.

A Jacobi field along geodesics of a given Riemannian manifold $(M, g)$ is an important role in the study of differential geometry. It satisfies a well known

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differential equation which inspires Jacobi operators. The Jacobi operator is defined by $(R_X(Y))(p) = (R(Y,X)X)(p)$, where $R$ denotes the curvature tensor of $M$ and $X, Y$ denote tangent vector fields on $M$. Then we see that $R_X$ is a self-adjoint endomorphism on the tangent space of $M$ and is related to the differential equation, so called Jacobi equation, which is given by $\nabla_{\gamma'}(\nabla'_\gamma Y) + R(Y,\gamma')\gamma' = 0$ along a geodesic $\gamma$ on $M$, where $\gamma'$ denotes the velocity vector along $\gamma$ on $M$.

When we study a real hypersurface $M$ in a complex space form $M_n(c)$, $c \neq 0$, we will call the Jacobi operator on $M$ with respect to the *Reeb* vector $\xi$ the *structure Jacobi operator* on $M$ and will denote it by $R_\xi$, where $R_\xi$ is defined by $R_\xi(X) = R(X,\xi)\xi$ for the curvature tensor $R$ of $M$ and any tangent vector field $X$ on $M$.

For a commuting problem in quaternionic space forms Berndt [2] has introduced the notion of normal Jacobi operator $\bar{R}(X,N)N \in \mathrm{End}T_xM$, $x \in M$ for real hypersurfaces $M$ in quaternionic projective space $\mathbb{Q}P^m$ or in quaternionic hyperbolic space $\mathbb{Q}H^m$, where $\bar{R}$ denotes the curvature tensor of a quaternionic projective space $\mathbb{Q}P^m$ and a quaternionic hyperbolic space $\mathbb{Q}H^m$. He [2] also has shown that the curvature adaptedness, that is, the normal Jacobi operator commutes with the shape operator $A$, is equivalent to the fact that the distributions $\mathfrak{D}$ and $\mathfrak{D}^\perp = \mathrm{Span}\{\xi_1, \xi_2, \xi_3\}$ are invariant by the shape operator $A$ of $M$, where $T_xM = \mathfrak{D} \oplus \mathfrak{D}^\perp$, $x \in M$.

Now let us consider a complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ which consists of all complex 2-dimensional linear subspaces in $\mathbb{C}^{m+2}$. Then the situation for real hypersurfaces in $G_2(\mathbb{C}^{m+1})$ with normal Jacobi operator $\bar{R}_N$ is not so simple and will be quite different from the cases mentioned above.

The ambient space $G_2(\mathbb{C}^{m+2})$ is known to be the unique compact irreducible Riemannian symmetric space equipped with both a Kähler structure $J$ and a quaternionic Kähler structure $\mathfrak{J}$ not containing $J$ (See Berndt and Suh [3]). So, in $G_2(\mathbb{C}^{m+2})$ we have the two natural geometric conditions for real hypersurfaces that $[\xi] = \mathrm{Span}\ \{\xi\}$ or $\mathfrak{D}^\perp = \mathrm{Span}\ \{\xi_1, \xi_2, \xi_3\}$ is invariant under the shape operator. By using such two conditions Berndt and Suh [3] have proved the following:

**Theorem A.** *Let $M$ be a connected real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then both $[\xi]$ and $\mathfrak{D}^\perp$ are invariant under the shape operator of $M$ if and only if either*

(A) *$M$ is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, or*

(B) *$m$ is even, say $m = 2n$, and $M$ is an open part of a tube around a totally geodesic $\mathbb{Q}P^n$ in $G_2(\mathbb{C}^{m+2})$.*

If the *Reeb* vector field $\xi$ of a real hypersurface $M$ in $G_2(\mathbb{C}^{m+2})$ is invariant by the shape operator, $M$ is said to be a *Hopf hypersurface*. In such a case the integral curves of the *Reeb* vector field $\xi$ are geodesics (See Berndt and Suh [4]). Moreover, the flow generated by the integral curves of the structure vector field $\xi$ for Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ is said to be a *geodesic Reeb flow.* Moreover, we say $M$ is with non-vanishing *geodesic Reeb flow* if the corresponding principal curvature $\alpha$ is non-vanishing.

On the other hand, we say that the Reeb vector field is *Killing* if the Lie derivative along the direction of the structure vector field $\xi$ vanishes, that is, $\mathcal{L}_\xi g = 0$, where $g$ denotes the Riemannian metric induced from $G_2(\mathbb{C}^{m+2})$. Then this is equivalent to the fact that the structure tensor $\phi$ commutes with the shape operator $A$ of $M$ in $G_2(\mathbb{C}^{m+2})$. This condition also has the geometric meaning that the flow of Reeb vector field is *isometric.* Moreover, Berndt and Suh [4] have proved that real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with isometric flow is of a tube over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.

Now let us introduce a structure Jacobi operator $R_\xi$ in such a way that

$$R_\xi(X) = R(X,\xi)\xi$$

for the curvature tensor $R(X,Y)Z$ of $M$ in $G_2(\mathbb{C}^{m+2})$, where $\xi$ denotes the structure vector, $X, Y$ and $Z$ any tangent vector fields of $M$ in $G_2(\mathbb{C}^{m+2})$. Then the structure Jacobi operator $R_\xi$ is said to be *commuting* if the structure Jacobi operator $R_\xi$ commutes with the structure tensor $\phi$, that is, $R_\xi o\phi = \phi o R_\xi$.

Recently, some geometric properties for such a structure Jacobi operator $R_\xi$ of real hypersurfaces in complex space forms $M_n(c)$ have been studied by many authors (See [5], [8], and [9]). Among them commuting and parallel properties of such a structure Jacobi operator was studied by Ki, Pérez, Santos and Suh [5]. Moreover, $\mathfrak{D}$-parallel or Lie $\xi$-parallel of the structure Jacobi operator are studied by Pérez, Santos, and Suh (See [8] and [9]).

Now let us put the structure vector $\xi = -JN$ into the curvature tensor $R$ of a real hypersurface $M$ in $G_2(\mathbb{C}^{m+2})$, where $N$ denotes a unit normal vector of $M$ in $G_2(\mathbb{C}^{m+2})$. Then for any tangent vector field $X$ on $M$ in $G_2(\mathbb{C}^{m+2})$ we calculate the structure Jacobi operator $R_\xi$ in such a way that

$$\begin{aligned} R_\xi(X) &= R(X,\xi)\xi \\ &= X - \eta(X)\xi - \sum\nolimits_{\nu=1}^{3}\{\eta_\nu(X)\xi_\nu - \eta(X)\eta_\nu(\xi)\xi_\nu \\ &\quad + 3g(\phi_\nu X,\xi)\phi_\nu\xi + \eta_\nu(\xi)\phi_\nu\phi X\} + \alpha AX - \eta(AX)A\xi, \end{aligned}$$

where $\alpha$ denotes the function defined by $g(A\xi,\xi)$.

Related to such a structure Jacobi operator $R_\xi$, in this paper we give a classification of real hypersurfaces $M$ in $G_2(\mathbb{C}^{m+2})$ with commuting structure Jacobi operator, that is $R_\xi o\phi = \phi o R_\xi$, as follows:

**Theorem.** *Let $M$ be a real hypersurface in $G_2(\mathbb{C}^{m+2})$ with non-vanishing Reeb flow and commuting structure Jacobi operator. If the $\mathfrak{D}$ component of the structure vector $\xi$ is invariant by the shape operator, then $M$ is congruent to a tube over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.*

## 1. Riemannian geometry of $G_2(\mathbb{C}^{m+2})$

In this section we summarize basic material about $G_2(\mathbb{C}^{m+2})$, for details we refer to [3] and [4]. By $G_2(\mathbb{C}^{m+2})$ we denote the set of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. The special unitary group $G = SU(m+2)$ acts transitively on $G_2(\mathbb{C}^{m+2})$ with stabilizer isomorphic to $K = S(U(2) \times U(m)) \subset G$. Then $G_2(\mathbb{C}^{m+2})$ can be identified with the homogeneous space $G/K$, which we equip with the unique analytic structure for which the natural action of $G$ on $G_2(\mathbb{C}^{m+2})$ becomes analytic. Denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebra of $G$ and $K$, respectively, and by $\mathfrak{m}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Cartan-Killing form $B$ of $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is an $Ad(K)$-invariant reductive decomposition of $\mathfrak{g}$. We put $o = eK$ and identify $T_oG_2(\mathbb{C}^{m+2})$ with $\mathfrak{m}$ in the usual manner. Since $B$ is negative definite on $\mathfrak{g}$, its negative restricted to $\mathfrak{m} \times \mathfrak{m}$ yields a positive definite inner product on $\mathfrak{m}$. By $Ad(K)$-invariance of $B$ this inner product can be extended to a $G$-invariant Riemannian metric $g$ on $G_2(\mathbb{C}^{m+2})$. In this way $G_2(\mathbb{C}^{m+2})$ becomes a Riemannian homogeneous space, even a Riemannian symmetric space. For computational reasons we normalize $g$ so that the maximal sectional curvature of $(G_2(\mathbb{C}^{m+2}), g)$ is eight. Since $G_2(\mathbb{C}^3)$ is isometric to the three-dimensional complex projective space $\mathbb{C}P^2$ with constant holomorphic sectional curvature eight we will assume $m \geq 2$ from now on. Note that the isomorphism $Spin(6) \simeq SU(4)$ yields an isometry between $G_2(\mathbb{C}^4)$ and the real Grassmann manifold $G_2^+(\mathbb{R}^6)$ of oriented two-dimensional linear subspaces of $\mathbb{R}^6$.

The Lie algebra $\mathfrak{k}$ has the direct sum decomposition $\mathfrak{k} = \mathfrak{su}(m) \oplus \mathfrak{su}(2) \oplus \mathfrak{R}$, where $\mathfrak{R}$ denotes the center of $\mathfrak{k}$. Viewing $\mathfrak{k}$ as the holonomy algebra of $G_2(\mathbb{C}^{m+2})$, the center $\mathfrak{R}$ induces a Kähler structure $J$ and the $\mathfrak{su}(2)$-part a quaternionic Kähler structure $\mathfrak{J}$ on $G_2(\mathbb{C}^{m+2})$. If $J_1$ is any almost Hermitian structure in $\mathfrak{J}$, then $JJ_1 = J_1J$, and $JJ_1$ is a symmetric endomorphism with $(JJ_1)^2 = I$ and $\mathrm{tr}(JJ_1) = 0$. This fact will be used frequently throughout this paper.

A canonical local basis $J_1, J_2, J_3$ of $\mathfrak{J}$ consists of three local almost Hermitian structures $J_\nu$ in $\mathfrak{J}$ such that $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1}J_\nu$, where the index is taken modulo three. Since $\mathfrak{J}$ is parallel with respect to the Riemannian connection $\bar{\nabla}$ of $(G_2(\mathbb{C}^{m+2}), g)$, there exist for any canonical local basis $J_1, J_2, J_3$ of $\mathfrak{J}$ three local one-forms $q_1, q_2, q_3$ such that

$$\bar{\nabla}_X J_\nu = q_{\nu+2}(X)J_{\nu+1} - q_{\nu+1}(X)J_{\nu+2} \tag{1.1}$$

for all vector fields $X$ on $G_2(\mathbb{C}^{m+2})$.

Let $p \in G_2(\mathbb{C}^{m+2})$ and $W$ a subspace of $T_pG_2(\mathbb{C}^{m+2})$. We say that $W$ is a quaternionic subspace of $T_pG_2(\mathbb{C}^{m+2})$ if $JW \subset W$ for all $J \in \mathfrak{J}_p$. And we say that $W$ is a totally complex subspace of $T_pG_2(\mathbb{C}^{m+2})$ if there exists a one-dimensional subspace $\mathfrak{V}$ of $\mathfrak{J}_p$ such that $JW \subset W$ for all $J \in \mathfrak{V}$ and $JW \perp W$ for all $J \in \mathfrak{V}^\perp \subset \mathfrak{J}_p$. Here, the orthogonal complement of $\mathfrak{V}$ in $\mathfrak{J}_p$ is taken with respect to the bundle metric and orientation on $\mathfrak{J}$ for which any local oriented orthonormal frame field of $\mathfrak{J}$ is a canonical local basis of $\mathfrak{J}$. A quaternionic (resp. totally complex) submanifold of $G_2(\mathbb{C}^{m+2})$ is a submanifold all of whose tangent spaces are quaternionic (resp. totally complex) subspaces of the corresponding tangent spaces of $G_2(\mathbb{C}^{m+2})$.

The Riemannian curvature tensor $\bar{R}$ of $G_2(\mathbb{C}^{m+2})$ is locally given by

$$\begin{aligned}(1.2)\qquad &\bar{R}(X,Y)Z \\ &= g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX \\ &\quad - g(JX,Z)JY - 2g(JX,Y)JZ \\ &\quad + \sum_{\nu=1}^{3}\{g(J_\nu Y,Z)J_\nu X - g(J_\nu X,Z)J_\nu Y - 2g(J_\nu X,Y)J_\nu Z\} \\ &\quad + \sum_{\nu=1}^{3}\{g(J_\nu JY,Z)J_\nu JX - g(J_\nu JX,Z)J_\nu JY\},\end{aligned}$$

where $J_1, J_2, J_3$ is any canonical local basis of $\mathfrak{J}$.

## 2. Some fundamental formulas for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$

In this section we want to derive the normal Jacobi operator from the curvature tensor of complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ given in (1.2) and the equation of Gauss. Moreover, in this section we derive some basic formulae from the Codazzi equation for a real hypersurface in $G_2(\mathbb{C}^{m+2})$ (See [3], [4], [11], [12], [14], and [15]).

Let $M$ be a real hypersurface of $G_2(\mathbb{C}^{m+2})$, that is, a hypersurface of $G_2(\mathbb{C}^{m+2})$ with real codimension one. The induced Riemannian metric on $M$ will also be denoted by $g$, and $\nabla$ denotes the Riemannian connection of $(M,g)$. Let $N$ be a local unit normal field of $M$ and $A$ the shape operator of $M$ with respect to $N$. The Kähler structure $J$ of $G_2(\mathbb{C}^{m+2})$ induces on $M$ an almost contact metric structure $(\phi, \xi, \eta, g)$. Furthermore, let $J_1, J_2, J_3$ be a canonical local basis of $\mathfrak{J}$. Then each $J_\nu$ induces an almost contact metric structure $(\phi_\nu, \xi_\nu, \eta_\nu, g)$ on $M$. Using the above expression for $\bar{R}$, the Codazzi equation becomes

$$\begin{aligned}(\nabla_X A)Y - (\nabla_Y A)X &= \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi \\ &\quad + \sum_{\nu=1}^{3}\{\eta_\nu(X)\phi_\nu Y - \eta_\nu(Y)\phi_\nu X - 2g(\phi_\nu X, Y)\xi_\nu\} \\ &\quad + \sum_{\nu=1}^{3}\{\eta_\nu(\phi X)\phi_\nu\phi Y - \eta_\nu(\phi Y)\phi_\nu\phi X\} \\ &\quad + \sum_{\nu=1}^{3}\{\eta(X)\eta_\nu(\phi Y) - \eta(Y)\eta_\nu(\phi X)\}\xi_\nu .\end{aligned}$$

The following identities can be proved in a straightforward method and will be used frequently in subsequent calculations:

$$\begin{aligned}&\phi_{\nu+1}\xi_\nu = -\xi_{\nu+2}, \quad \phi_\nu\xi_{\nu+1} = \xi_{\nu+2},\\&\phi\xi_\nu = \phi_\nu\xi, \quad \eta_\nu(\phi X) = \eta(\phi_\nu X),\\&\phi_\nu\phi_{\nu+1}X = \phi_{\nu+2}X + \eta_{\nu+1}(X)\xi_\nu,\\&\phi_{\nu+1}\phi_\nu X = -\phi_{\nu+2}X + \eta_\nu(X)\xi_{\nu+1}.\end{aligned} \tag{2.1}$$

Now let us put

$$JX = \phi X + \eta(X)N, \quad J_\nu X = \phi_\nu X + \eta_\nu(X)N \tag{2.2}$$

for any tangent vector $X$ of a real hypersurface $M$ in $G_2(\mathbb{C}^{m+2})$, where $N$ denotes a normal vector of $M$ in $G_2(\mathbb{C}^{m+2})$. Then from this and the formulae (1.1) and (2.1) we have that

$$(\nabla_X\phi)Y = \eta(Y)AX - g(AX,Y)\xi, \quad \nabla_X\xi = \phi AX, \tag{2.3}$$

$$\nabla_X\xi_\nu = q_{\nu+2}(X)\xi_{\nu+1} - q_{\nu+1}(X)\xi_{\nu+2} + \phi_\nu AX, \tag{2.4}$$

$$\begin{aligned}(\nabla_X\phi_\nu)Y = &- q_{\nu+1}(X)\phi_{\nu+2}Y + q_{\nu+2}(X)\phi_{\nu+1}Y + \eta_\nu(Y)AX\\&- g(AX,Y)\xi_\nu.\end{aligned} \tag{2.5}$$

Summing up these formulae, we find the following

$$\begin{aligned}\nabla_X(\phi_\nu\xi) &= \nabla_X(\phi\xi_\nu)\\&= (\nabla_X\phi)\xi_\nu + \phi(\nabla_X\xi_\nu)\\&= q_{\nu+2}(X)\phi_{\nu+1}\xi - q_{\nu+1}(X)\phi_{\nu+2}\xi + \phi_\nu\phi AX\\&\quad - g(AX,\xi)\xi_\nu + \eta(\xi_\nu)AX.\end{aligned} \tag{2.6}$$

Moreover, from $JJ_\nu = J_\nu J$, $\nu = 1,2,3$, it follows that

$$\phi\phi_\nu X = \phi_\nu\phi X + \eta_\nu(X)\xi - \eta(X)\xi_\nu. \tag{2.7}$$

Then from (1.2) and the above formulae, the equation of Gauss is given by

$$\begin{aligned}&R(X,Y)Z\\&= g(Y,Z)X - g(X,Z)Y\\&\quad + g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y - 2g(\phi X,Y)\phi Z\\&\quad + \sum_{\nu=1}^3 \{g(\phi_\nu Y,Z)\phi_\nu X - g(\phi_\nu X,Z)\phi_\nu Y - 2g(\phi_\nu X,Y)\phi_\nu Z\}\\&\quad + \sum_{\nu=1}^3 \{g(\phi_\nu\phi Y,Z)\phi_\nu\phi X - g(\phi_\nu\phi X,Z)\phi_\nu\phi Y\}\\&\quad - \sum_{\nu=1}^3 \{\eta(Y)\eta_\nu(Z)\phi_\nu\phi X - \eta(X)\eta_\nu(Z)\phi_\nu\phi Y\}\\&\quad - \sum_{\nu=1}^3 \{\eta(X)g(\phi_\nu\phi Y,Z) - \eta(Y)g(\phi_\nu\phi X,Z)\}\,\xi_\nu\\&\quad + g(AY,Z)AX - g(AX,Z)AY.\end{aligned} \tag{2.8}$$

## 3. Proof of our main theorem

Let $M$ be a real hypersurface in $G_2(\mathbb{C}^{m+2})$ with commuting structure Jacobi operator, that is, $R_\xi \circ \phi = \phi \circ R_\xi$.

Now by the equation of Gauss (2.8), we define a structure Jacobi operator $R_\xi$ in such a way that

$$\begin{aligned} R_\xi(X) &= R(X,\xi)\xi \\ &= X - \eta(X)\xi - \sum_{\nu=1}^{3}\{\eta_\nu(X)\xi_\nu - \eta(X)\eta_\nu(\xi)\xi_\nu \\ &\quad + 3g(\phi_\nu X,\xi)\phi_\nu\xi + \eta_\nu(\xi)\phi_\nu\phi X\} \\ &\quad + \alpha AX - \eta(AX)A\xi. \end{aligned}$$

Then it follows that

$$\begin{aligned} R_\xi(\phi X) = \phi X &- \sum_{\nu=1}^{3}\{\eta_\nu(\phi X)\xi_\nu - 3\eta_\nu(X)\phi_\nu\xi \\ &+ 3\eta(\xi_\nu)\eta(X)\phi_\nu\xi - \eta_\nu(\xi)\phi_\nu X \\ &+ \eta_\nu(\xi)\eta(X)\phi_\nu\xi\} - \eta(A\phi X)A\xi + \alpha A\phi X, \end{aligned}$$

$$\begin{aligned} \phi R_\xi(X) = \phi X &- \sum_{\nu=1}^{3}\{\eta_\nu(X)\phi\xi_\nu \\ &- \eta(X)\eta_\nu(\xi)\phi\xi_\nu - 3\eta(\phi_\nu X)\xi_\nu + 4\eta_\nu(\xi)\eta(\phi_\nu X)\xi \\ &- \eta_\nu(\xi)\phi_\nu X + \eta_\nu(\xi)\eta(X)\phi\xi_\nu\} \\ &+ \alpha\phi AX - \eta(AX)\phi A\xi. \end{aligned}$$

Then the commuting Jacobi structure operator, $R_\xi \circ \phi = \phi \circ R_\xi$ is given by

$$\begin{aligned} &4\sum_{\nu=1}^{3}\{\eta_\nu(\phi X)\xi_\nu - \eta_\nu(X)\phi\xi_\nu + \eta(X)\eta_\nu(\xi)\phi\xi_\nu - \eta_\nu(\xi)\eta(\phi_\nu X)\xi\} \\ &= \alpha(A\phi - \phi A)X + \eta(AX)\phi A\xi - \eta(A\phi X)A\xi. \end{aligned} \tag{3.1}$$

Since we have assumed that $M$ is Hopf, by differentiating $A\xi = \alpha\xi$ (See Berndt and Suh [3]) we have

$$\begin{aligned} &\alpha(A\phi + \phi A)X - 2A\phi AX + 2\phi X \\ &= 2\sum_{\nu=1}^{3}\{-\eta_\nu(X)\phi\xi_\nu - \eta_\nu(\phi X)\xi_\nu \\ &\quad - \eta_\nu(\xi)\phi_\nu X + 2\eta_\nu(\xi)\eta(X)\phi\xi_\nu + 2\eta_\nu(\xi)\eta_\nu(\phi X)\xi\}. \end{aligned} \tag{3.2}$$

By the assumption that the structure vector $\xi$ is principal in (3.1) and (3.2), we also have

$$\begin{aligned} A\phi AX - \alpha A\phi X - \phi X = \sum_{\nu=1}^{3}&\{3\eta_\nu(X)\phi\xi_\nu - \eta_\nu(\phi X)\xi_\nu \\ &+ \eta_\nu(\xi)\phi_\nu X - 4\eta_\nu(\xi)\eta(X)\phi\xi_\nu\}. \end{aligned} \tag{3.3}$$

Now in this paper we prove the following

**Proposition 3.1.** *Let $M$ be a hypersurface in $G_2(\mathbb{C}^{m+2})$ with commuting structure Jacobi operator and non-vanishing geodesic Reeb flow. If the $\mathfrak{D}$ component of the structure vector $\xi$ is invariant by the shape operator, then the structure vector $\xi$ belongs to either the distribution $\mathfrak{D}$ or the distribution $\mathfrak{D}^\perp$.*

*Proof.* Let us put $\xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1$ for some unit $X_0 \in \mathfrak{D}$ and $\xi_1 \in \mathfrak{D}^\perp$. Since we have assumed $M$ is Hopf, $A\xi = \alpha\xi$ gives that

$$\eta(X_0)AX_0 + \eta(\xi_1)A\xi_1 = \alpha\eta(X_0)X_0 + \alpha\eta(\xi_1)\xi_1.$$

Then it follows that

$$(*) \qquad A\xi_1 = \alpha\xi_1, AX_0 = \alpha X_0 \quad \text{and} \quad \eta_2(\xi) = \eta_3(\xi) = 0.$$

Then putting $X = X_0$ into (3.1) and using $g(X_0, \phi_\nu\xi) = -g(\phi_\nu X_0, \xi) = 0$, we have

$$(3.4) \qquad 4\eta(X_0)\eta_1(\xi)\phi\xi_1 = \alpha(A\phi - \phi A)X_0 = \alpha A\phi X_0 - \alpha^2\phi X_0.$$

Also by putting $X = X_0$ into (3.3), we have

$$(3.5) \qquad \begin{aligned} & A\phi AX_0 - \alpha A\phi X_0 - \phi X_0 \\ &= \eta_1(\xi)\phi_1 X_0 - 4\eta_1(\xi)\eta(X_0)\phi\xi_1 \\ &= \eta_1(\xi)\phi_1 X_0 - 4\eta_1(\xi)\eta(X_0)\phi\xi_1 \\ &= \eta_1(\xi)\phi_1 X_0 - 4\eta_1(\xi)\eta(X_0)^2\phi_1 X_0 \\ &= \eta_1(\xi)(1 - 4\eta(X_0)^2)\phi_1 X_0. \end{aligned}$$

Then from (3.5) and $(*)$ we have

$$(3.6) \qquad \eta(\xi_1)^2 = 1 - \eta(X_0)^2 = g(\phi X_0, \phi X_0) = \eta_1(\xi)^2(1 - 4\eta(X_0)^2)^2.$$

If $\eta(\xi_1) = 0$, then by $(*)$ we know that the structure vector $\xi$ belongs to the distribution $\mathfrak{D}$. If $\eta_1(\xi) \neq 0$, then (3.6) gives $1 - 4\eta(X_0)^2 = \pm 1$. Then we consider the following two subcases.

Sub. 1) $1 - 4\eta(X_0)^2 = 1$.
In such a subcase $\eta(X_0) = 0$. Then it follows that $\xi = \eta(\xi_1)\xi_1 \in \mathfrak{D}^\perp$.

Sub. 2) $1 - 4\eta(X_0)^2 = -1$.
It follows that $\eta(X_0) = \eta(\xi_1) = \pm\frac{1}{\sqrt{2}}$. Then without loss of generality we consider that

$$\xi = \frac{1}{\sqrt{2}}X_0 + \frac{1}{\sqrt{2}}\xi_1.$$

Then (3.5) and $(*)$ give the following

$$(3.7) \qquad -\phi X_0 = -\frac{1}{\sqrt{2}}\phi_1 X_0.$$

On the other hand, (3.4) implies the following

$$(3.8) \qquad A\phi X_0 = (\alpha + \frac{2}{\alpha})\phi X_0.$$

Then by putting $X = \xi_1$ into (3.1) and using (3.7), (3.8), we have

$$
\begin{aligned}
&4\sum_{\nu=1}^{3}\{\eta_\nu(\phi\xi_1)\xi_\nu - \phi\xi_1 + \eta(\xi_1)^2\phi_1\xi - \eta_1(\xi)\eta(\phi_1\xi)\}\\
&= \alpha(A\phi - \phi A)\xi_1\\
&= \alpha(A\phi_1\xi - \phi A\xi_1)\\
&= \alpha(A\eta(X_0)\phi_1 X_0 - \alpha\eta(X_0)\phi_1 X_0)\\
&= \alpha\eta(X_0)(A\phi_1 X_0 - \alpha\phi_1 X_0)\\
&= \alpha\left\{((\alpha + \frac{2}{\alpha})\phi X_0 - \alpha\phi X_0)\right\}\\
&= 2\phi X_0.
\end{aligned} \tag{3.9}
$$

On the other hand, the left side of (3.9) becomes

$$4\sum_{\nu=1}^{3}\{\eta_\nu(\phi\xi_1)\xi_\nu - \phi\xi_1 + \eta(\xi_1)^2\phi_1\xi - \eta_1(\xi)\eta(\phi_1\xi)\} = -4\eta(X_0)^3\phi_1 X_0 = -2\phi X_0.$$

From this, together with (3.9), we have $\phi X_0 = 0$. This gives

$$X_0 = \eta(X_0)\xi = \frac{1}{\sqrt{2}}(\frac{1}{\sqrt{2}}X_0 + \frac{1}{\sqrt{2}}\xi_1).$$

This implies $X_0 = \xi_1$, which makes a contradiction. So we complete the proof of our proposition. □

## 4. Key propositions

In this section we want to give a complete proof of our main theorem. In order to do this, let us use Proposition 3.1. First we consider the case that $\xi\in\mathfrak{D}^\perp$. Accordingly, we may put $\xi = \xi_1$. Then (3.1) implies the following

**Proposition 4.1.** *Let us consider the same assumptions as in Proposition* 3.1. *If $\xi\in\mathfrak{D}^\perp$, then $M$ is congruent to a tube over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.*

*Proof.* Since we have assumed $\xi\in\mathfrak{D}^\perp$, we may put $\xi = \xi_1$. Then from (3.1) we know that

$$\alpha(A\phi - \phi A)X = 4\sum_{\nu=1}^{3}\{\eta_\nu(\phi X)\xi_\nu - \eta_\nu(X)\phi\xi_\nu - \eta_\nu(\xi)\eta(\phi_\nu X)\xi\} = 0$$

for any vector field $X$ on $M$ in $G_2(\mathbb{C}^{m+2})$. Then by a non-vanishing Reeb flow we know that the structure tensor $\phi$ commutes with the shape operator $A$ of $M$ in $G_2(\mathbb{C}^{m+2})$. This means that the Reeb flow is isometric. Then by a theorem due to Berndt and Suh [4] $M$ is locally congruent to a tube over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$ □

On the other hand, we introduce the following derived from $A\xi = \alpha\xi$ in Berndt and Suh [3]

$$\begin{aligned}
&\alpha g((A\phi + \phi A)X, Y) - 2g(A\phi AX, Y) + 2g(\phi X, Y) \\
&= 2\sum_{\nu=1}^{3} \{\eta_\nu(X)\eta_\nu(\phi Y) - \eta_\nu(Y)\eta_\nu(\phi X) \\
&\quad - g(\phi_\nu X, Y)\eta_\nu(\xi) - 2\eta(X)\eta_\nu(\phi Y)\eta_\nu(\xi) \\
&\quad + 2\eta(Y)\eta_\nu(\phi X)\eta_\nu(\xi)\}.
\end{aligned} \tag{4.1}$$

Next also by Proposition 3.1 we are able to consider the case that $\xi\in\mathfrak{D}$. Now we assert the following

**Proposition 4.2.** *Let us consider the same assumption as in Proposition* 3.1. *If* $\xi\in\mathfrak{D}$, *then* $g(A\mathfrak{D}, \mathfrak{D}^\perp) = 0$.

*Proof.* The formula (4.1) for $\xi\in\mathfrak{D}$ gives the following

$$\alpha(A\phi + \phi A) - 2A\phi AX + 2\phi X = -2\sum_{\nu=1}^{3} \{\eta_\nu(X)\phi\xi_\nu + \eta_\nu(\phi X)\xi_\nu\}. \tag{4.2}$$

Moreover, from (3.1) and $\xi\in\mathfrak{D}$ we have

$$4\sum_{\nu=1}^{3} \eta_\nu(\phi X)\xi_\nu - 4\sum_{\nu=1}^{3} \eta_\nu(X)\phi\xi_\nu = \alpha(A\phi - \phi A)X, \tag{4.3}$$

where we have used that $M$ is with geodesic Reeb flow, that is, $A\xi = \alpha\xi$. Then from (4.2) and (4.3) we have the following

**Lemma 4.3.** *For* $\xi\in\mathfrak{D}$ *we have*

$$\begin{aligned}
\alpha A\phi X - A^2\phi X + \phi X = &\sum_{\nu=1}^{3} \eta_\nu(\phi X)\xi_\nu - 3\sum_{\nu=1}^{3} \eta_\nu(X)\phi\xi_\nu \\
&- \frac{4}{\alpha}\sum_{\nu=1}^{3} \eta_\nu(\phi X)A\xi_\nu + \frac{4}{\alpha}\sum_{\nu=1}^{3} \eta_\nu(X)A\phi\xi_\nu.
\end{aligned}$$

Now let us consider the maximal distribution $\mathfrak{h}$ spanned by the orthogonal complement of the structure vector $\xi$ of $M$ in $G_2(\mathbb{C}^{m+2})$. Then by replacing $\phi X$ of $X\in\mathfrak{h}$ we have

$$\begin{aligned}
A^2X - \alpha AX - X = &-\sum_{\nu=1}^{3} \eta_\nu(X)\xi_\nu - 3\sum_{\nu=1}^{3} \eta_\nu(\phi X)\phi\xi_\nu \\
&+ \frac{4}{\alpha}\sum_{\nu=1}^{3} \eta_\nu(X)A\xi_\nu + \frac{4}{\alpha}\sum_{\nu=1}^{3} \eta_\nu(\phi X)A\phi\xi_\nu.
\end{aligned} \tag{4.4}$$

Then for any $\xi_\mu\in\mathfrak{D}^\perp$ and $\xi_\mu\in\mathfrak{h}$, we have

$$\alpha A^2\xi_\mu - (\alpha^2 + 4)A\xi_\mu = 0. \tag{4.5}$$

From this, let us take an inner product (4.4) with $X\in\mathfrak{D}$. Then it follows that

$$\begin{aligned}(4.6)\qquad &\alpha g(A^2\xi_\mu, X)\\ &= \alpha g(\xi_\mu, A^2X)\\ &= \alpha g(\xi_\mu, \alpha AX + X - \sum_{\nu=1}^3 \eta_\nu(X)\xi_\nu - 3\sum_{\nu=1}^3 \eta_\nu(\phi X)\phi\xi_\nu\\ &\quad + \frac{4}{\alpha}\sum_{\nu=1}^3 \eta_\nu(X)A\xi_\nu + \frac{4}{\alpha}\sum_{\nu=1}^3 \eta_\nu(\phi X)A\phi\xi_\nu)\\ &= \alpha^2 g(\xi_\mu, AX) + 4\sum_{\nu=1}^3 g(\xi_\mu, A\phi\xi_\nu)\eta_\nu(\phi X).\end{aligned}$$

For any $X\in\mathfrak{D}$ orthogonal to $\phi_1\xi, \phi_2\xi$ and $\phi_3\xi$ we have

$$\alpha g(A^2\xi_\mu, X) = \alpha^2 g(\xi_\mu, AX).$$

From this, together with (4.5), it follows that

$$\begin{aligned}(4.7)\qquad 0 &= \alpha g(A^2\xi_\mu, X) - (\alpha^2+4)g(A\xi_\mu, X)\\ &= -4g(A\xi_\mu, X)\end{aligned}$$

for any $X\in\mathfrak{D}$ orthogonal to $\phi_1\xi, \phi_2\xi$ and $\phi_3\xi$.

Let us put $X$ in (4.6) by $\phi_\lambda\xi\in\mathfrak{D}$, $\lambda = 1, 2, 3$. Then it follows that

$$\begin{aligned}\alpha g(A^2\xi_\mu, \phi_\lambda\xi) &= \alpha^2 g(\xi_\mu, A\phi_\lambda\xi) + 4\sum_{\nu=1}^3 g(\xi_\mu, A\phi\xi_\nu)\eta_\nu(\phi^2\xi_\lambda)\\ &= \alpha^2 g(\xi_\mu, A\phi_\lambda\xi) - 4g(\xi_\mu, A\phi\xi_\lambda).\end{aligned}$$

Comparing this one with the formula obtained from (4.5) by taking an inner product with $\phi_\lambda\xi$ gives

$$(\alpha^2+4)g(A\xi_\mu, \phi_\lambda\xi) = \alpha g(A^2\xi_\mu, \phi_\lambda\xi) = \alpha^2 g(\xi_\mu, A\phi_\lambda\xi) - 4g(\xi_\mu, A\phi_\lambda\xi).$$

From this it follows that

$$(4.8)\qquad 8g(A\xi_\mu, \phi_\lambda\xi) = 0.$$

Summing up (4.7) and (4.8), we conclude that

$$g(A\xi_\mu, X) = 0$$

for any $X\in\mathfrak{D}$. That is, $g(A\mathfrak{D}, \mathfrak{D}^\perp) = 0$. This completes the proof of our Proposition 4.2. □

By Proposition 4.2 and a theorem due to Berndt and Suh [3] we know that $M$ is congruent to a tube over a totally geodesic and totally real quaternionic projective space $QP^n$, $n = 2m$, in $G_2(\mathbb{C}^{m+2})$.

It remains to check whether such kind of hypersurfaces satisfy *commuting structure Jacobi operator* or not.

Let us recall $R_\xi\circ\phi = \phi\circ R_\xi$ for $\xi\in\mathfrak{D}$ and $\xi$ is principal. Then we have

$$(4.9)\qquad \alpha(A\phi - \phi A)X = 4\sum_{\nu=1}^3 \{\eta_\nu(\phi X)\xi_\nu - \eta_\nu(X)\phi\xi_\nu\}.$$

We introduce a proposition due to Berndt and Suh [3] as follows:

**Proposition B.** *Let $M$ be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $A\mathfrak{D} \subset \mathfrak{D}$, $A\xi = \alpha\xi$, and $\xi$ is tangent to $\mathfrak{D}$. Then the quaternionic dimension $m$ of $G_2(\mathbb{C}^{m+2})$ is even, say $m = 2n$, and $M$ has five distinct constant principal curvatures*

$$\alpha = -2\tan(2r) \ , \ \beta = 2\cot(2r) \ , \ \gamma = 0 \ , \ \lambda = \cot(r) \ , \ \mu = -\tan(r)$$

*with some $r \in (0, \pi/4)$. The corresponding multiplicities are*

$$m(\alpha) = 1 \ , \ m(\beta) = 3 = m(\gamma) \ , \ m(\lambda) = 4n - 4 = m(\mu)$$

*and the corresponding eigenspaces are*

$$T_\alpha = \mathbb{R}\xi \ , \ T_\beta = \mathfrak{J}J\xi \ , \ T_\gamma = \mathfrak{J}\xi \ , \ T_\lambda \ , \ T_\mu \ ,$$

*where*

$$T_\lambda \oplus T_\mu = (\mathbb{HC}\xi)^\perp \ , \ \mathfrak{J}T_\lambda = T_\lambda \ , \ \mathfrak{J}T_\mu = T_\mu \ , \ JT_\lambda = T_\mu \ .$$

In Proposition B we consider a vector $X \in T_\lambda$ such that $AX = \lambda X = \cot rX$. From this we have

$$\alpha(A\phi X - \phi AX) = \alpha(\mu - \lambda)\phi X = 0.$$

This means $\cot^2 r + 1 = 0$, which makes a contradiction. So a real hypersurface $M$ in $G_2(\mathbb{C}^{m+2})$ with commuting structure Jacobi operator does not exist. From this we give a complete proof of our theorem in the introduction.

*Remark* 4.1. A tube over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$ in Theorem A has commuting shape operator on the distributions $\mathfrak{D}$ and $\mathfrak{D}^\perp$. Of course, it is Hopf and naturally in Section 3 we have asserted that such a hypersurface satisfy $R_\xi \circ \phi = \phi \circ R_\xi$.

*Remark* 4.2. A tube over a totally real totally geodesic $\mathbb{Q}P^n$ in $G_2(\mathbb{C}^{m+2})$ has not commuting shape operator on the distributions $\mathfrak{D}$ and $\mathfrak{D}^\perp$. In Section 4 we have proved that such a hypersurface is Hopf but can not satisfy $R_\xi \circ \phi = \phi \circ R_\xi$.

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