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# STABILITY OF A QUADRATIC FUNCTIONAL EQUATION IN QUASI-BANACH SPACES

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Abstract. In this paper we establish the general solution and investigate the Hyers–Ulam–Rassias stability of the following functional equation in quasi-Banach spaces.

$$\sum_{\substack{1\le i<j\le 4\\ 1\le k<l\le 4\\ k,l\in I_{ij}}} f(x_i+x_j-x_k-x_l)=2\sum_{1\le i<j\le 4} f(x_i-x_j),$$

where $I_{ij}=\{1,2,3,4\}\setminus\{i,j\}$ for all $1\le i<j\le 4$. The concept of Hyers-Ulam-Rassias stability originated from Th. M. Rassias' stability theorem that appeared in his paper: On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc. **72** (1978), 297–300.

## 1. Introduction and preliminaries

In 1940, S. M. Ulam [16] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following question concerning the stability of homomorphisms.

*Let $(G_1,*)$ be a group and let $(G_2,\diamond,d)$ be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon>0$, does there exist a $\delta(\epsilon)>0$ such that if a mapping $h:G_1\to G_2$ satisfies the inequality*

$$d(h(x*y),h(x)\diamond h(y))<\delta$$

*for all $x,y\in G_1$, then there is a homomorphism $H:G_1\to G_2$ with*

$$d(h(x),H(x))<\epsilon$$

*for all $x\in G_1$?*

In 1941, D. H. Hyers [8] considered the case of approximately additive mappings $f:E\to E'$, where $E$ and $E'$ are Banach spaces and $f$ satisfies *Hyers inequality*

$$\|f(x+y)-f(x)-f(y)\|\le\epsilon$$

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for all $x, y \in E$. It was shown that the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in E$ and that $L : E \to E'$ is the unique additive mapping satisfying

$$\|f(x) - L(x)\| \le \epsilon.$$

In 1978, Th. M. Rassias [13] provided a generalization of Hyers' theorem which allows the *Cauchy difference to be unbounded.*

Quadratic functional equation was used to characterize inner product spaces [1, 2, 9]. Several other functional equations were also to characterize inner product spaces. A square norm on an inner product space satisfies the important parallelogram equality

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.1}$$

is related to a symmetric bi-additive function [1, 11]. It is natural that each equation is called a quadratic functional equation. In particular, every solution of the quadratic equation (1.1) is said to be a quadratic function. It is well known that a function $f$ between real vector spaces is quadratic if and only if there exists a unique symmetric biadditive function $B$ such that $f(x) = B(x,x)$ for all $x$ (see [1, 11]). The biadditive function $B$ is given by

$$B(x,y) = \frac{1}{4}\big[f(x+y) - f(x-y)\big]. \tag{1.2}$$

A Hyers–Ulam stability problem for the quadratic functional equation (1.1) was proved by Skof for functions $f : E_1 \to E_2$, where $E_1$ is a normed space and $E_2$ a Banach space (see [15]). Cholewa [4] noticed that the theorem of Skof is still true if the relevant domain $E_1$ is replaced by an Abelian group. In the paper [5], Czerwik proved the Hyers–Ulam–Rassias stability of the quadratic functional equation (1.1). Grabiec [7] has generalized these results mentioned above. Jun and Lee [10] proved the Hyers–Ulam–Rassias stability of the Pexiderized quadratic equation (1.1).

Throughout this paper $I_{ij} = \{1,2,3,4\} \setminus \{i,j\}$ for all $1 \le i < j \le 4$. In this paper, we deal with the next functional equation deriving from quadratic function:

$$\sum_{\substack{1\le i<j\le 4\\ 1\le k<l\le 4\\ k,l\in I_{ij}}} f(x_i + x_j - x_k - x_l) = 2\sum_{1\le i<j\le 4} f(x_i - x_j). \tag{1.3}$$

It is easy to see that the function $f(x) = ax^2$ is a solution of the functional equation (1.3). The main purpose of this paper is to establish the general solution of Eq. (1.3) and investigate the Hyers–Ulam–Rassias stability for Eq. (1.3).

We recall some basic facts concerning quasi-Banach spaces and some preliminary results.

**Definition 1.1** ([3], [14]). Let $X$ be a real linear space. A *quasi-norm* is a real-valued function on $X$ satisfying the following:

(i) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$.
(ii) $\|\lambda x\| = |\lambda|\|x\|$ for all $\lambda \in \mathbb{R}$ and all $x \in X$.
(iii) There is a constant $K \geq 1$ such that $\|x + y\| \leq K(\|x\| + \|y\|)$ for all $x, y \in X$.

It follows from condition (iii) that

$$\left\|\sum_{i=1}^{2n} x_i\right\| \leq K^n \sum_{i=1}^{2n} \|x_i\|, \quad \left\|\sum_{i=1}^{2n+1} x_i\right\| \leq K^{n+1} \sum_{i=1}^{2n+1} \|x_i\|$$

for all integers $n \geq 1$ and all $x_1, x_2, \ldots, x_{2n+1} \in X$.

The pair $(X, \|\cdot\|)$ is called a *quasi-normed space* if $\|\cdot\|$ is a quasi-norm on $X$. The smallest possible $K$ is called the *modulus of concavity* of $\|\cdot\|$. A *quasi-Banach space* is a complete quasi-normed space.

A quasi-norm $\|\cdot\|$ is called a *p-norm* $(0 < p \leq 1)$ if

$$\|x + y\|^p \leq \|x\|^p + \|y\|^p$$

for all $x, y \in X$. In this case, a quasi-Banach space is called a *p-Banach space.*

By the Aoki–Rolewicz theorem [14] (see also [3]), each quasi-norm is equivalent to some $p$-norm. Since it is much easier to work with $p$-norms than quasi-norms, henceforth we restrict our attention mainly to $p$-norms.

## 2. Solutions of Eq. (1.3)

Throughout this section, $X$ and $Y$ will be real vector spaces. Before proceeding the proof of Theorem 2.3 which is the main result in this section, we shall need the following lemmas.

**Lemma 2.1.** *A function $f : X \to Y$ satisfies* (1.3) *for all $x_1, x_2, x_3, x_4 \in X$, if and only if the function $f$ is quadratic.*

*Proof.* Let $f$ satisfy (1.3). Letting $x_1 = x_2 = x_3 = x_4 = 0$ in (1.3), we get that $f(0) = 0$. Setting $x_1 = x$ and $x_2 = x_3 = x_4 = 0$ in (1.3), we conclude that $f(-x) = f(x)$ for all $x \in X$. This means that $f$ is an even function.

Letting $x_1 = x_2 = x$ and $x_3 = x_4 = 0$ in (1.3), and using the evenness of $f$, we get $f(2x) = 4f(x)$ for all $x \in X$. Letting $x_3 = x_4 = 0$ in (1.3), and using the evenness of $f$, we get

$$2f(x_1 + x_2) + 4f(x_1 - x_2) = 2f(x_1 - x_2) + 4f(x_1) + 4f(x_2)$$

for all $x_1, x_2 \in X$. Therefore

$$f(x_1 + x_2) + f(x_1 - x_2) = 2f(x_1) + 2f(x_2)$$

for all $x_1, x_2 \in X$. Therefore the function $f : X \to Y$ is quadratic.

Conversely, let $f$ be a quadratic function. So

$$f(x_1+x_2-x_3-x_4)+f(x_1+x_4-x_2-x_3)=2f(x_1-x_3)+2f(x_2-x_4), \tag{2.1}$$

$$f(x_1+x_3-x_2-x_4)+f(x_1+x_4-x_2-x_3)=2f(x_1-x_2)+2f(x_3-x_4), \tag{2.2}$$

$$f(x_1+x_2-x_3-x_4)+f(x_1+x_3-x_2-x_4)=2f(x_1-x_4)+2f(x_2-x_3) \tag{2.3}$$

for all $x_1,x_2,x_3,x_4 \in X$. Since $f$ is even, then we conclude from (2.1), (2.2), and (2.3) that the function $f$ satisfies (1.3). □

**Lemma 2.2.** *A function $f : X \to Y$ satisfies* (1.3) *for all $x_1,x_2,x_3,x_4 \in X \setminus \{0\}$, if and only if the function $f$ is quadratic.*

*Proof.* Suppose that the function $f$ satisfies (1.3) for all $x_1,x_2,x_3,x_4 \in X\setminus\{0\}$. Letting $x_1=x_2=x_3=x_4$ in (1.3), we get that $f(0)=0$. So by letting $x_1=x_2=x$ and $x_3=x_4=-x$ in (1.3), we get

$$f(4x)+f(-4x)=8f(2x) \tag{2.4}$$

for all $x \in X\setminus\{0\}$. It follows from (2.4) that the function $f$ is even. So if we put $x_3=x_4=x$ in (1.3), we have

$$f(x_1+x_2-2x)+f(x_1-x_2)=2f(x_1-x)+2f(x_2-x) \tag{2.5}$$

for all $x,x_1,x_2 \in X\setminus\{0\}$. Let $u,v \in X$ and let $z \in X\setminus\{0,-u,-v\}$. Putting $x=z, x_1=u+z$ and $x_2=v+z$ in (2.5), we get that

$$f(u+v)+f(u-v)=2f(u)+2f(v).$$

Therefore the function $f$ is quadratic.

The converse is evident by Lemma 2.1. □

Now we are ready to find out the general solution of (1.3).

**Theorem 2.3.** *A function $f : X \to Y$ satisfies* (1.3) *for all $x_1,x_2,x_3,x_4 \in X$, $(x_1,x_2,x_3,x_4 \in X\setminus\{0\})$ if and only if there exists a symmetric bi-additive function $B : X\times X \to Y$ such that $f(x)=B(x,x)$ for all $x \in X$.*

*Proof.* The result follows from Lemma 2.1, Lemma 2.2, and Proposition 1, p. 166 of [1]. □

## 3. Hyers–Ulam–Rassias stability of Eq. (1.3)

Throughout this section, assume that $X$ is a quasi-normed space with quasi-norm $\|\cdot\|_X$ and that $Y$ is a $p$-Banach space with $p$-norm $\|\cdot\|_Y$. Let $K$ be the modulus of concavity of $\|.\|_Y$.

In this section, using an idea of Găvruta [6] we prove the stability of Eq. (1.3) in the spirit of Hyers, Ulam, and Rassias. For convenience, we use the following abbreviation for a given function $f : X \to Y$ :

$$Df(x_1,x_2,x_3,x_4) = \sum_{\substack{1\le i<j\le 4\\ 1\le k<l\le 4\\ k,l\in I_{ij}}} f(x_i + x_j - x_k - x_l) - 2\sum_{1\le i<j\le 4} f(x_i - x_j)$$

for all $x_1, x_2, x_3, x_4 \in X$.

**Notation.** Let $X$ be a linear space. $x \in X^\star$ means $x \in X$ or $x \in X \setminus \{0\}$.

We will use the following lemma in this section.

**Lemma 3.1** ([12]). *Let $0 < p \le 1$ and let $x_1, x_2, \ldots, x_n$ be non-negative real numbers. Then*

$$\Big(\sum_{i=1}^{n} x_i\Big)^p \le \sum_{i=1}^{n} x_i^p.$$

**Theorem 3.2.** *Let $\varphi : X^4 \to [0,\infty)$ be a function such that*

$$\lim_{n\to\infty} 4^n \varphi\big(\frac{x_1}{2^n}, \frac{x_2}{2^n}, \frac{x_3}{2^n}, \frac{x_4}{2^n}\big) = 0, \tag{3.1}$$

$$\widetilde{\varphi}(x) := \sum_{i=2}^{\infty} 4^{ip} \varphi^p\big(\frac{x}{2^i}, \frac{x}{2^i}, -\frac{x}{2^i}, -\frac{x}{2^i}\big) < \infty \tag{3.2}$$

*for all $x, x_1, x_2, x_3, x_4 \in X$. Suppose that a function $f : X \to Y$ satisfies the inequality*

$$\|Df(x_1,x_2,x_3,x_4)\|_Y \le \varphi(x_1,x_2,x_3,x_4) \tag{3.3}$$

*for all $x_1, x_2, x_3, x_4 \in X$. Then the limit*

$$Q(x) := \lim_{n\to\infty} 4^n f\big(\frac{x}{2^n}\big) \tag{3.4}$$

*exists for all $x \in X$ and $Q : X \to Y$ is a unique quadratic function satisfying*

$$\|f(x) - Q(x)\|_Y \le \frac{K}{32}\Big\{\Big(\frac{K}{8}\Big)^p [\widetilde{\varphi}(2x) + \widetilde{\varphi}(-2x)] + \widetilde{\varphi}(x)\Big\}^{\frac{1}{p}} \tag{3.5}$$

*for all $x \in X$.*

*Proof.* Letting $x_1 = x_2 = x$ and $x_3 = x_4 = -x$ in (3.3), we get

$$\|f(4x) + f(-4x) - 8f(2x)\|_Y \le \varphi(x,x,-x,-x) \tag{3.6}$$

for all $x \in X$. Replacing $x$ by $-x$ in (3.6) and using (3.6), we get

$$\|f(2x) - f(-2x)\|_Y \le \frac{K}{8}\big[\varphi(x,x,-x,-x) + \varphi(-x,-x,x,x)\big] \tag{3.7}$$

for all $x \in X$. Replacing $x$ by $2x$ in (3.7) and using (3.6), we get

$$\|f(4x) - 4f(2x)\|_Y \le \gamma(x) \tag{3.8}$$

for all $x \in X$, where

$$\gamma(x) = \frac{K^2}{16}[\varphi(2x, 2x, -2x, -2x) + \varphi(-2x, -2x, 2x, 2x)] + \frac{K}{2}\varphi(x, x, -x, -x).$$

If we replace $x$ in (3.8) by $\frac{x}{2^{n+2}}$ and multiply both sides of (3.8) to $4^n$, then we have

$$\left\|4^{n+1}f\left(\frac{x}{2^{n+1}}\right) - 4^n f\left(\frac{x}{2^n}\right)\right\|_Y \le 4^n\gamma\left(\frac{x}{2^{n+2}}\right) \tag{3.9}$$

for all $x \in X$ and all non-negative integers $n$. Since $Y$ is a $p$-Banach space, we have

$$\begin{aligned}\left\|4^{n+1}f\left(\frac{x}{2^{n+1}}\right) - 4^m f\left(\frac{x}{2^m}\right)\right\|_Y^p &\le \sum_{i=m}^{n}\left\|4^{i+1}f\left(\frac{x}{2^{i+1}}\right) - 4^i f\left(\frac{x}{2^i}\right)\right\|_Y^p \\ &\le 16^{-p}\sum_{i=m+2}^{n+2} 4^{ip}\gamma^p\left(\frac{x}{2^i}\right)\end{aligned} \tag{3.10}$$

for all $x \in X$ and all non-negative integers $m$ and $n$ with $n \ge m$. It follows from Lemma 3.1 that

$$\sum_{i=2}^{\infty} 4^{ip}\gamma^p\left(\frac{x}{2^i}\right) \le \left(\frac{K^2}{16}\right)^p[\widetilde{\varphi}(2x) + \widetilde{\varphi}(-2x)] + \left(\frac{K}{2}\right)^p\widetilde{\varphi}(x)$$

for all $x \in X$. Therefore we conclude from (3.2) and (3.10) that the sequence $\{4^n f(\frac{x}{2^n})\}$ is a Cauchy sequence in $Y$ for all $x \in X$. Since $Y$ is complete, the sequence $\{4^n f(\frac{x}{2^n})\}$ converges in $Y$ for all $x \in X$. So one can define the function $Q : X \to Y$ by (3.4) for all $x \in X$. Letting $m = 0$ and passing the limit $n \to \infty$ in (3.10), we get (3.5). Now, we show that $Q$ is quadratic. It follows from (3.1), (3.3), and (3.4) that

$$\begin{aligned}\|DQ(x_1, x_2, x_3, x_4)\|_Y &= \lim_{n\to\infty} 4^n\left\|Df\left(\frac{x_1}{2^n}, \frac{x_2}{2^n}, \frac{x_3}{2^n}, \frac{x_4}{2^n}\right)\right\|_Y \\ &\le \lim_{n\to\infty} 4^n\varphi\left(\frac{x_1}{2^n}, \frac{x_2}{2^n}, \frac{x_3}{2^n}, \frac{x_4}{2^n}\right) = 0\end{aligned}$$

for all $x_1, x_2, x_3, x_4 \in X$. Therefore the function $Q : X \to Y$ satisfies (1.3). So by Lemma 2.1, we get that the function $Q : X \to Y$ is quadratic.

To prove the uniqueness of $Q$, let $T : X \to Y$ be another quadratic function satisfying (3.5). Then

$$\begin{aligned}\|Q(x) - T(x)\|_Y^p &= \lim_{n\to\infty} 4^{np}\left\|f\left(\frac{x}{2^n}\right) - T\left(\frac{x}{2^n}\right)\right\|_Y^p \\ &\le \frac{K^p}{32^p}\lim_{n\to\infty} 4^{np}\left\{\left(\frac{K}{8}\right)^p\left[\widetilde{\varphi}\left(\frac{x}{2^{n-1}}\right) + \widetilde{\varphi}\left(\frac{-x}{2^{n-1}}\right)\right] + \widetilde{\varphi}\left(\frac{x}{2^n}\right)\right\} = 0\end{aligned}$$

for all $x \in X$. So $Q = T$. □

**Corollary 3.3.** *Let $\psi_i : [0, \infty) \to [0, \infty)$ be a family of functions such that*

(1) $\psi_i(ts) \le \psi_i(t)\psi_i(s)$ *for all* $t, s \ge 0$;
(2) $\psi_i(1/2) < 1/4$

*for all* $1 \le i \le 4$. *Suppose that a function* $f : X \to Y$ *satisfies the inequality*

$$\|Df(x_1, x_2, x_3, x_4)\|_Y \le \sum_{i=1}^{4} \psi_i(\|x_i\|) \tag{3.11}$$

*for all* $x_1, x_2, x_3, x_4 \in X$. *Then there exists a unique quadratic function* $Q : X \to Y$ *satisfying*

$$\|f(x) - Q(x)\|_Y \le \frac{K}{16}\Big\{\sum_{j=1}^{4} \frac{2K^p + 8^p\psi_j^p(\frac{1}{2})}{1 - 4^p\psi_j^p(\frac{1}{2})}\psi_j^p(\frac{1}{2})\psi_j^p(\|x\|)\Big\}^{\frac{1}{p}}$$

*for all* $x \in X$. *The function* $Q : X \to Y$ *is given by* (3.4).

*Proof.* Let $\varphi : X^4 \to [0, \infty)$ be a function defined by

$$\varphi(x_1, x_2, x_3, x_4) = \sum_{i=1}^{4} \psi_i(\|x_i\|).$$

It is clear that the function $\varphi$ satisfies (3.1) and (3.2). It follows from Lemma 3.1 and conditions (1), (2) that

$$\widetilde{\varphi}(x) \le 4^{2p}\sum_{j=1}^{4} \frac{\psi_j^{2p}(\frac{1}{2})\psi_j^p(\|x\|)}{1 - 4^p\psi_j^p(\frac{1}{2})}, \qquad \widetilde{\varphi}(-2x) = \widetilde{\varphi}(2x) \le 4^{2p}\sum_{j=1}^{4} \frac{\psi_j^p(\frac{1}{2})\psi_j^p(\|x\|)}{1 - 4^p\psi_j^p(\frac{1}{2})}$$

for all $x \in X$. Therefore the result follows from Theorem 3.2. □

The following theorem is an alternative result of Theorem 3.2.

**Theorem 3.4.** *Let* $\varphi : X^4 \to [0, \infty)$ *be a function such that*

$$\lim_{n\to\infty} \frac{1}{4^n}\varphi(2^n x_1, 2^n x_2, 2^n x_3, 2^n x_4) = 0,$$

$$\widetilde{\varphi}(x) := \sum_{i=-1}^{\infty} \frac{1}{4^{ip}}\varphi^p(2^i x, 2^i x, -2^i x, -2^i x) < \infty \tag{3.12}$$

*for all* $x, x_1, x_2, x_3, x_4 \in X$. *Suppose that a function* $f : X \to Y$ *satisfies the inequality* (3.3) *for all* $x_1, x_2, x_3, x_4 \in X$. *Then the limit*

$$Q(x) := \lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$$

*exists for all* $x \in X$ *and* $Q : X \to Y$ *is a unique quadratic function satisfying* (3.5) *for all* $x \in X$.

**Corollary 3.5.** *Let* $\psi_i : [0, \infty) \to [0, \infty)$ *be a family of functions such that*

(1)′ $\psi_i(ts) \le \psi_i(t)\psi_i(s)$ *for all* $t, s \ge 0$;
(2)′ $\psi_i(2) < 4$

*for all* $1 \le i \le 4$. *Suppose that a function* $f : X \to Y$ *satisfies the inequality* (3.11) *for all* $x_1, x_2, x_3, x_4 \in X$. *Then there exists a unique quadratic function* $Q : X \to Y$ *satisfying*

$$\|f(x) - Q(x)\|_Y \le \frac{K}{16}\Big\{\sum_{j=1}^{4} \frac{2K^p\psi_j^p(2) + 8^p}{\psi_j^p(2)[4^p - \psi_j^p(2)]}\psi_j^p(\|x\|)\Big\}^{\frac{1}{p}}$$

*for all* $x \in X$. *The function* $Q : X \to Y$ *is given by*

$$Q(x) := \lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$$

*for all* $x \in X$.

**Corollary 3.6.** *Let* $\theta \ge 0$ *and* $\{r_i\}_{i\in J}$ *be non-zero real numbers such that* $r_i > 2$ (*respectively,* $r_i < 2$) *for all* $i \in J$, *where* $J$ *is a subset of* $\{1, 2, 3, 4\}$ *with* $|J| \ge 3$. *Suppose that a function* $f : X \to Y$ *satisfies the inequality*

$$\|Df(x_1, x_2, x_3, x_4)\|_Y \le \theta \sum_{i\in J} \|x_i\|_X^{r_i} \tag{3.13}$$

*for all* $x_1, x_2, x_3, x_4 \in X^\star$. *Then there exists a unique quadratic function* $Q : X \to Y$ *satisfying*

$$\|f(x) - Q(x)\|_Y \le \frac{K\theta}{16}\Big\{\sum_{i\in J} \frac{2K^p.2^{pr_i} + 8^p}{2^{pr_i}|2^{pr_i} - 4^p|}\|x\|_X^{pr_i}\Big\}^{\frac{1}{p}}$$

*for all* $x \in X^\star$.

**Corollary 3.7.** *Let* $\theta$ *be a non-negative real number. Suppose that a function* $f : X \to Y$ *satisfies the inequality*

$$\|Df(x_1, x_2, x_3, x_4)\|_Y \le \theta$$

*for all* $x_1, x_2, x_3, x_4 \in X$. *Then there exists a unique quadratic function* $Q : X \to Y$ *satisfies*

$$\|f(x) - Q(x)\|_Y \le \frac{K\theta}{256}\Big[\frac{2K^p + 8^p}{(4^p - 1)}\Big]^{\frac{1}{p}}$$

*for all* $x \in X$.

**Theorem 3.8.** *Let* $\varphi : X^4 \to [0, \infty)$ *be a function satisfying* (3.1) *and*

$$\widetilde{\varphi}(x) := \sum_{i=1}^{\infty} 4^{ip}\varphi^p\Big(\frac{x}{2^i}, \frac{x}{2^i}, 0, 0\Big) < \infty \tag{3.14}$$

*for all* $x \in X$. *Suppose that a function* $f : X \to Y$ *satisfies the inequality* (3.3) *for all* $x_1, x_2, x_3, x_4 \in X$. *Then there exists a unique quadratic function* $Q : X \to Y$ *satisfying*

$$\|f(x) - Q(x)\|_Y \le \frac{K}{8}\Big\{\Big(\frac{K}{8}\Big)^p[\widetilde{\varphi}(2x) + \widetilde{\varphi}(-2x)] + \widetilde{\varphi}(x)\Big\}^{\frac{1}{p}} \tag{3.15}$$

*for all* $x \in X$. *The function* $Q : X \to Y$ *is given by* (3.4).

*Proof.* Letting $x_1 = x_2 = x$ and $x_3 = x_4 = 0$ in (3.3), we get

$$\|f(2x) + f(-2x) - 8f(x)\|_Y \le \varphi(x, x, 0, 0) \tag{3.16}$$

for all $x \in X$. Replacing $x$ by $-x$ in (3.16) and using (3.16), we get that

$$\|f(x) - f(-x)\|_Y \le \frac{K}{8}\big[\varphi(x, x, 0, 0) + \varphi(-x, -x, 0, 0)\big]$$

for all $x \in X$. Hence (3.16) implies that

$$\|f(2x) - 4f(x)\|_Y \le \psi(x) \tag{3.17}$$

for all $x \in X$, where

$$\psi(x) := \frac{K^2}{16}\big[\varphi(2x, 2x, 0, 0) + \varphi(-2x, -2x, 0, 0)\big] + \frac{K}{2}\varphi(x, x, 0, 0).$$

If we replace $x$ in (3.17) by $\frac{x}{2^{n+1}}$ and multiply both sides of (3.17) to $4^n$, then we have

$$\Big\|4^{n+1} f\big(\frac{x}{2^{n+1}}\big) - 4^n f\big(\frac{x}{2^n}\big)\Big\|_Y \le 4^n \psi\big(\frac{x}{2^{n+1}}\big) \tag{3.18}$$

for all $x \in X$ and all non-negative integers $n$. Since $Y$ is a $p$-Banach space, we have

$$\begin{aligned} &\Big\|4^{n+1} f\big(\frac{x}{2^{n+1}}\big) - 4^m f\big(\frac{x}{2^m}\big)\Big\|_Y^p \\ &\le \sum_{i=m}^{n} \Big\|4^{i+1} f\big(\frac{x}{2^{i+1}}\big) - 4^i f\big(\frac{x}{2^i}\big)\Big\|_Y^p \\ &\le 4^{-p} \sum_{i=m+1}^{n+1} 4^{ip}\psi^p\big(\frac{x}{2^i}\big) \end{aligned} \tag{3.19}$$

for all $x \in X$ and all non-negative integers $m$ and $n$ with $n \ge m$. Since

$$\sum_{i=1}^{\infty} 4^{ip}\psi^p\big(\frac{x}{2^i}\big) \le \Big(\frac{K^2}{16}\Big)^p \big[\widetilde{\varphi}(2x) + \widetilde{\varphi}(-2x)\big] + \Big(\frac{K}{2}\Big)^p \widetilde{\varphi}(x)$$

for all $x \in X$, therefore we conclude from (3.14) and (3.19) that the sequence $\{4^n f(\frac{x}{2^n})\}$ is a Cauchy sequence in $Y$ for all $x \in X$. Since $Y$ is complete, the sequence $\{4^n f(\frac{x}{2^n})\}$ converges in $Y$ for all $x \in X$. So one can define the function $Q : X \to Y$ by (3.4) for all $x \in X$. Letting $m = 0$ and passing the limit $n \to \infty$ in (3.19), we get (3.15). Now, we show that $Q$ is quadratic. It follows from (3.1), (3.3) and (3.4),

$$\begin{aligned} \|DQ(x_1, x_2, 0, 0)\|_Y &= \lim_{n\to\infty} 4^n \Big\|Df\big(\frac{x_1}{2^n}, \frac{x_2}{2^n}, 0, 0\big)\Big\|_Y \\ &\le \lim_{n\to\infty} 4^n \varphi\big(\frac{x_1}{2^n}, \frac{x_2}{2^n}, 0, 0\big) = 0 \end{aligned}$$

for all $x_1, x_2 \in X$. Therefore the function $Q : X \to Y$ satisfies (1.3). So by the proof of Lemma 2.1 the function $Q : X \to Y$ is quadratic.

To prove the uniqueness of $Q$, let $T : X \to Y$ be another quadratic function satisfying (3.15). Since

$$\lim_{n\to\infty} 4^{np}\widetilde{\varphi}(\frac{x}{2^n}) = \lim_{n\to\infty} 4^{np}\sum_{i=1}^{\infty} 4^{ip}\varphi^p(\frac{x}{2^{n+i}}, \frac{x}{2^{n+i}}, 0, 0) = \lim_{n\to\infty} \sum_{i=n+1}^{\infty} 4^{ip}\varphi^p(\frac{x}{2^i}, \frac{x}{2^i}, 0, 0) = 0$$

for all $x \in X$, then it follows from (3.15) that

$$\|Q(x) - T(x)\|_Y^p = \lim_{n\to\infty} 4^{np}\Big\|f(\frac{x}{2^n}) - T(\frac{x}{2^n})\Big\|_Y^p = 0$$

for all $x \in X$. So $Q = T$. □

**Corollary 3.9.** *Let $\psi_i : [0,\infty) \to [0,\infty)$ be a family of functions such that*

(1) *$\psi_i(ts) \leq \psi_i(t)\psi_i(s)$ for all $t, s \geq 0$;*
(2) *$\psi_i(1/2) < 1/4$, $\psi_3(0) = \psi_4(0) = 0$*

*for all $1 \leq i \leq 4$. Suppose that a function $f : X \to Y$ satisfies the inequality (3.11) for all $x_1, x_2, x_3, x_4 \in X$. Then there exists a unique quadratic function $Q : X \to Y$ satisfying*

$$\|f(x) - Q(x)\|_Y \leq \frac{K}{16}\Big\{\sum_{j=1}^{2} \frac{2K^p + 8^p\psi_j^p(\frac{1}{2})}{1 - 4^p\psi_j^p(\frac{1}{2})}\psi_j^p(\|x\|)\Big\}^{\frac{1}{p}}$$

*for all $x \in X$. The function $Q : X \to Y$ is given by (3.4).*

**Theorem 3.10.** *Let $\varphi : X^4 \to [0,\infty)$ be a function such that*

$$\lim_{n\to\infty} \frac{1}{4^n}\varphi(2^n x_1, 2^n x_2, 2^n x_3, 2^n x_4) = 0 \tag{3.20}$$

*and*

$$\widetilde{\varphi}(x) := \sum_{i=0}^{\infty} \frac{1}{4^{ip}}\varphi^p(2^i x, 2^i x, 0, 0) < \infty \tag{3.21}$$

*for all $x, x_1, x_2, x_3, x_4 \in X$. Suppose that a function $f : X \to Y$ satisfies the inequality (3.3) for all $x_1, x_2, x_3, x_4 \in X$. Then there exists a unique quadratic function $Q : X \to Y$ satisfying (3.15). The function $Q : X \to Y$ is given by*

$$Q(x) := \lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$$

*for all $x \in X$.*

*Proof.* If we replace $x$ in (3.17) by $2^n x$ and divide both sides of (3.17) by $4^{n+1}$, then we have

$$\Big\|\frac{1}{4^{n+1}}f(2^{n+1}x) - \frac{1}{4^n}f(2^n x)\Big\|_Y \leq \frac{1}{4^{n+1}}\psi(2^n x) \tag{3.22}$$

for all $x \in X$ and all non-negative integers $n$, where

$$\psi(x) := \frac{K^2}{16}\left[\varphi(2x, 2x, 0, 0) + \varphi(-2x, -2x, 0, 0)\right] + \frac{K}{2}\varphi(x, x, 0, 0).$$

Since $Y$ is a $p$-Banach space, we have

$$\begin{aligned} &\left\|\frac{1}{4^{n+1}}f(2^{n+1}x) - \frac{1}{4^m}f(2^m x)\right\|_Y^p \\ &\le \sum_{i=m}^{n}\left\|\frac{1}{4^{i+1}}f(2^{i+1}x) - \frac{1}{4^i}f(2^i x)\right\|_Y^p \qquad (3.23) \\ &\le 4^{-p}\sum_{i=m}^{n}\frac{1}{4^{ip}}\psi^p(2^i x) \end{aligned}$$

for all $x \in X$ and all non-negative integers $m$ and $n$ with $n \ge m$. Since

$$\sum_{i=0}^{\infty}\frac{1}{4^{ip}}\psi^p(2^i x) \le \left(\frac{K^2}{16}\right)^p\left[\widetilde{\varphi}(2x) + \widetilde{\varphi}(-2x)\right] + \left(\frac{K}{2}\right)^p\widetilde{\varphi}(x)$$

for all $x \in X$, therefore we conclude from (3.21) and (3.23) that the sequence $\{\frac{1}{4^n}f(2^n x)\}$ is a Cauchy sequence in $Y$ for all $x \in X$. Since $Y$ is complete, the sequence $\{\frac{1}{4^n}f(2^n x)\}$ converges in $Y$ for all $x \in X$. So one can define the function $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty}\frac{1}{4^n}f(2^n x)$$

for all $x \in X$. Letting $m = 0$ and passing the limit $n \to \infty$ in (3.23), we get (3.15).

The rest of the proof is similar to the proof of Theorem 3.8. □

**Corollary 3.11.** *Let $\psi_i : [0, \infty) \to [0, \infty)$ be a family of functions such that*

(1)′ $\psi_i(ts) \le \psi_i(t)\psi_i(s)$ *for all* $t, s \ge 0$;

(2)′ $\psi_i(2) < 4$, $\psi_3(0) = \psi_4(0) = 0$

*for all $1 \le i \le 4$. Suppose that a function $f : X \to Y$ satisfies the inequality (3.11) for all $x_1, x_2, x_3, x_4 \in X$. Then there exists a unique quadratic function $Q : X \to Y$ satisfying*

$$\|f(x) - Q(x)\|_Y \le \frac{K}{16}\left\{\sum_{j=1}^{2}\frac{2K^p\psi_j^p(2) + 8^p}{4^p - \psi_j^p(2)}\psi_j^p(\|x\|)\right\}^{\frac{1}{p}}$$

*for all $x \in X$. The function $Q : X \to Y$ is given by*

$$Q(x) := \lim_{n\to\infty}\frac{1}{4^n}f(2^n x).$$

**Corollary 3.12.** *Let $\theta \geq 0$ and $\{r_i\}_{i=1}^4$ be non-zero real numbers such that $r_i > 2$ (respectively, $r_1, r_2 < 2$ and $0 < r_3, r_4 < 2$) for all $1 \leq i \leq 4$. Suppose that a function $f : X \to Y$ satisfies the inequality*

$$\|Df(x_1, x_2, x_3, x_4)\|_Y \leq \theta \sum_{i=1}^{4} \|x_i\|_X^{r_i}$$

*for all $x_1, x_2, x_3, x_4 \in X^\star$. Then there exists a unique quadratic function $Q : X \to Y$ satisfying*

$$\|f(x) - Q(x)\|_Y \leq \frac{K\theta}{16}\Big\{\sum_{i=1}^{2} \frac{2K^p.2^{pr_i} + 8^p}{|2^{pr_i} - 4^p|}\|x\|_X^{pr_i}\Big\}^{\frac{1}{p}}$$

*for all $x \in X^\star$.*

*Remark* 3.13. If we replace the condition (3.14) (respectively, (3.21)) by one of the following conditions

- $\sum_{i=1}^{\infty} 4^{ip}\varphi^p(0, 0, \frac{x}{2^i}, \frac{x}{2^i}) < \infty$ (respectively, $\sum_{i=0}^{\infty} \frac{1}{4^{ip}}\varphi^p(0, 0, 2^i x, 2^i x) < \infty$),
- $\sum_{i=1}^{\infty} 4^{ip}\varphi^p(0, \frac{x}{2^i}, 0, \frac{x}{2^i}) < \infty$ (respectively, $\sum_{i=0}^{\infty} \frac{1}{4^{ip}}\varphi^p(0, 2^i x, 0, 2^i x) < \infty$),
- $\sum_{i=1}^{\infty} 4^{ip}\varphi^p(\frac{x}{2^i}, 0, \frac{x}{2^i}, 0) < \infty$ (respectively, $\sum_{i=0}^{\infty} \frac{1}{4^{ip}}\varphi^p(2^i x, 0, 2^i x, 0) < \infty$)

for all $x \in X$, then we achieve alternative results of Theorem 3.8 (respectively, Theorem 3.10) and their corollaries.

## 4. Quadratic functions

**Theorem 4.1.** *Let $\theta, r, s$ be positive real numbers. Suppose that a function $f : X \to Y$ satisfies the inequality*

$$\|Df(x_1, x_2, x_3, x_4)\|_Y \leq \theta(\|x_i\|_X^r + \|x_j\|_X^s) \tag{4.1}$$

*for all $x_1, x_2, x_3, x_4 \in X$, where $1 \leq i < j \leq 4$. Then the function $f : X \to Y$ is quadratic.*

*Proof.* It follows from (4.1) that $f(0) = 0$. Let $k \in I_{ij}$ and $k > 1$. Letting $x_k = x$ and $x_l = 0$ in (4.1) for all $l \neq k$, we get that $f(x) = f(-x)$ for all $x \in X$. So the function $f$ is even. Therefore

$$\begin{aligned} &Df(x_1, x_2, x_3, x_4) \\ &= 2f(x_i + x_j - x_k - x_l) + 2f(x_i - x_j + x_k - x_l) \\ &\quad + 2f(x_i - x_j - x_k + x_l) - 2f(x_i - x_j) - 2\sum_{\substack{1\leq p<q\leq 4 \\ (p,q)\neq(i,j)}} f(x_p - x_q) \end{aligned}$$

for all $x_1, x_2, x_3, x_4 \in X$, where $k, l \in I_{ij}$ and $k \neq l$. So by letting $x_i = x_j = 0$ in (4.1), we get

$$f(x_k + x_l) + f(x_k - x_l) = 2f(x_k) + 2f(x_l)$$

for all $x_k, x_l \in X$. Hence the function $f$ is quadratic. $\square$

**Corollary 4.2.** *Let $\theta, r$ be positive real numbers. Suppose that a function $f : X \to Y$ satisfies the inequality*

$$\|Df(x_1, x_2, x_3, x_4)\|_Y \le \theta \|x_i\|_X^r$$

*for all $x_1, x_2, x_3, x_4 \in X$ and for some $1 \le i \le 4$. Then the function $f : X \to Y$ is quadratic.*

The proof of the following theorem is similar to the proof of Theorem 4.1.

**Theorem 4.3.** *Let $\theta$ and $\{r_i\}_{i\in J}$ be positive real numbers, where $J$ is a non-empty subset of $\{1, 2, 3, 4\}$. Suppose that a function $f : X \to Y$ satisfies the inequality*

$$\|Df(x_1, x_2, x_3, x_4)\|_Y \le \theta \prod_{i\in J} \|x_i\|_X^{r_i}$$

*for all $x_1, x_2, x_3, x_4 \in X$. Then the function $f : X \to Y$ is quadratic.*

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