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SOME RESULTS RELATED TO DISTRIBUTION FUNCTIONS OF CHI-SQUARE TYPE RANDOM VARIABLES WITH RANDOM DEGREES OF FREEDOM

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Abstract. The main aim of this paper is to present some results related to asymptotic behavior of distribution functions of random variables of chi-square type $\chi_N^2 = \sum_{i=1}^N X_i^2$ with degrees of freedom N, where N is a positive integer-valued random variable independent on all standard normally distributed random variables X_i. Two ways for computing the distribution functions of chi-square type random variables with random degrees of freedom are considered. Moreover, some tables concerning considered distribution functions are demonstrated in Appendix.

1. Introduction

Let $X_1, X_2, \ldots, X_n$ be n independent identically distributed (i.i.d.) random variables of standard normal law $\mathcal{N}(0,1)$. The sum $X_1^2 + X_2^2 + \cdots + X_n^2$ is said to be a chi-square random variable of n degrees of freedom, denoted by χ_n^2.

The density function of random variable χ_n^2 is defined by (we refer the reader to [2] and [3].)

$$f_n(x) = \begin{cases} 0 & \text{if} \quad x \leq 0, \\ \frac{x^{\frac{n}{2}-1}}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})} e^{-\frac{x}{2}} & \text{if} \quad x > 0, \end{cases} \tag{1}$$

where $\Gamma(s) = \int_0^\infty e^{-x}x^{s-1}dx$ denotes the Gamma function.

It has long been known that in probability theory and statistics, the chi-square distribution of random variable χ_n^2 (also chi-squared or χ^2-distribution) is one of the most widely used theoretical probability distributions in inferential statistics, for instance in chi-square tests and in estimating variances. This distribution enters the problem of estimating the mean of a normally distributed population and the problem of estimating the slope of a regression line via its

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role in Student's t-distribution. Moreover, it also enters all analysis of variance problems via its role in the F-distribution, which is the distribution of the ratio of two independent chi-square random variables divided by their respective degrees of freedom (see [2] and [3] for more details).

During the last several decades the random-index approach has risen to become one of the most important tools available for investigating some problems of Applied Statistics (for a deeper discussion of this we refer the reader to [4] and [8]).

The question arises as to what happens if the degrees of freedom n of a chi-square random variable χ_n^2 shall be replayed by a positive integer-valued random variable N. The answer of above question is main aim of this paper. The applications of probability distributions of the random variable of chi-square type χ_N^2 with random degrees of freedom N in some applied problems of statistics will be taken up in the next research results.

This paper is divided into five main sections. The second section deals with the asymptotic behaviors of distributions of chi-square random variable χ_n^2 (Theorem 2.1) and of chi-square type χ_N^2 with random degrees of freedom N (Theorems 2.2-2.8). The third section gives the proofs of all results in second section. The fourth section describes two approaches to computation of the distribution functions of the random variable of chi-square type χ_N^2 with random degrees of freedom N. The Appendix (last section) devotes to some tables concerning the distribution functions of chi-square type χ_N^2 with random degrees of freedom N in some concrete cases.

2. Main results

Throughout this paper, we denote by Z a random variable degenerated at point 1, and by $\varphi_Z(t) = e^{it}$ its characteristic function.

From now on, the notation $\xrightarrow{d}$ will mean the convergence in distribution and $\xrightarrow{\mathbb{P}}$ will denote the convergence in probability.

Theorem 2.1. *Let χ_n^2 be a chi-square random variable of n degrees of freedom (n is positive integer number). Then*

$$\frac{\chi_n^2}{n} \xrightarrow{d} Z,$$

as $n \to \infty$.

Theorem 2.2. *Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables with chi-square distribution function χ_n^2. Suppose that N is a positive integer-valued random variable independent of all X_i. Furthermore, let us consider the random sum $S_N := \sum_{i=1}^N X_i$. Then*

$$\frac{S_N}{n} \xrightarrow{d} N,$$

as $n \to \infty$.

We now return to some interesting results concerning to random variable $\chi^2_{N_n} := X_1^2 + \cdots + X^2_{N_n}$, if for the random variable χ^2_n with degrees of freedom n, the fixed number n will be replaced by the positive integer-valued random variables $N_n, n \geq 1$, independent of all $X_i \sim \mathcal{N}(0,1), i = 1, 2, \ldots, N_n$. The following theorems will be main results of this paper.

Theorem 2.3. *Let* $N_n \sim \text{Binomial}(n, p)$, $p \in (0,1)$. *Then*

$$\frac{\chi^2_{N_n}}{np} \xrightarrow{d} Z,$$

as $n \to \infty$.

Theorem 2.4. *Let* $N_n \sim \text{Poisson}(\lambda_n)$, $\lambda_n > 0$, $n = 1, 2, \ldots$ *and* $\lambda_n \to \infty$ *as* $n \to \infty$. *Then*

$$\frac{\chi^2_{N_n}}{\lambda_n} \xrightarrow{d} Z,$$

as $n \to \infty$.

Theorem 2.5. *Let* $\{N_n, n \geq 1\}$ *be a sequence of positive integer-valued random variables, independent from all* $X_i \sim \mathcal{N}(0,1), i = 1, 2, \ldots$. *Furthermore, assume that the following conditions are satisfied*

$$E(N_n) \to \infty; \quad \frac{E|N_n - E(N_n)|}{E(N_n)} \to 0 \quad as \quad n \to \infty. \tag{2}$$

Then, as $n \to \infty$ *we get*

$$\frac{\chi^2_{N_n}}{E(N_n)} \xrightarrow{d} Z. \tag{3}$$

Remark 1. (a) Theorem 2.5 is generalization of Theorems 2.1, 2.3, and 2.4.

(b) It is easily seen that $E(\chi^2(1)) = 1$. Thus, as a fundamental result from random sum, it follows $E(\chi^2_{N_n}) = E(\chi^2(1))E(N_n)$. Then, (3) in Theorem 2.5 can be formulated as follows

$$\frac{\chi^2_{N_n}}{E(\chi^2_{N_n})} \xrightarrow{d} Z, \quad \text{as} \quad n \to \infty. \tag{4}$$

Theorem 2.6. *Let* $N_n \sim \text{Uniform}(n)$. *Then, as* $n \to \infty$, *we have*

$$\frac{\chi^2_{N_n}}{n} \xrightarrow{d} U \sim \text{Uniform}[0,1].$$

Theorem 2.7. *Let* $N_n \sim \text{Geometric}(p_n)$ *and suppose that* $p_n \to 0$ *as* $n \to \infty$. *Then*

$$p_n \chi^2_{N_n} \xrightarrow{d} Y \sim \text{Exp}(1),$$

as $n \to \infty$.

Remark 2. According to Theorems 2.6 and 2.7 we can see the condition $E(N_n) \to \infty$ in (2) (as $n \to \infty$) is not sufficient to confirm conclusion as in Remark 1.b.

Theorem 2.8. *Let $\{N_n, n \geq 1\}$ be a sequence of positive integer-valued random variables, independent from all $X_i \sim \mathcal{N}(0,1), i = 1, 2, \ldots$. Moreover, if*

$$\frac{N_n}{n} \xrightarrow{\mathbb{P}} 1, \quad n \to \infty \tag{5}$$

then, as $n \to \infty$, we get

$$\frac{\chi^2_{N_n}}{n} \xrightarrow{d} Z.$$

Remark 3. (a) The condition (5) in Theorem 2.8 is Feller's condition (see [1]).

(b) According to the results of paper [7], we will be right in a conjecture: if random variables N_n have asymptotic normal distribution function, then the $\chi^2_{N_n}$ has asymptotic generate distribution function at point one.

3. Proofs

Proof of Theorem 2.1. It is easily seen that, the characteristic function of the χ^2_n is defined by

$$\varphi(t) = \frac{1}{(1-2it)^{\frac{n}{2}}} = \Big(\frac{1+2it}{1+4t^2}\Big)^{\frac{n}{2}} = \Big[1 + \frac{2it - 4t^2}{1+4t^2}\Big]^{\frac{n}{2}}.$$

Then

$$\varphi_{\frac{\chi^2_n}{n}}(t) = \varphi(t/n) = \Big[1 + \frac{2}{n}\Big(\frac{int - 2t^2}{n + 4t^2/n}\Big)\Big]^{\frac{n}{2}}.$$

By letting $n \to \infty$, we have $\varphi_{\frac{\chi^2_n}{n}}(t) \to e^{it}$. It follows the proof of theorem. □

Proof of Theorem 2.2. Denote by g the generating function of N, and by φ the characteristic function of χ^2_n. According to proof of Theorem 2.1 we obtained $\varphi(t/n) \to e^{it}$, as $n \to \infty$. Then, characteristic function of $\frac{S_N}{n}$ will be defined by

$$\varphi_{\frac{S_N}{n}}(t) = g(\varphi(t/n)) \to g(e^{it}) = E(e^{itN}), \quad \text{as} \quad n \to \infty.$$

It completes the proof of theorem. □

Proof of Theorem 2.3. Under assumption of theorem, it is easily seen that the random variable N_n has generating function $g(t) = [1 + p(t-1)]^n$ and the random variable X^2_k has characteristic function $\varphi(t) = \frac{1}{\sqrt{1-2it}}$. Then, the characteristic function of $\frac{\chi^2_{N_n}}{np}$ is given by

$$\begin{aligned}
\varphi_{\frac{\chi^2_{N_n}}{np}} &= g(\varphi(t/np)) = [1 + p(\varphi(t/np) - 1)]^n \\
&= \Big[1 + p\Big(\frac{\sqrt{np}}{\sqrt{np - 2it}} - 1\Big)\Big]^n = \Big[1 + \frac{1}{n}\frac{2itnp}{(\sqrt{np} + \sqrt{np-2it})\sqrt{np-2it}}\Big]^n \\
&= \Big[1 + \frac{1}{n}\frac{2it}{(1 + \sqrt{1 - 2it/np})\sqrt{1 - 2it/np}}\Big]^n.
\end{aligned}$$

By letting $n \to \infty$, then $\varphi_{\frac{\chi^2_{N_n}}{np}} \to e^{it}$. It follows the proof. □

Proof of Theorem 2.4. Obviously, from the assumption of theorem, we can check that the random variable N_n has generating function $g(t) = e^{\lambda_n(t-1)}$ and the random variable X_k^2 has characteristic function $\varphi(t) = \frac{1}{\sqrt{1-2it}}$. Then, characteristic function of $\chi^2_{N_n}/\lambda_n$ is given by

$$\begin{aligned}\varphi_{\frac{\chi^2_{N_n}}{\lambda_n}} &= g(\varphi(t/\lambda_n)) = e^{\lambda_n[\varphi(t/\lambda_n)-1]} \\ &= e^{\lambda_n\left[\frac{\sqrt{\lambda_n}-\sqrt{\lambda_n-2it}}{\sqrt{\lambda_n-2it}}\right]} = e^{\frac{2it}{(1+\sqrt{1-2it/\lambda_n})\sqrt{1-2it/\lambda_n}}}.\end{aligned}$$

By letting $n \to \infty$, we obtain $\varphi_{\frac{\chi^2_{N_n}}{\lambda_n}} \to e^{it}$. This completes the proof. □

Proof of Theorem 2.5. Put $a_n = E(N_n)$ and $p_k = P(N_n = k)$. It is easily seen that the generating function of random variable N_n is $g(t) = E(t^{N_n})$. Because of all random variables $X_k, k = 1, 2, \ldots, n$ belong to standard normal law, the characteristic function of random variables $X_k^2, k = 1, 2, \ldots n$, is given by $\varphi(t) = \frac{1}{\sqrt{1-2it}}$. Then, the characteristic function ψ_n of $\frac{\chi^2_{N_n}}{a_n}$ will be given by

$$\begin{aligned}\psi_n(t) &= g(\varphi(t/a_n)) = \sum_{k=0}^{\infty} p_k \varphi^k(t/a_n) = \sum_{k=0}^{\infty} p_k (1 - 2it/a_n)^{-k/2} \\ &= \sum_{k=0}^{\infty} p_k \Big[1 + \frac{2}{a_n}\Big(\frac{it - 2t^2/a_n}{1 + 4t^2/a_n^2}\Big)\Big]^{k/2}.\end{aligned}$$

Putting

$$\delta_n = \Big[1 + \frac{2}{a_n}\Big(\frac{it - 2t^2/a_n}{1 + 4t^2/a_n^2}\Big)\Big]^{a_n/2}.$$

We have

$$|\delta_n| \le 1; \qquad \delta_n \to e^{it} \quad \text{as} \quad n \to \infty. \tag{6}$$

It is easily seen that

$$|\psi_n(t) - \delta_n| = \Big|\sum_{k=0}^{\infty} p_k \big[\delta_n^{k/a_n} - \delta_n\big]\Big| \le \sum_{k=0}^{\infty} p_k \big|\delta_n^{k/a_n} - \delta_n\big|.$$

Let us consider the continuous function $h(x) = \delta_n^x$ on $[k/a_n, 1]$ or $[1, k/a_n]$. According to Lagrange's Theorem and (6), we can obtain

$$\begin{aligned}\big|\delta_n^{k/a_n} - \delta_n\big| &= |h(k/a_n) - h(1)| = |k/a_n - 1||h'(c)| \quad (c > 0) \\ &= |k/a_n - 1||\ln\delta_n||\delta_n|^c \le |k/a_n - 1||\ln\delta_n|.\end{aligned}$$

Thus, our task is to estimate

$$|\psi_n(t) - \delta_n| \le \sum_{k=0}^{\infty} p_k |\ln\delta_n| \frac{|k - a_n|}{a_n} = |\ln\delta_n| \frac{E|N_n - a_n|}{a_n}.$$

From this we deduce that

$$|\psi_n(t) - e^{it}| \leq |\psi_n(t) - \delta_n| + |\delta_n - e^{it}| \leq |\ln \delta_n| \frac{E|N_n - a_n|}{a_n} + |\delta_n - e^{it}|.$$

By virtue of the condition (2) and from results in (6), if $n \to \infty$, we can get $|\psi_n(t) - e^{it}| \to 0$. The proof is complete. □

Proof of Theorem 2.6. Evidently, the generating function of random variables N_n is $g(t) = \frac{t(t^n-1)}{n(t-1)}$ and the characteristic function of random variables $X_k^2, k = 1, 2, \ldots, n$ is $\varphi(t) = \frac{1}{\sqrt{1-2it}}$. Then, the characteristic function of random variable $\chi^2_{N_n}/n$ is given by

$$\begin{aligned}\varphi_{\frac{\chi^2_{N_n}}{n}}(t) &= g(\varphi(t/n)) = \frac{\varphi(t/n)[\varphi^n(t/n) - 1]}{n[\varphi(t/n) - 1]} \\ &= \frac{\frac{1}{\sqrt{1-2it/n}}\left[\frac{1}{(1-2it/n)^{\frac{n}{2}}} - 1\right]}{n\left[\frac{\sqrt{n}-\sqrt{n-2it}}{\sqrt{n-2it}}\right]} = \frac{(1 + \sqrt{1-2it/n})\left[\frac{1}{(1-2it/n)^{\frac{n}{2}}} - 1\right]}{2it}.\end{aligned}$$

In the proof of Theorem 2.1, we have

$$\frac{1}{(1-2it/n)^{\frac{n}{2}}} \to e^{it} \quad \text{as} \quad n \to \infty.$$

By letting $n \to \infty$, we can conclude that

$$\varphi_{\frac{\chi^2_{N_n}}{n}}(t) \to \frac{e^{it} - 1}{it} = \frac{e^{it1} - e^{it0}}{(1-0)it}.$$

This finishes the proof. □

Proof of Theorem 2.7. Obviously, the generating functions of random variables N_n is $g(t) = \frac{p_n t}{1-(1-p_n)t}$ and the characteristic function of random variables $X_k^2, k = 1, 2, \ldots, n$ is $\varphi(t) = \frac{1}{\sqrt{1-2it}}$. Then, the characteristic function of $p_n.\chi^2_{N_n}$ is defined by

$$\begin{aligned}\varphi_{p_n \cdot \chi^2_{N_n}}(t) &= g(\varphi(p_n t)) = \frac{p_n \varphi(p_n t)}{1 - (1-p_n)\varphi(p_n t)} = \frac{p_n}{\sqrt{1-2ip_n t} - (1-p_n)} \\ &= \frac{p_n[\sqrt{1-2ip_n t} + (1-p_n)]}{1 - 2ip_n t - (1-p_n)^2} = \frac{\sqrt{1-2ip_n t} + 1 - p_n}{2 - 2it - p_n}.\end{aligned}$$

By letting $n \to \infty$, we can assert that

$$\varphi_{p_n \cdot \chi^2_{N_n}}(t) \to \frac{1}{1-it}.$$

The proof is straightforward. □

Proof of Theorem 2.8. According to assumptions of theorem and from Theorem 2.1, we have

$$N_n \xrightarrow{d} \infty \quad \text{and} \quad \frac{\chi^2_n}{n} \xrightarrow{d} Z$$

as $n \to \infty$. It is easily seen that

$$\frac{\chi^2_{N_n}}{N_n} \xrightarrow{d} Z.$$

Because of the random variable Z is generated at point one, we conclude

$$\frac{\chi^2_{N_n}}{N_n} \xrightarrow{P} 1, \quad \text{as} \quad n \to \infty.$$

Then, we can see that

$$\frac{\chi^2_{N_n}}{n} = \frac{\chi^2_{N_n}}{N_n} \cdot \frac{N_n}{n} \xrightarrow{P} 1, \quad \text{as} \quad n \to \infty.$$

Thus, the proof is complete

$$\frac{\chi^2_{N_n}}{n} \xrightarrow{d} Z, \quad \text{as} \quad n \to \infty.$$

□

4. Two approaches to computation of distribution functions of chi-square type χ^2_N with random degrees of freedom N.

Firstly, using results of Lebedev in [6], we can construct an algorithm to find the values of distribution functions of chi-square type with random degrees of freedom $\chi^2_N(x)$.

Theorem 4.1 (We recall from [6]). *Denote by $\chi^2_n(x)$ a distribution function of chi-square random variable χ^2_n with n degrees of freedom. Then, for every $n = 1, 2, \ldots$,*

$$\chi^2_{n+2}(x) = \chi^2_n(x) - \delta_n \frac{x^{n/2}}{n!!} e^{-x/2},$$

with $\chi^2_1(x) = 2\Phi(\sqrt{x}) - 1, \quad \chi^2_2(x) = 1 - e^{-x/2}$,
where

$$\delta_n = \begin{cases} 1, & n = 2k \\ \sqrt{\frac{2}{\pi}}, & n = 2k+1 \end{cases}$$

and

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} x^{-t/2} dt.$$

Corollary 4.2. *If $n = 2m$, $m = 1, 2, \ldots$, then*

$$\chi^2_{2m}(x) = 1 - e^{-x/2} \sum_{p=0}^{m-1} \frac{x/2^p}{p!}.$$

If $n = 2m + 1$, $m = 0, 1, 2, \ldots$, then

$$\chi^2_{2m+1}(x) = 2\Phi(\sqrt{x}) - 1 - \sqrt{\frac{2}{\pi}} e^{-x/2} \sum_{p=1}^{m} \frac{x^{p-1/2}}{(2p-1)!!}.$$

Theorem 4.3. *Let N be a positive integer-valued random variable with probability distribution $P(N = k) = p_k$. Assume that N is independent from all $X_i \sim \mathcal{N}(0,1)$, $i = 1, 2, \ldots$. Put $\chi_n^2 = \sum_{i=1}^n {X_i}^2$ and $\chi_N^2 = \sum_{i=1}^N {X_i}^2$. Denote by $\chi_n^2(x)$ and $F_N(x)$ the distribution functions of random variables χ_n^2 and χ_N^2. Then*

$$F_N(x) = \sum_{k=1}^{\infty} \chi_k^2(x) p_k.$$

Proof. By virtue of formula of total probability and independence of N for all X_i, $i = 1, 2, \ldots$, we have

$$\begin{aligned} F_N(x) &= P(\chi_N^2 \le x) = \sum_{k=1}^{\infty} P(\chi_N^2 \le x | N = k) P(N = k) \\ &= \sum_{k=1}^{\infty} P(\chi_k^2 \le x) P(N = k) = \sum_{k=1}^{\infty} \chi_n^2(x) p_k. \end{aligned}$$

The proof is straightforward. □

An algorithm for computing of distribution function $\chi_N^2(x)$ with random degrees of freedom

(1) Construct distribution function $\chi_n^2(x) := P(\chi_n^2 < x)$ (based on result from Corollary 4.2).
(2) Compute $P(N = k) = p_k, k = 1, 2, \ldots$ of discrete random variable N.
(3) Compute distribution function of random variable χ_N^2 (based on results from Theorem 4.3).

According to above algorithm, by using Maple, we can get computations of distribution functions of random variable χ_N^2, where N is a positive integer-valued random variable. The received results from algorithm will be given in Appendix.

From now we can show another way for approaching to distribution functions of χ_N^2 by directly computing its density function. Here, we compute the density function of χ_N^2, where N be a random variable from negative binomial law, i.e.,

$$P(N = k) = C_{k-1}^{r-1} p^r q^{k-r} \qquad (p + q = 1, k = r, r + 1, \ldots). \tag{7}$$

Let us consider the series

$$\varphi(x) = \sum_{k=r}^{\infty} C_{k-1}^{r-1} \frac{x^n}{\Gamma(\frac{k}{2})} \tag{8}$$

with convergent domain $\mathbb{R}$. Let $f_k(x)$ be a density function of χ_k^2 with k degrees of freedom (see (1)). Then, by virtue of (7), it is easily seen that the random sum χ_N^2 will have density function $f(x), x \geq 0$, denoted by

$$f(x) = \sum_{k=r}^{\infty} \frac{x^{\frac{k}{2}-1}e^{-\frac{x}{2}}}{2^{\frac{k}{2}}\Gamma(\frac{k}{2})} C_{k-1}^{r-1} p^r q^{k-r} = x^{-1}e^{-\frac{x}{2}}(p/q)^r \sum_{k=r}^{\infty} C_{k-1}^{r-1} \frac{[q\sqrt{x/2}]^k}{\Gamma(\frac{k}{2})}$$
$$= x^{-1}e^{-\frac{x}{2}}(p/q)^r \varphi(q\sqrt{x/2}).$$

Thus, we firstly must compute the series $\varphi(x)$ in (8). Let us consider the series

$$\phi_n(x) = \sum_{k=1}^{\infty} \frac{k^n x^k}{\Gamma(\frac{k}{2})} \qquad n = 0, 1, 2, \ldots$$

with following properties:

i. $\phi_0(x) = \dfrac{x}{\sqrt{\pi}} + 2x^2 e^{x^2} \Phi(x\sqrt{2})$, where $\Phi(x)$ is Laplace's function.

ii. $\phi_{n+1}(x) = x \dfrac{\partial}{\partial x} \phi_n(x)$.

By virtue of two above properties and by using Maple, we can get the series $\phi_n(x)$. For example

$$\begin{aligned}
\phi_1(x) &= \frac{x(1+2x^2)}{\sqrt{\pi}} + 4x^2(1+x^2)e^{x^2}\Phi(x\sqrt{2}), \\
\phi_2(x) &= \frac{x(1+10x^2+4x^4)}{\sqrt{\pi}} + 8x^2(1+3x^2+x^4)e^{x^2}\Phi(x\sqrt{2}), \\
&\vdots
\end{aligned}$$

Thus, if $r = 1$ then $\varphi(x) = \phi_0(x)$. And if $r \geq 2$, then

$$\varphi(x) = \frac{1}{(r-1)!} \sum_{k=r}^{\infty} (k-r+1)(k-r+2)\cdots(k-1) \frac{x^k}{\Gamma(\frac{k}{2})}.$$

And we can demonstrate $\varphi(x)$ through the series $\phi_n(x)$. For example

$$\begin{aligned}
r &= 2, & \varphi(x) &= \phi_1(x) - \phi_0(x) \\
r &= 3, & \varphi(x) &= \frac{1}{2!}[\phi_2(x) - 3\phi_1(x) + 2\phi_0(x)] \\
r &= 4, & \varphi(x) &= \frac{1}{3!}[\phi_3(x) - 6\phi_2(x) + 11\phi_1(x) - 6\phi_0(x)] \\
&\vdots
\end{aligned}$$

The values of distribution function $F(x) = \int_0^x f(t)dt$ in some cases will be illustrated in Appendix.

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Appendix

In this section, three tables concerning the distribution functions of random variable χ^2_N with random degrees of freedom N are established.

Table 1. Distribution function of chi square-geometry with parameter $p = 1/2$.

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.00000	03999	05651	06925	08006	08959	09818	10608	11354	12049
0.1	0.12716	13349	13952	14525	15089	15632	16157	16663	17157	17643
0.2	0.18109	18572	19021	19460	19890	20317	20733	21140	21538	21937
0.3	0.22323	22712	23081	23461	23827	24187	24543	24897	25245	25590
0.4	0.25930	26267	26596	26927	27251	27572	27895	28211	28520	28829
0.5	0.29141	29441	29745	30041	30338	30635	30922	31213	31503	31786
0.6	0.32066	32348	32625	32901	33178	33447	33716	33985	34253	34512
0.7	0.34777	35039	35297	35552	35809	36060	36318	36569	36815	37059
0.8	0.37309	37555	37797	38042	38283	38517	38753	38992	39232	39459
0.9	0.39694	39922	40152	40378	40609	40840	41061	41287	41506	41730
1.0	0.41949	42167	42390	42600	42814	43032	43247	43461	43671	43882
1.1	0.44088	44295	44507	44715	44922	45127	45327	45528	45728	45935
1.2	0.46127	46329	46526	46726	46925	47115	47312	47508	47705	47892
1.3	0.48082	48278	48462	48648	48837	49027	49209	49400	49577	49768
1.4	0.49948	50131	50312	50488	50673	50856	51031	51210	51391	51560
1.5	0.51729	51910	52086	52259	52432	52605	52775	52947	53113	53288
1.6	0.53451	53623	53788	53958	54119	54282	54450	54607	54778	54942
1.7	0.55100	55267	55429	55582	55742	55901	56058	56217	56373	56538
1.8	0.56685	56845	56995	57152	57308	57457	57611	57760	57910	58066
1.9	0.58217	58357	58503	58657	58802	58952	59102	59242	59385	59537
2.0	0.59683	59821	59967	60104	60253	60397	60533	60675	60817	60955
2.1	0.61090	61232	61373	61511	61648	61779	61919	62047	62187	62322
2.2	0.62454	62595	62721	62859	62990	63122	63243	63378	63513	63642
2.3	0.63774	63900	64031	64158	64282	64413	64537	64660	64787	64916
2.4	0.65035	65161	65287	65405	65533	65651	65771	65899	66017	66135
2.5	0.66255	66375	66498	66617	66734	66857	66972	67088	67202	67320
2.6	0.67439	67554	67662	67787	67894	68007	68125	68238	68352	68461
2.7	0.68579	68678	68792	68905	69019	69126	69239	69339	69453	69558
2.8	0.69668	69782	69887	69992	70099	70206	70307	70415	70521	70623
2.9	0.70732	70834	70940	71040	71143	71247	71350	71451	71553	71652
3.0	0.71755	71851	71949	72052	72151	72248	72342	72451	72547	72641
3.1	0.72736	72833	72930	73027	73123	73216	73310	73409	73501	73602
3.2	0.73686	73783	73869	73973	74067	74153	74243	74339	74428	74516
3.3	0.74609	74699	74787	74873	74968	75057	75149	75233	75318	75411
3.4	0.75496	75582	75672	75757	75848	75926	76012	76101	76187	76266
3.5	0.76352	76443	76520	76605	76684	76772	76849	76936	77015	77102
3.6	0.77176	77257	77344	77426	77503	77584	77665	77747	77824	77898
3.7	0.77978	78057	78134	78210	78286	78365	78442	78523	78595	78673
3.8	0.78751	78834	78894	78974	79050	79129	79193	79272	79353	79417
3.9	0.79497	79573	79639	79715	79784	79863	79929	80002	80069	80137
4.0	0.80215	80284	80357	80423	80492	80570	80632	80698	80772	80841
4.1	0.80906	80978	81045	81110	81173	81244	81308	81384	81438	81508

4.2	0.81579	81642	81704	81774	81838	81900	81961	82038	82100	82161
4.3	0.82220	82287	82344	82417	82476	82540	82601	82663	82718	82785
4.4	0.82851	82905	82968	83032	83090	83155	83213	83276	83331	83392
4.5	0.83449	83515	83569	83629	83683	83750	83809	83856	83917	83978
4.7	0.84030	84095	84148	84198	84260	84318	84376	84425	84490	84542
4.8	0.84597	84647	84709	84762	84819	84867	84923	84971	85028	85082
4.9	0.85134	85194	85249	85303	85353	85406	85454	85513	85561	85613
5.0	0.85666	85719	85763	85811	85866	85917	85974	86020	86065	86116
5.1	0.86164	86210	86262	86311	86369	86410	86458	86509	86561	86609
5.2	0.86651	86700	86756	86798	86841	86892	86937	86987	87034	87074
5.3	0.87118	87178	87212	87266	87307	87351	87398	87445	87493	87531
5.4	0.87584	87623	87671	87707	87760	87800	87845	87890	87935	87971
5.5	0.88016	88061	88106	88149	88185	88234	88273	88324	88361	88395
5.6	0.88447	88482	88532	88567	88609	88650	88694	88728	88770	88811
5.7	0.88850	88893	88935	88976	89013	89052	89092	89131	89166	89203
5.8	0.89252	89292	89327	89363	89396	89435	89483	89513	89552	89590
5.9	0.89620	89660	89706	89745	89777	89816	89851	89888	89922	89956
6.0	0.89992	90030	90080	90112	90141	90178	90214	90248	90278	90314
6.1	0.90347	90386	90421	90456	90494	90524	90555	90591	90631	90660
6.2	0.90694	90729	90764	90798	90832	90858	90894	90927	90960	90992
6.3	0.91020	91059	91086	91126	91155	91179	91210	91249	91283	91314
6.4	0.91346	91378	91414	91434	91466	91499	91532	91562	91593	91616
6.5	0.91651	91685	91710	91739	91772	91802	91832	91861	91886	91913
6.6	0.91947	91981	92004	92033	92063	92095	92112	92150	92180	92209
6.7	0.92230	92259	92288	92316	92347	92377	92402	92429	92459	92486
6.8	0.92506	92541	92562	92589	92616	92644	92673	92699	92726	92752
6.9	0.92774	92806	92826	92859	92880	92912	92932	92958	92984	93017
7	0.93284	93525	93757	93982	94188	94405	94602	94791	94983	95158
8	0.95340	95505	95662	95812	95965	96119	96247	96382	96510	96633
9	0.96758	96875	96990	97091	97197	97305	97399	97492	97584	97664
10	0.97766	97839	97920	97988	98057	98130	98199	98265	98323	98389
11	0.98448	98496	98555	98605	98654	98707	98754	98796	98844	98883
12	0.98932	98963	99001	99030	99074	99107	99132	99171	99200	99230
13	0.99254	99279	99311	99336	99355	99380	99401	99419	99451	99467
14	0.99488	99500	99520	99536	99553	99570	99586	99606	99616	99636
15	0.99645	99655	99669	99678	99690	99704	99714	99722	99732	99743
16	0.99756	99760	99768	99777	99785	99800	99807	99816	99818	99830
17	0.99831	99840	99842	99845	99851	99860	99863	99867	99871	99875
18	0.99881	99884	99888	99893	99896	99902	99903	99906	99912	99912
19	0.99917	99918	99924	99924	99927	99932	99933	99934	99935	99937
20	0.99948	99942	99944	99946	99957	99950	99951	99953	99955	99956

Table 2. Distribution function of chi square-negative binomial with parameter $(r = 1, p = 1/3)$.

Density function is

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ pqe^{-x(1-q^2)/2}\Phi(q\sqrt{x}) + \frac{pe^{-x/2}}{\sqrt{2\pi x}} & \text{if } x > 0 \end{cases}$$

x	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0	0	0.0949	0.1401	0.1768	0.2089	0.2379	0.2647	0.2897	0.3132	0.3355
1	0.3566	0.3768	0.3961	0.4146	0.4324	0.4494	0.4659	0.4817	0.4970	0.5117
2	0.5260	0.5397	0.5531	0.5660	0.5784	0.5905	0.6022	0.6135	0.6245	0.6352
3	0.6455	0.6555	0.6652	0.6747	0.6838	0.6927	0.7013	0.7097	0.7178	0.7257
4	0.7334	0.7408	0.7480	0.7550	0.7619	0.7685	0.7749	0.7812	0.7872	0.7931
5	0.7989	0.8045	0.8099	0.8151	0.8202	0.8252	0.8300	0.8347	0.8393	0.8437
6	0.8481	0.8523	0.8563	0.8603	0.8641	0.8679	0.8715	0.8751	0.8785	0.8819
7	0.8851	0.8883	0.8913	0.8943	0.8972	0.9001	0.9028	0.9055	0.9081	0.9106
8	0.9131	0.9155	0.9178	0.9200	0.9222	0.9244	0.9265	0.9285	0.9304	0.9323
9	0.9342	0.9360	0.9378	0.9395	0.9411	0.9428	0.9443	0.9459	0.9473	0.9488
10	0.9502	0.9516	0.9529	0.9542	0.9554	0.9567	0.9578	0.9590	0.9601	0.9612
11	0.9623	0.9633	0.9643	0.9653	0.9663	0.9672	0.9681	0.9690	0.9698	0.9706
12	0.9714	0.9722	0.9730	0.9737	0.9744	0.9751	0.9758	0.9765	0.9771	0.9778
13	0.9784	0.9790	0.9795	0.9801	0.9806	0.9812	0.9817	0.9822	0.9827	0.9832
14	0.9836	0.9841	0.9845	0.9849	0.9853	0.9857	0.9861	0.9865	0.9869	0.9872
15	0.9876	0.9879	0.9883	0.9886	0.9889	0.9892	0.9895	0.9898	0.9901	0.9903
16	0.9906	0.9909	0.9911	0.9914	0.9916	0.9918	0.9920	0.9923	0.9925	0.9927
17	0.9929	0.9931	0.9933	0.9935	0.9936	0.9938	0.9940	0.9941	0.9943	0.9945
18	0.9946	0.9948	0.9949	0.9950	0.9952	0.9953	0.9954	0.9956	0.9957	0.9958
19	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964	0.9965	0.9966	0.9967	0.9968
20	0.9969	0.9970	0.9971	0.9972	0.9972	0.9973	0.9974	0.9975	0.9975	0.9976
21	0.9977	0.9977	0.9978	0.9978	0.9979	0.9980	0.9980	0.9981	0.9981	0.9982
22	0.9982	0.9983	0.9983	0.9984	0.9984	0.9985	0.9985	0.9985	0.9986	0.9986
23	0.9987	0.9987	0.9987	0.9988	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990
24	0.9990	0.9990	0.9990	0.9991	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992
25	0.9992	0.9992	0.9993	0.9993	0.9993	0.9993	0.9993	0.9994	0.9994	0.9994
26	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995	0.9995	0.9995	0.9995	0.9995
27	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
28	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997
29	0.9997	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998

Table 3. Distribution function of chi square-negative binomial with parameter $(r = 2, p = 1/3)$.

Density function is

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ p^2 e^{-x(1-q^2)/2}\Phi(q\sqrt{x})(1+q^2x) + \frac{p^2 q\sqrt{x}e^{-x/2}}{\sqrt{2\pi}} & \text{if } x > 0 \end{cases}$$

x	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0	0	0.0068	0.0147	0.0233	0.0324	0.0419	0.0517	0.0618	0.0722	0.0829
1	0.0937	0.1046	0.1157	0.1269	0.1382	0.1495	0.1609	0.1724	0.1839	0.1954
2	0.2069	0.2184	0.2298	0.2413	0.2527	0.2641	0.2754	0.2866	0.2978	0.3090
3	0.3200	0.3310	0.3419	0.3527	0.3634	0.3740	0.3845	0.3949	0.4052	0.4154
4	0.4255	0.4355	0.4453	0.4551	0.4647	0.4742	0.4836	0.4929	0.5021	0.5111
5	0.5200	0.5289	0.5375	0.5461	0.5545	0.5629	0.5710	0.5791	0.5871	0.5949
6	0.6026	0.6103	0.6177	0.6251	0.6324	0.6395	0.6465	0.6534	0.6602	0.6669
7	0.6735	0.6799	0.6863	0.6925	0.6987	0.7047	0.7106	0.7165	0.7222	0.7278

8	0.7333	0.7388	0.7441	0.7493	0.7545	0.7595	0.7645	0.7693	0.7741	0.7788
9	0.7834	0.7879	0.7923	0.7967	0.8009	0.8051	0.8092	0.8132	0.8172	0.8211
10	0.8248	0.8286	0.8322	0.8358	0.8393	0.8427	0.8461	0.8494	0.8527	0.8558
11	0.8589	0.8620	0.8650	0.8679	0.8708	0.8736	0.8763	0.8790	0.8817	0.8843
12	0.8868	0.8893	0.8917	0.8941	0.8964	0.8987	0.9010	0.9032	0.9053	0.9074
13	0.9095	0.9115	0.9134	0.9154	0.9173	0.9191	0.9209	0.9227	0.9244	0.9261
14	0.9278	0.9294	0.9310	0.9325	0.9341	0.9356	0.9370	0.9384	0.9398	0.9412
15	0.9425	0.9438	0.9451	0.9464	0.9476	0.9488	0.9500	0.9511	0.9522	0.9533
16	0.9544	0.9554	0.9565	0.9575	0.9584	0.9594	0.9603	0.9612	0.9621	0.9630
17	0.9639	0.9647	0.9655	0.9663	0.9671	0.9679	0.9686	0.9693	0.9701	0.9708
18	0.9714	0.9721	0.9727	0.9734	0.9740	0.9746	0.9752	0.9758	0.9764	0.9769
19	0.9775	0.9780	0.9785	0.9790	0.9795	0.9800	0.9805	0.9809	0.9814	0.9818
20	0.9822	0.9827	0.9831	0.9835	0.9839	0.9842	0.9846	0.9850	0.9853	0.9857
21	0.9860	0.9864	0.9867	0.9870	0.9873	0.9876	0.9879	0.9882	0.9885	0.9887
22	0.9890	0.9893	0.9895	0.9898	0.9900	0.9903	0.9905	0.9907	0.9910	0.9912
23	0.9914	0.9916	0.9918	0.9920	0.9922	0.9924	0.9926	0.9927	0.9929	0.9931
24	0.9932	0.9934	0.9936	0.9937	0.9939	0.9940	0.9942	0.9943	0.9944	0.9946
25	0.9947	0.9948	0.9950	0.9951	0.9952	0.9953	0.9954	0.9955	0.9957	0.9958
26	0.9959	0.9960	0.9961	0.9962	0.9963	0.9963	0.9964	0.9965	0.9966	0.9967
27	0.9968	0.9968	0.9969	0.9970	0.9971	0.9971	0.9972	0.9973	0.9974	0.9974
28	0.9975	0.9975	0.9976	0.9977	0.9977	0.9978	0.9978	0.9979	0.9979	0.9980
29	0.9980	0.9981	0.9981	0.9982	0.9982	0.9983	0.9983	0.9983	0.9984	0.9984

References

[1] W. Feller, *An Introduction to Probability Theory and Its Applications, vol II*, 2nd edition, John Wiley and Sons, New York, 1971.

[2] N. L. Johnson and S. Kotz, *Continuous Univariate Distribution-2*, A Wiley-interscience publication, 1970.

[3] R. V. Hogg, J. W. McKean, and A. T. Craig, *Introduction to Mathematical Statistics*, Sixth Edition, International Edition, 2005.

[4] L. Hung Tran and T. T Tran, *Some results on random sum of independent random variables*, Submitted to Statistics and Probability Letters, 2007.

[5] L. H. Tran, T. T. Tran, and Q. V. Bui, *On distribution function of chi-square with random degrees of freedom*, (in vietnamese), to appear in Vietnam Journal of Applied Mathematics, 2007.

[6] E. A. Lebedev, *On new properties distributions of mathematical statistics, Probability Theory, Random Processes, Mathematical Statistics and Applications*, Proceedings of the Intertional Conference in honor of 70 years Jubilee of Professor, Doctor of Physical and Mathematical Sciences Gennady Medvedev, Minsk February 21-25, 2005, pp. 154–160.

[7] H. Robbins, *The asymptotic distribution of the sum of a random number of random variables*, Bull. Amer. Math. Soc. **54** (1948), 1151–1161.

[8] Y. B. Shvetsov, *Random sum estimators and their efficiency*, Technical Report, Department of Mathematical Sciences, Montana State University, 2004, pp.1–20.

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