AN EXTREMAL PROBLEM ON POTENTIALLY $K_{r,r} - ke$-GRAPHIC SEQUENCES

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ABSTRACT. For $1 \leq k \leq r$, let $\sigma(K_{r,r} - ke, n)$ be the smallest even integer such that every $n$-term graphic sequence $\pi = (d_1, d_2, \ldots, d_n)$ with term sum $\sigma(\pi) = d_1 + d_2 + \cdots + d_n \geq \sigma(K_{r,r} - ke, n)$ has a realization $G$ containing $K_{r,r} - ke$ as a subgraph, where $K_{r,r} - ke$ is the graph obtained from the $r \times r$ complete bipartite graph $K_{r,r}$ by deleting $k$ edges which form a matching. In this paper, we determine $\sigma(K_{r,r} - ke, n)$ for even $r \geq 4$ and $n \geq 7r^2 + \frac{1}{2}r - 22$ and for odd $r \geq 5$ and $n \geq 7r^2 + 9r - 26$.

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1. Introduction

The set of all sequences $\pi = (d_1, d_2, \ldots, d_n)$ of nonnegative integers with $d_i \leq n-1$ for each $i$ is denoted by $NS_n$. A sequence $\pi \in NS_n$ is said to be graphic if it is the degree sequence of a simple graph $G$ on $n$ vertices, and such a graph $G$ is called a realization of $\pi$. The set of all graphic non-increasing sequences in $NS_n$ is denoted by $GS_n$. For a sequence $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$, denote $\sigma(\pi) = d_1 + d_2 + \cdots + d_n$. For given a graph $H$, a sequence $\pi \in GS_n$ is potentially $H$-graphic if there exists a realization of $\pi$ containing $H$ as a subgraph. Gould et al. [3] considered the following variation of the classical Turán-type extremal problems: for given a graph $H$, determine the smallest even integer $\sigma(H, n)$ such that every sequence $\pi \in GS_n$ with $\sigma(\pi) \geq \sigma(H, n)$ is potentially $H$-graphic. If $H = K_{r+1}$, the complete graph on $r+1$ vertices, this problem was considered by Erdős et al. [2] where they showed that $\sigma(K_3, n) = 2n$ for $n \geq 6$ and conjectured that $\sigma(K_{r+1}, n) = (r-1)(2n-r) + 2$ for sufficiently large $n$. Gould et al. [3] and Li et al. [5] independently proved it for $r = 3$. Recently, Li et al. [6,7] proved that the conjecture is true for $r = 4$ and $n \geq 10$ and for $r \geq 5$ and

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n ≥ \binom{s}{2} + 3. For H = K_{r,s}, the r \times s complete bipartite graph, Gould et al. [3] determined \( \sigma(K_{2,2},n) \) for \( n \geq 4 \). Yin et al. [8] determined \( \sigma(K_{3,3},n) \) for \( n \geq 6 \) and \( \sigma(K_{4,4},n) \) for \( n \geq 8 \). Yin et al. [9] also determined \( \sigma(K_{r,r},n) \) for even \( r \geq 4 \) and \( n \geq 4r^2 - r - 6 \) and for odd \( r \geq 3 \) and \( n \geq 4r^2 + 3r - 8 \). Recently, Yin et al. [10,11] further determined \( \sigma(K_{r,s},n) \) for \( s \geq r \geq 1 \) and sufficiently large \( n \). The purpose of this paper is to determine \( \sigma(K_{r,r} - ke,n) \) for even \( r \geq 4 \) and \( n \geq 7r^2 + \frac{1}{2}r - 22 \) (Theorem 6) and for odd \( r \geq 5 \) and \( n \geq 7r^2 + 9r - 26 \) (Theorem 7), where \( 1 \leq k \leq r \) and \( K_{r,r} - ke \) is the graph obtained from \( K_{r,r} \) by deleting \( k \) edges which form a matching.

2. Preliminaries

In order to prove our main results, we need the following known theorems.

Let \( \pi = (d_1, d_2, \ldots, d_n) \in NS_n \) be a non-increasing sequence. Denote \( f(\pi) = \max \{i \mid d_i \geq i\} \) and define an \( n \)-by-\( n \) matrix \( A = (a_{ij}) \) as follows: if \( d_i \geq i \), then
\[
a_{ij} = \begin{cases} 
1 & \text{if } 1 \leq j \leq d_i + 1 \text{ and } j \neq i, \\
0 & \text{otherwise}, 
\end{cases}
\]
and if \( d_i < i \), then
\[
a_{ij} = \begin{cases} 
1 & \text{if } 1 \leq j \leq d_i, \\
0 & \text{otherwise}. 
\end{cases}
\]

\( f(\pi) \) and \( \bar{A} \) are called the trace and the left-most off-diagonal matrix of \( \pi \), respectively. The column sum vector of \( \bar{A} \), denoted by \( \bar{\pi} = (\bar{d}_1, \bar{d}_2, \ldots, \bar{d}_n) \), is called the corrected conjugate vector of \( \pi \). Clearly, the row sum vector of \( \bar{A} \) is \( \pi \) and \( \bar{\sigma}(\bar{\pi}) = \sigma(\pi) \).

**Theorem 1.** [1] Let \( \pi = (d_1, d_2, \ldots, d_n) \in NS_n \) be a non-increasing sequence with even \( \sigma(\pi) \). Then \( \pi \) is graphic if and only if \( d_1 + d_2 + \cdots + d_i \leq \bar{d}_1 + \bar{d}_2 + \cdots + \bar{d}_i \) for each \( i = 1, 2, \ldots, f(\pi) \).

For a non-increasing sequence \( \pi = (d_1, d_2, \ldots, d_n) \in NS_n \), let \( d'_1 \geq d'_2 \geq \cdots \geq d'_{n-1} \) be the rearrangement of \( d_1 - 1, d_2 - 1, \ldots, d_{n-1} - 1 \). Then \( \pi' = (d'_1, d'_2, \ldots, d'_{n-1}) \) is called the residual sequence of \( \pi \). It is easy to see that if \( \pi' \) is graphic then \( \pi \) is also graphic, since a realization \( G \) of \( \pi \) can be obtained from a realization \( G' \) of \( \pi' \) by adding a new vertex of degree \( d_n \) and joining it to the vertices whose degrees are reduced by one in going from \( \pi \) to \( \pi' \). In fact, more is true:

**Theorem 2.** [4] Let \( \pi = (d_1, d_2, \ldots, d_n) \in NS_n \) be a non-increasing sequence. Then \( \pi \) is graphic if and only if \( \pi' \) is graphic.

**Theorem 3.** [7] If \( r \geq 5 \), then \( \sigma(K_{r+1,n}) \leq 2n(r-2)+8 \) for \( 2r+2 \leq n \leq \binom{r}{2} + 3 \) and \( \sigma(K_{r+1,n}) = (r-1)(2n-r) + 2 \) for \( n \geq \binom{r}{2} + 3 \).

**Theorem 4.** [8] Let \( \pi = (d_1, \ldots, d_r, d_{r+1}, \ldots, d_{r+s}, d_{r+s+1}, \ldots, d_n) \in GS_n \), where \( d_{r+s} \geq r + s - 1 \) and \( d_n \geq r \). Then \( \pi \) is potentially \( K_{r,s} \)-graphic.
Theorem 5. [8] Let \( \pi = (d_1, \ldots, d_r, d_{r+1}, \ldots, d_{r+s}, d_{r+s+1}, \ldots, d_n) \in GS_n \), where \( d_s \geq r + s - 1 \), \( d_{r+s} \leq r + s - 2 \) and \( d_n \geq r \). If \( n \geq (r + s)(s - 1) \), then \( \pi \) is potentially \( K_{r,s} \)-graphic.

In order to prove our main results, we also borrow an idea from [8,11]. Let

\[
\pi' = (d_1, \ldots, d_r, d_{r+1}, \ldots, d_{2r}, d_{2r+1}, \ldots, d_n)
\]

where \( d_1 \geq \cdots \geq d_{r-1} \geq r \), \( d_r \geq r - 1 \), \( d_{r+1} \geq \cdots \geq d_{2r} \geq r \) and \( d_{2r+1} \geq \cdots \geq d_n \geq r \). Let

\[
\pi'_1 = \begin{cases} 
(d_2, \ldots, d_r, d_{r+1} - 1, \ldots, d_{r+d_1} - 1, d_{r+d_1+1}, \ldots, d_n) \\
(d_3 - 1, \ldots, d_{d_1+r-n+1} - 1, d_{d_1+r-n+2}, \ldots, d_r, d_{r+1} - 1, \ldots, d_n - 1)
\end{cases}
\]

and \( \pi''_1 = (d_2^{(1)}, \ldots, d_r^{(1)}, d_{r+1}^{(1)}, \ldots, d_{2r}^{(1)}, d_{2r+1}^{(1)}, \ldots, d_n^{(1)}) \), where \( d_2^{(1)} \geq \cdots \geq d_n^{(1)} \) is the rearrangement of the first \( r - 1 \) terms in \( \pi'_1 \), \( d_{r+i}^{(1)} = d_r^{(1)} - 1 \) for \( 1 \leq i \leq r \) and \( d_{2r+i}^{(1)} \geq \cdots \geq d_{2n}^{(1)} \) is the rearrangement of the final \( n - 2r \) terms in \( \pi'_1 \).

For \( \pi'_2 = (d_2^{(1)}, \ldots, d_r^{(1)}, d_{r+1}^{(1)}, \ldots, d_{2r}^{(1)}, d_{2r+1}^{(1)}, \ldots, d_n^{(1)}) \), if \( d_2^{(1)} \geq \cdots \geq d_{r-1}^{(1)} \geq r \) and \( d_r^{(1)} \geq r - 1 \), we can similarly define \( \pi''_2 \) as follows:

\[
\pi''_2 = \begin{cases} 
(d_3^{(1)}, \ldots, d_r^{(1)}, d_{r+1}^{(1)} - 1, \ldots, d_{r+d_2^{(1)} - 1}, d_1^{(1)}, d_{r+d_2^{(1)} + 1}^{(1)}, \ldots, d_n^{(1)}) \\
(d_3^{(1)} - 1, \ldots, d_{d_2^{(1)}+r-n+2}^{(1)} - 1, d_1^{(1)}, d_2^{(1)} - 1, \ldots, d_n^{(1)} - 1)
\end{cases}
\]

and \( \pi''_2 = (d_3^{(2)}, \ldots, d_r^{(2)}, d_{r+1}^{(2)}, \ldots, d_{2r}^{(2)}, d_{2r+1}^{(2)}, \ldots, d_n^{(2)}) \), where \( d_3^{(2)} \geq \cdots \geq d_{r}^{(2)} \) is the rearrangement of the first \( r - 2 \) terms in \( \pi''_2 \), \( d_{r+i}^{(2)} = d_{r+i}^{(1)} - 1 \) for \( 1 \leq i \leq r \) and \( d_{2r+i}^{(2)} \geq \cdots \geq d_{2n}^{(2)} \) is the rearrangement of the final \( n - 2r \) terms in \( \pi''_2 \). For \( k = 3, 4, \ldots, r - 1 \) in turn, if \( d_{k}^{(k-1)} \geq \cdots \geq d_{r-1}^{(k-1)} \geq r \) and \( d_{r}^{(k-1)} \geq r - 1 \), the definitions of \( \pi''_k \) and \( \pi''_k \) are similar.

For \( \pi''_{r-1} = (d_r^{(r-1)}, d_{r+1}^{(r-1)}, \ldots, d_{2r}^{(r-1)}, d_{2r+1}^{(r-1)}, \ldots, d_n^{(r-1)}) \), if \( d_r^{(r-1)} \geq r - 1 \), we define \( \pi''_r \) as follows:

\[
\pi''_r = (d_{r+1}^{(r-1)} - 1, \ldots, d_{r+1}^{(r-1)} - 1, d_1^{(r-1)}, d_{r+1}^{(r-1)} + 1, \ldots, d_n^{(r-1)}),
\]

and \( \pi''_r = (d_r^{(r)}, d_{2r}^{(r)}, \ldots, d_n^{(r)}) \), where \( d_r^{(r)} \geq \cdots \geq d_{2r}^{(r)} \) is the rearrangement of the first \( r \) terms in \( \pi''_r \) and \( d_{2r+1}^{(r)} \geq \cdots \geq d_n^{(r)} \) is the rearrangement of the final \( n - 2r \) terms in \( \pi''_r \). By the definition of \( \pi''_r \), the following Proposition 1 is obvious.

Proposition 1. Let \( \pi = (d_1, \ldots, d_r, d_{r+1}, \ldots, d_{2r}, d_{2r+1}, \ldots, d_n) \in NS_n \), where \( d_1 \geq \cdots \geq d_{r-1} \geq r \), \( d_r \geq r - 1 \), \( d_{r+1} \geq \cdots \geq d_{2r} \geq r \) and \( d_{2r+1} \geq \cdots \geq d_n \geq r \). Let \( \pi''_r \) be defined as above. If \( \pi''_r \) is graphic, then \( \pi \) is potentially \( K_{r,r} \)-e-graphic.
For the defined sequence \( \pi''_r = (d^{(r)}_{r+1}, \ldots, d^{(r)}_{2r}, d^{(r)}_{2r+1}, \ldots, d^{(r)}_n) \) in Proposition 1, if \( d^{(r)}_{r+1} \geq \cdots \geq d^{(r)}_{2r} \geq 1 \) and \( d^{(r)}_{2r+1} \geq \cdots \geq d^{(r)}_n \geq 1 \), we define \( \pi''_{r+1} = (d^{(r)}_{r+2} - 1, \ldots, d^{(r)}_{r+d^{(r)}_{2r+1}+1} - 1, d^{(r)}_{r+d^{(r)}_{2r+1}+2} - 1, \ldots, d^{(r)}_n) \), and \( \pi''_{r+1} = (d^{(r+1)}_{r+2}, \ldots, d^{(r+1)}_{2r+1}, d^{(r+1)}_{2r+1}, \ldots, d^{(r+1)}_n) \), where \( d^{(r+1)}_{r+2} \geq \cdots \geq d^{(r+1)}_{2r} \) is the rearrangement of the first \( r-1 \) terms in \( \pi''_{r+1} \) and \( d^{(r+1)}_{2r+1} \geq \cdots \geq d^{(r+1)}_n \) is the rearrangement of the final \( n-2r \) terms in \( \pi''_{r+1} \). For \( k = 2, 3, \ldots, r \) in turn, if \( d^{(r+k-1)}_{r+k} \geq \cdots \geq d^{(r+k-1)}_{2r+k} \geq 1 \) and \( d^{(r+k-1)}_{2r+k+1} \geq \cdots \geq d^{(r+k-1)}_n \geq 1 \), the definitions of \( \pi''_{r+k} \) and \( \pi''_{n+k} \) are similar.

**Proposition 2.** [11] Let \( \pi''_r = (d^{(r)}_{r+1}, \ldots, d^{(r)}_{2r}, d^{(r)}_{2r+1}, \ldots, d^{(r)}_n) \) be a defined sequence as in Proposition 1, \( 1 \leq k \leq r \) and let \( \pi''_{r+k} \) be defined as above. If \( \pi''_{r+k} \) is graphic, then \( \pi''_r \) is also graphic.

**Lemma 1.** [8] Let \( \pi = (d_1, d_2, \ldots, d_n) \in NS_n \), \( m = \max\{d_1, d_2, \ldots, d_n\} \) and \( \sigma(\pi) \) be even. Let \( \pi^* = (d^*_1, d^*_2, \ldots, d^*_n) \) be the rearrangement sequence of \( \pi \), where \( m = d^*_1 \geq d^*_2 \geq \cdots \geq d^*_n \) is the rearrangement of \( d_1, d_2, \ldots, d_n \). If there exists an integer \( n_1 \) \((\leq n)\) such that \( d^*_n \geq h \geq 1 \) and \( n_1 \geq \frac{[\frac{n+1}{2}]}{4} \), then \( \pi \) is graphic.

We now prove the following:

**Lemma 2.** Let \( r \geq 4 \) and \( n \geq \frac{r^2}{4} + \frac{5}{2}r \). Let \( \pi = (d_1, \ldots, d_r, d_{r+1}, \ldots, d_{2r}, d_{2r+1}, \ldots, d_n) \in GS_n \) with \( d_r \leq 2r-2 \) and \( d_n \geq r \). If there exists an integer \( t \in \{1, 2, \ldots, \left[ \frac{r}{2} \right] - 1\} \) such that \( d_{r+t} \geq 2r - 2 - t \) and \( d_{2r} \geq r + t - 1 \), then \( \pi \) is potentially \( K_{r,r} \)-e-graphic.

**Proof.** Rearrange the terms in \( \pi \) in order to get that

\[
(p_1, \ldots, p_{r-t}, p_{r-t+1}, \ldots, p_r, p_{r+1}, \ldots, p_{r+t}, p_{r+t+1}, \ldots, p_n),
\]

where \( p_1 = d_1, \ldots, p_{r-t} = d_{r-t}; p_{r-t+1} = d_{r+1}, \ldots, p_r = d_{r+t}; p_{r+1} = d_{r+1}, \ldots, p_{r+t} = d_r; p_{r+t+1} = d_{r+t+1}, \ldots, p_n = d_n \). For convenience, the new sequence is still denoted by \( \pi \). Clearly, \( p_1 \geq \cdots \geq p_r \geq 2r - 2 - t, p_{r+1} \geq \cdots \geq p_{2r} \geq r + t - 1 \) and \( 2r - 2 \geq p_{2r+1} \geq \cdots \geq p_n \geq r \). By Proposition 1, it is enough to prove that \( \pi''_r \) is graphic.

Since \( \pi''_r = (p^{(r-t-2)}_{r-t-2}, \ldots, p^{(r-t-2)}_{r-t-2}, p^{(r-t-2)}_{r-t-2}, p^{(r-t-2)}_{2r}, p^{(r-t-2)}_{2r}, \ldots, p^{(r-t-2)}_n) \) satisfies

1. \( 2r - 2 \geq p^{(r-t-2)}_{r-t-1} \geq \cdots \geq p^{(r-t-2)}_1 \geq d_{r+t} - (r - t - 2) \geq r, \)
2. \( 2r - 2 \geq p^{(r-t-2)}_{r-t+1} \geq \cdots \geq p^{(r-t-2)}_n \geq t + 2, \)
3. \( p^{(r-t-2)}_{r-t} = r \) and \( p^{(r-t-2)}_{r-t} = r \) and \( p^{(r-t-2)}_{r-t} = r \) and \( p^{(r-t-2)}_{r-t} = r \) and \( p^{(r-t-2)}_{r-t} = r \) and \( p^{(r-t-2)}_{r-t} = r \) and \( p^{(r-t-2)}_{r-t} \), we get that \( \pi''_r = (p^{(r)}_{r-t+1}, \ldots, p^{(r)}_{r-t}, p^{(r)}_{r-t}, p^{(r)}_{2r}, p^{(r)}_{2r}, \ldots, p^{(r)}_n) \) satisfies
(4) \( n - r - 1 \geq p_{r+1}^{(r)} \geq \cdots \geq p_{r+t-1}^{(r)} \),
(5) \( 2r - 2 \geq p_{2r}^{(r)} \geq \cdots \geq p_{2r+1}^{(r)} \geq t - 1 \),
(6) \( 2r - 2 \geq p_{2r+1}^{(r)} \geq \cdots \geq p_{n}^{(r)} \geq t + 1 \).

Hence \( \pi_{r+t-1}^\prime \) satisfies
(7) \( 2r - 2 \geq p_{r+t}^{(r+t-1)} \geq \cdots \geq p_{2r}^{(r+t-1)} \geq 0 \),
(8) \( 2r - 2 \geq p_{2r+1}^{(r+t-1)} \geq \cdots \geq p_{n}^{(r+t-1)} \geq 2 \).

Thus, \( \frac{1}{2} \left( \frac{(2r-2+2+1)^2}{4} \right) \leq \frac{r^2}{2} + \frac{r}{2} \leq n - 2r \). By Lemma 1, \( \pi_{r+t-1}^\prime \) is graphic, and hence \( \pi_{r}^\prime \) is also graphic by Proposition 2.

\[ \square \]

3. Main results

For convenience, we first introduce the following notations. Let \( r = 8k + t \), where \( k \geq 0 \) and \( 0 \leq t \leq 7 \). If \( t \in \{2, 3, 6, 7\} \), let \( F_t = \{ (8k + t, n) \mid k \geq 0 \text{ and } n \geq 16k + 2t \} \). If \( t \in \{0, 1, 4, 5\} \), let \( F'_t = \{ (8k + t, n) \mid k \geq 0, n \geq 16k + 2t \text{ and } n \text{ is odd} \} \) and \( F''_t = \{ (8k + t, n) \mid k \geq 0, n \geq 16k + 2t \text{ and } n \text{ is even} \} \). Denote \( E_1 = F_2 \cup F'_6 \cup F''_4, E_2 = F_0 \cup F'_4 \cup F''_6, E_3 = F'_7 \cup F''_6 \cup F''_0 \) and \( E_4 = F_3 \cup F'_6 \cup F''_1 \).

3.1 \( \sigma(K_{r,r} - ke, n) \) for even \( r \) and \( n \geq 7r^2 + \frac{1}{2}r - 22 \).

Lemma 3. Let \( r \) be even, \( r \geq 4 \) and \( n \geq 2r \). Then
\[
\sigma(K_{r,r} - re, n) \geq \begin{cases} 
(\frac{5}{2}r - 2)n - \frac{11}{8}r^2 + \frac{5}{4}r + 2 - (n - r + 1), & \text{if } (r, n) \in E_1, \\
(\frac{5}{2}r - 2)n - \frac{11}{8}r^2 + \frac{5}{4}r + 1 - (n - r + 1), & \text{if } (r, n) \in E_2.
\end{cases}
\]

Proof. Suppose \( (r, n) \in E_1 \). Consider \( \pi = ((n - 1)^{r-1}, 2r - 3, 2r - 4, \ldots, \frac{3}{2}r - 1, (\frac{5}{2}r - 2)^n - \frac{3}{2}r + 2) \), where \( x^y \) stands for \( y \) consecutive terms, each equal to \( x \).

Then \( \sigma(\pi) = (\frac{5}{2}r - 2)n - \frac{11}{8}r^2 + \frac{5}{4}r - (n - r + 1) \) is even and \( f(\pi) = \frac{5}{2}r - 2 \). It follows from the left-most off-diagonal matrix \( \overline{A} \) of \( \pi \) that \( \overline{A} = (\overline{A}_1, \overline{A}_2, \ldots, \overline{A}_n) \) satisfies \( \overline{A}_1 = \cdots = \overline{A}_{\frac{3}{2}r - 2} = n - 1 \). Clearly, \( d_1 + d_2 + \cdots + d_i \leq \frac{3}{2}r - 1 \) for each \( i = 1, 2, \ldots, f(\pi) \).

By Theorem 1, \( \pi \in GS_n \). Let \( \pi_1 = (r - 2, r - 3, \ldots, \frac{5}{2}r - 2, (\frac{5}{2}r - 2)^n - \frac{3}{2}r + 2) \). If \( \pi \) is potentially \( K_{r,r} - re \)-graphic, then there exist integers \( t \) and \( s \), \( t \geq s \geq 1 \) and \( t + s = r + 1 \) such that \( \pi_1 \) is potentially \( K_{s,t} - se \)-graphic, where \( K_{s,t} - se \) is the graph obtained from \( K_{s,t} \) by deleting \( s \) edges which form a matching. Hence, there are at least \( s \) terms in \( \pi_1 \) which are greater than or equal to \( r - s \), a contradiction. So \( \pi \) is not potentially \( K_{r,r} - re \)-graphic.

Thus \( \sigma(K_{r,r} - re, n) \geq \sigma(\pi) + 2 = (\frac{5}{2}r - 2)n - \frac{11}{8}r^2 + \frac{5}{4}r + 2 - (n - r + 1) \).

Now assume \( (r, n) \in E_2 \). Consider \( \pi = ((n - 1)^{r-1}, 2r - 3, 2r - 4, \ldots, \frac{3}{2}r - 1, (\frac{5}{2}r - 2)^n - \frac{3}{2}r + 3) \). Then \( \sigma(\pi) = (\frac{5}{2}r - 2)n - \frac{11}{8}r^2 + \frac{5}{4}r - 1 - (n - r + 1) \) is even and \( f(\pi) = \frac{5}{2}r - 2 \). By the left-most off-diagonal matrix \( \overline{A} \) of \( \pi \) that \( \overline{A} = (\overline{A}_1, \overline{A}_2, \ldots, \overline{A}_n) \) satisfies \( \overline{A}_1 = \cdots = \overline{A}_{\frac{3}{2}r - 3} = n - 1, \overline{A}_{\frac{3}{2}r - 2} = n - 2 \). Clearly, \( d_1 + d_2 + \cdots + d_i \leq \frac{3}{2}r - 1 \) for each \( i = 1, 2, \ldots, f(\pi) \).

By Theorem 1, \( \pi \in GS_n \). Similarly, we also can prove that \( \pi \) is not potentially \( K_{r,r} - re \)-graphic.

Thus \( \sigma(K_{r,r} - re, n) \geq \sigma(\pi) + 2 = (\frac{5}{2}r - 2)n - \frac{11}{8}r^2 + \frac{5}{4}r + 1 - (n - r + 1) \). \[ \square \]
Lemma 4. Let \( r \) be even, \( r \geq 4 \) and \( n = r^2 + r - 2 \). Then
\[
\sigma(K_{r,r} - e, n) \leq \left( \frac{5}{2} r - 2 \right) n - \frac{11}{8} r^2 + \frac{5}{4} r + 2 - (n - r + 1) + \left( \frac{3}{2} r^3 - \frac{1}{8} r^2 - 5r \right).
\]

Proof. By Theorem 3,
\[
\sigma(K_{r,r} - e, n) \leq \sigma(K_{2r}, n) \leq (4r - 6)n + 8
= \left( \frac{5}{2} r - 2 \right) n - \frac{11}{8} r^2 + \frac{5}{4} r + 2 - (n - r + 1) + \left( \frac{3}{2} r^3 - \frac{1}{8} r^2 - \frac{33}{4} r + 13 \right)
\leq \left( \frac{5}{2} r - 2 \right) n - \frac{11}{8} r^2 + \frac{5}{4} r + 2 - (n - r + 1) + \left( \frac{3}{2} r^3 - \frac{1}{8} r^2 - 5r \right).
\]

Lemma 5. Let \( r \) be even, \( r \geq 4 \) and \( n = r^2 + r - 2 \). Let \( \pi = (d_1, d_2, \ldots, d_n) \in GS_n \) with \( d_n \geq r \). If \( \sigma(\pi) \geq \left( \frac{5}{2} r - 2 \right) n - \frac{11}{8} r^2 + \frac{5}{4} r + 2 - (n - r + 1) \), then \( \pi \) is potentially \( K_{r,r} - e \)-graphic.

Proof. If \( d_r \geq 2r - 1 \), then by \( n \geq (r + 2)(r - 1) \) and Theorems 4 and 5, \( \pi \) is potentially \( K_{r,r} - e \)-graphic, and hence \( \pi \) is potentially \( K_{r,r} - e \)-graphic.

Now assume \( d_r \leq 2r - 2 \). If \( d_{r+t} \leq 2r - 3 - t \) for any \( t \in \{1, 2, \ldots, \frac{r}{2} - 1\} \), then
\[
\sigma(\pi) \leq (n - 1)(r - 1) + (2r - 2) + (2r - 4) + \cdots + (\frac{3}{2} r - 1) + (\frac{3}{2} r - 2)(n - r) + 3)
= \left( \frac{5}{2} r - 2 \right) n - \frac{11}{8} r^2 + \frac{5}{4} r + 2 - (n - r + 1) < \left( \frac{5}{2} r - 2 \right) n - \frac{11}{8} r^2 + \frac{5}{4} r + 2 - (n - r + 1) \leq \sigma(\pi),
\]
a contradiction. Hence there exists an integer \( t \in \{1, 2, \ldots, \frac{r}{2} - 1\} \) such that \( d_{r+t} \geq 2r - 2 - t \). If \( d_{2r} \leq \frac{5}{2} r - 3 \), then \( \sigma(\pi) \leq (n - 1)(r - 1) + (2r - 2)r + (\frac{5}{2} r - 3)(n - 2r + 1) \leq \left( \frac{5}{2} r - 4 \right) n - r^2 + \frac{9}{2} r - 2 < \left( \frac{5}{2} r - 2 \right) n - \frac{11}{8} r^2 + \frac{5}{4} r + 2 - (n - r + 1) \leq \sigma(\pi),
\]
a contradiction. Hence \( d_{2r} \geq \frac{5}{2} r - 2 \). By Lemma 2, \( \pi \) is potentially \( K_{r,r} - e \)-graphic.

Lemma 6. Let \( r \) be even, \( r \geq 4 \) and \( n = r^2 + r - 2 + t \), where \( 0 \leq t \leq 6r^2 - \frac{5}{2} r - 20 \). Then
\[
\sigma(K_{r,r} - e, n) \leq \left( \frac{5}{2} r - 2 \right) n - \frac{11}{8} r^2 + \frac{5}{4} r + 2 - (n - r + 1) + \frac{r}{4} (6r^2 - r - 20) - \frac{1}{4} rt.
\]

Proof. Use induction on \( t \). It is known from Lemma 4 that the result holds for \( t = 0 \). Now assume that the result holds for \( t - 1 \), \( 0 \leq t - 1 \leq 6r^2 - \frac{5}{2} r - 21 \).

Let \( n = r^2 + r - 2 + t \), and \( \pi = (d_1, d_2, \ldots, d_n) \in GS_n \) with \( \sigma(\pi) \geq \left( \frac{5}{2} r - 2 \right) n - \frac{11}{8} r^2 + \frac{5}{4} r + 2 - (n - r + 1) + \frac{t}{4} (6r^2 - \frac{5}{2} r - 20) - \frac{1}{4} rt \). We only need to prove that \( \pi \) is potentially \( K_{r,r} - e \)-graphic. Obviously, \( \sigma(\pi) \geq \left( \frac{5}{2} r - 2 \right) n - \frac{11}{8} r^2 + \frac{5}{4} r + 2 - (n - r + 1) \). If \( d_n \geq r \), then by Lemma 5, \( \pi \) is potentially \( K_{r,r} - e \)-graphic. If \( d_n \leq r - 1 \), then the residual sequence \( \pi' \) of \( \pi \) satisfies \( \sigma(\pi') = \sigma(\pi) - 2d_n \geq \left( \frac{5}{2} r - 2 \right) (n - 1) - \frac{11}{8} r^2 + \frac{5}{4} r + 2 - ((n - 1) - r + 1) + \frac{t}{4} (6r^2 - \frac{5}{2} r - 20) - \frac{1}{4} r(t - 1) \). By the induction hypothesis, \( \pi' \) is potentially \( K_{r,r} - e \)-graphic, and hence so is \( \pi \).

Lemma 7. Let \( r \) be even, \( r \geq 4 \) and \( n = 7r^2 + \frac{1}{2} r - 22 \). Then
\[
\sigma(K_{r,r} - e, n) \leq \left( \frac{5}{2} r - 2 \right) n - \frac{11}{8} r^2 + \frac{5}{4} r + 2 - (n - r + 1).
\]
Proof. It is enough to prove that (•): if \( \pi = (d_1, \ldots, d_n) \in GS_n \) and \( \sigma(\pi) \geq (\frac{5}{2}r - 2)n - \frac{11}{8}r^2 + \frac{5}{2}r + 2 - (n - r + 1) \), then \( \pi \) is potentially \( K_{r,r} - e \)-graphic. Apply induction on \( n \). By Lemma 6, (•) holds for \( n = 7r^2 + \frac{1}{2}r - 22 \). Now suppose that (•) holds for \( n - 1 \geq 7r^2 + \frac{1}{2}r - 22 \). We will prove that (•) holds for \( n \). If \( d_n \geq r \), then by Lemma 5, \( \pi \) is potentially \( K_{r,r} - e \)-graphic. If \( d_n \leq r - 1 \), then the residual sequence \( \pi' \) satisfies \( \sigma(\pi') = \sigma(\pi) - 2d_n \geq (\frac{5}{2}r - 2)(n - 1) - \frac{11}{8}r^2 + \frac{5}{2}r + 2 - ((n - 1) - r + 1) \). By the induction hypothesis, \( \pi' \) and \( \pi \) are potentially \( K_{r,r} - e \)-graphic.

\[ \square \]

**Theorem 6.** Let \( r \geq 4 \) be even, \( 1 \leq k \leq r \) and \( n \geq 7r^2 + \frac{1}{2}r - 22 \). Then

\[
\sigma(K_{r,r} - ke, n) = \begin{cases} 
\left( \frac{5}{2}r - 2 \right)n - \frac{11}{8}r^2 + \frac{5}{2}r + 2 - (n - r + 1), & \text{if } (r, n) \in E_1, \\
\left( \frac{5}{2}r - 2 \right)n - \frac{11}{8}r^2 + \frac{5}{2}r + 1 - (n - r + 1), & \text{if } (r, n) \in E_2.
\end{cases}
\]

Proof. Since \( K_{r,r} - e \) contains \( K_{r,r} - ke \) as a subgraph and \( K_{r,r} - re \) as a subgraph, it is well known that \( \sigma(K_{r,r} - re, n) \leq \sigma(K_{r,r} - ke, n) \leq \sigma(K_{r,r} - e, n) \). By Lemmas 3 and 7, for \( (r, n) \in E_1 \),

\[
\sigma(K_{r,r} - ke, n) = \left( \frac{5}{2}r - 2 \right)n - \frac{11}{8}r^2 + \frac{5}{4}r + 2 - (n - r + 1),
\]

and for \( (r, n) \in E_2 \),

\[
\sigma(K_{r,r} - ke, n) = \left( \frac{5}{2}r - 2 \right)n - \frac{11}{8}r^2 + \frac{5}{2}r + 2 - (n - r + 1).
\]

Since \( \sigma(K_{r,r} - ke, n) \) is even, we have

\[
\sigma(K_{r,r} - ke, n) = \left( \frac{5}{2}r - 2 \right)n - \frac{11}{8}r^2 + \frac{5}{2}r + 1 - (n - r + 1)
\]

for \( (r, n) \in E_2 \). \( \square \)

**3.2 \( \sigma(K_{r,r} - ke, n) \) for odd \( r \) and \( n \geq 7r^2 + 9r - 26 \).**

**Lemma 8.** Let \( r \) be odd, \( r \geq 5 \) and \( n \geq 2r \). Then

\[
\sigma(K_{r,r} - re, n) \geq \begin{cases} 
\left( \frac{5}{2}r - \frac{5}{2} \right)n - \frac{11}{8}r^2 + \frac{5}{2}r + \frac{15}{8} - (n - r + 1), & \text{if } (r, n) \in E_3, \\
\left( \frac{5}{2}r - \frac{5}{2} \right)n - \frac{11}{8}r^2 + \frac{5}{4}r + \frac{3}{2} - (n - r + 1), & \text{if } (r, n) \in E_4.
\end{cases}
\]

Proof. Suppose that \( (r, n) \in E_3 \). Consider \( \pi = ((n - 1)^{-1}, 2r - 3, 2r - 4, \ldots, \frac{3}{2}r - 1, \frac{3}{2}r - \frac{3}{2}, \frac{3}{2}r - \frac{5}{2}, \frac{3}{2}r - \frac{5}{2}n^{-2r+1}) \). Then \( \sigma(\pi) = (\frac{5}{2}r - \frac{5}{2}n) - \frac{11}{8}r^2 + \frac{5}{2}r - \frac{5}{2}r - \frac{1}{2} - (n - r + 1) \) is even and \( f(\pi) = \frac{5}{2}r - \frac{3}{2} \). It follows from the left-most off-diagonal matrix \( A \) of \( \pi \) that \( \pi = (\overline{d_1}, \overline{d_2}, \ldots, \overline{d_n}) \) satisfies \( \overline{d_1} = \overline{d_2} = \cdots = \overline{d_{\frac{3}{2}r - \frac{1}{2}}} = n - 1, \overline{d_{\frac{3}{2}r - \frac{1}{2}}} = 2r - 2 \). Clearly, \( d_1 + d_2 + \cdots + d_i \leq \overline{d_1} + \overline{d_2} + \cdots + \overline{d_i} \) for each \( i = 1, 2, \ldots, f(\pi) \). By Theorem 1, \( \pi \in GS_n \). Let \( \pi_1 = (r - 2, r - 3, \ldots, \frac{r}{2} + \frac{1}{2}, \frac{3}{2}r - \frac{1}{2}, \frac{3}{2}r - \frac{3}{2}, \frac{3}{2}r - \frac{5}{2}, \frac{3}{2}r - \frac{5}{2}n^{-2r+1}) \). If \( \pi \) is potentially \( K_{r,r} - re \)-graphic, then there exist integers \( t \) and \( s, t \geq s \geq 1 \) and \( t + s = r + 1 \) such that \( \pi_1 \) is potentially \( K_{s,t} - se \)-graphic. If \( s \leq \frac{r}{2} + \frac{1}{2} \), then there are at least \( s \) terms in \( \pi_1 \) which are greater than or equal to \( r - s \), which is impossible. If \( s = t = \frac{r}{2} + \frac{1}{2} \), then there are at least \( r + 1 \) terms in \( \pi_1 \) which are greater than or equal to \( \frac{r}{2} - \frac{1}{2} \), which is also impossible. Thus, \( \pi \) is not potentially \( K_{r,r} - re \)-graphic. Hence

\[
\sigma(K_{r,r} - re, n) \geq \sigma(\pi) + 2 = \left( \frac{5}{2}r - \frac{5}{2}n \right)n - \frac{11}{8}r^2 + \frac{5}{2}r + \frac{15}{8} - (n - r + 1).
\]

Now suppose \( (r, n) \in E_4 \). Consider \( \pi = ((n - 1)^{-1}, 2r - 3, 2r - 4, \ldots, \frac{3}{2}r - 1, \frac{3}{2}r - \frac{3}{2}, \frac{3}{2}r - \frac{5}{2}, \frac{3}{2}r - \frac{5}{2}n^{-2r}, \frac{3}{2}r - \frac{7}{2}) \). Then \( \sigma(\pi) = (\frac{5}{2}r - \frac{5}{2}n)n - \frac{11}{8}r^2 + \frac{5}{2}r -
\[
\left(\frac{9}{8} - (n - r + 1)\right) \text{ is even and } f(\pi) = \frac{3}{2}r - \frac{3}{2}. \text{ By the left-most off-diagonal matrix } \\
A \text{ of } \pi, \bar{d} = (\bar{d}_1, \bar{d}_2, \ldots, \bar{d}_n) \text{ satisfies } \bar{d}_1 = \bar{d}_2 = \cdots = \bar{d}_{\frac{3}{2}r - \frac{3}{2}} = n - 1, \ bar{d}_{\frac{3}{2}r - \frac{3}{2}} = n - 2, \bar{d}_{\frac{3}{2}r - \frac{3}{2}} = 2r - 2. \text{ Clearly, } d_1 + d_2 + \cdots + d_i \leq d_1 + d_2 + \cdots + d_i \text{ for each } i = 1, 2, \ldots, f(\pi). \text{ By Theorem 1, } \pi \in GS_n. \text{ Similarly, we can prove that } \\
\pi \text{ is not potentially } K_{r,r} - re-graphic. \text{ Thus, } \sigma(K_{r,r} - re, n) \geq \sigma(\pi) + 2 = \left(\frac{5}{2}r - \frac{5}{2}\right)n - \frac{11}{8}r^2 + \frac{5}{2}r + \frac{7}{8} - (n - r + 1). \text{ } \Box
\]

**Lemma 9.** Let \( r \) be odd, \( r \geq 5 \) and \( n = r^2 + r - 2 \). Then
\[
\sigma(K_{r,r} - e, n) \leq \left(\frac{5}{2}r - \frac{5}{2}\right)n - \frac{11}{8}r^2 + \frac{9}{2}r + \frac{15}{8} - (n - r + 1) + \frac{1}{4}(6r^3 + 2r^2 - 32r + 24).
\]

**Proof.** By Theorem 3,
\[
\sigma(K_{r,r} - e, n) \leq \sigma(K_{2r}, n) \leq (4r - 6)n + 8
\]
\[
= \left(\frac{5}{2}r - \frac{5}{2}\right)n + \left(\frac{3}{2}r - \frac{3}{2}\right)(r^2 + r - 2) + 8
\]
\[
= \left(\frac{5}{2}r - \frac{5}{2}\right)n - \frac{11}{8}r^2 + \frac{5}{2}r + \frac{15}{8} - (n - r + 1) + \left(\frac{3}{2}r^3 + \frac{3}{2}r^2 - 9r + \frac{97}{8}\right)
\]
\[
= \left(\frac{5}{2}r - \frac{5}{2}\right)n - \frac{11}{8}r^2 + \frac{5}{2}r + \frac{15}{8} - (n - r + 1) + \frac{1}{4}(6r^3 + 3r^2 - 36r + \frac{97}{2})
\]
\[
= 2r - \frac{5}{2}n - \frac{11}{8}r^2 + \frac{5}{2}r + \frac{15}{8} - (n - r + 1) + \frac{1}{4}(6r^3 + \frac{9}{2}r^2 - 36r + 44)
\]
\[
\leq \frac{5}{2}(r - \frac{5}{2})n - \frac{11}{8}r^2 + \frac{5}{2}r + \frac{15}{8} - (n - r + 1) + \frac{1}{4}(6r^3 + 2r^2 - 32r + 24). \text{ } \Box
\]

**Lemma 10.** Let \( r \geq 5 \) be odd, \( n \geq r^2 + r - 2 \) and \( \pi = (d_1, d_2, \ldots, d_n) \in GS_n \) with \( d_n \geq r \). If \( \sigma(\pi) \geq \left(\frac{5}{2}r - \frac{5}{2}\right)n - \frac{11}{8}r^2 + \frac{5}{2}r + \frac{15}{8} - (n - r + 1) \), then \( \pi \) is potentially \( K_{r,r} - re-graphic \).

**Proof.** If \( d_r \geq 2r - 1 \), then by \( n \geq (r + 2)(r - 1) \) and Theorems 4 and 5, \( \pi \) is potentially \( K_{r,r} \)-graphic, and hence \( \pi \) is potentially \( K_{r,r} - e \)-graphic. Assume \( d_r \leq 2r - 2 \). If \( d_{r+t} \leq 2r - 3 - t \) for any \( t \in \{1, 2, \ldots, \frac{r}{2} - \frac{3}{2}\} \), then \( \sigma(\pi) \leq (n - 1)(r - 1) + (2r - 2) + (2r - 4) + \cdots + \left(\frac{3}{2}r - \frac{3}{2}\right)n - \frac{3}{2}r^2 + \frac{3}{2}r + \frac{3}{2} < (\frac{3}{2}r - \frac{3}{2})n - \frac{11}{8}r^2 + \frac{5}{2}r + \frac{15}{8} - (n - r + 1) \leq \sigma(\pi) \), a contradiction. Hence there exists \( t \in \{1, 2, \ldots, \frac{r}{2} - \frac{3}{2}\} \) such that \( d_{r+t} \geq 2r - 2 - t \). There are two cases.

**Case 1.** There exists \( t \in \{1, 2, \ldots, \frac{r}{2} - \frac{3}{2}\} \) such that \( d_{r+t} \geq 2r - 2 - t \). If \( d_{2r} \leq \frac{3}{2}r - \frac{7}{2} \), then \( \sigma(\pi) \leq (n - 1)(r - 1) + (2r - 2) + (\frac{3}{2}r - \frac{7}{2})(n - 2r + 1) = (\frac{3}{2}r - \frac{7}{2})n - r^2 + \frac{11}{8}r - \frac{5}{2} < \left(\frac{3}{2}r - \frac{5}{2}\right)n - \frac{11}{8}r^2 + \frac{5}{2}r + \frac{15}{8} - (n - r + 1) \leq \sigma(\pi) \), a contradiction. Hence \( d_{2r} \geq \frac{3}{2}r - \frac{5}{2} \). By Lemma 2, \( \pi \) is potentially \( K_{r,r} - e \)-graphic.

**Case 2.** \( d_{r+t} \leq 2r - 3 - t \) for any \( t \in \{1, 2, \ldots, \frac{r}{2} - \frac{3}{2}\} \) and \( d_{\frac{3}{2}r - \frac{1}{2}} = \frac{3}{2}r - \frac{3}{2} \). If \( d_{2r} \leq \frac{3}{2}r - \frac{5}{2} \), then \( \sigma(\pi) \leq (n - 1)(r - 1) + (2r - 2) + (2r - 4) + \cdots + \left(\frac{3}{2}r - \frac{5}{2}\right)n - \frac{5}{2}r - \frac{5}{2} < \left(\frac{3}{2}r - \frac{5}{2}\right)n - \frac{11}{8}r^2 + \frac{7}{2}r - \frac{1}{2} < (\frac{3}{2}r - \frac{5}{2})n - \frac{11}{8}r^2 + \frac{5}{2}r + \frac{15}{8} - (n - r + 1) \leq \sigma(\pi) \), a contradiction. Hence \( d_{2r} \geq \frac{3}{2}r - \frac{3}{2} \). By Lemma 2, \( \pi \) is potentially \( K_{r,r} - e \)-graphic. \( \Box \)
Lemma 11. Let \( r \) be odd, \( r \geq 5 \) and \( n = r^2 + r - 2 + t \), where \( 0 \leq t \leq 6r^2 + 8r - 24 \). Then
\[
\sigma(K_{r,r} - e, n) \leq \left( \frac{5}{2} r - \frac{5}{2} \right) n - \frac{11}{8} r^2 + \frac{5}{8} r + \frac{15}{8} - (n-r+1) + \frac{r-1}{4} (6r^2 + 8r - 24) - \frac{r-1}{4} t.
\]

Proof. Use induction on \( t \). It follows from Lemma 9 that Lemma 11 holds for \( t = 0 \). Now suppose that Lemma 11 holds for \( 0 \leq t-1 \leq 6r^2 + 8r - 25 \). We will prove that Lemma 11 holds for \( t \). Let \( n = r^2 + r - 2 + t \), and \( \pi = (d_1, d_2, \ldots, d_n) \in GS_n \) with \( \sigma(\pi) \geq \left( \frac{5}{2} r - \frac{5}{2} \right) n - \frac{11}{8} r^2 + \frac{5}{8} r + \frac{15}{8} - (n-r+1) + \frac{r-1}{4} (6r^2 + 8r - 24) - \frac{r-1}{4} t \). Clearly, \( \sigma(\pi) \geq \left( \frac{5}{2} r - \frac{5}{2} \right) n - \frac{11}{8} r^2 + \frac{5}{8} r + \frac{15}{8} - (n-r+1) \). If \( d_n \geq r \), then by Lemma 10, \( \pi \) is potentially \( K_{r,r} - e \)-graphic. If \( d_n \leq r - 1 \), then the residual sequence \( \pi' \) satisfies \( \sigma(\pi') = \sigma(\pi) - 2d_n \geq \left( \frac{5}{2} r - \frac{5}{2} \right) (n-1) - \frac{11}{8} r^2 + \frac{5}{8} r + \frac{15}{8} - \left( (n-1) - r + 1 \right) + \frac{r-1}{4} (6r^2 + 8r - 24) - \frac{r-1}{4} (t-1). \) By the induction hypothesis, \( \pi' \) and \( \pi \) are potentially \( K_{r,r} - e \)-graphic. Thus, \( \sigma(K_{r,r} - e, n) \leq \left( \frac{5}{2} r - \frac{5}{2} \right) n - \frac{11}{8} r^2 + \frac{5}{8} r + \frac{15}{8} - (n-r+1) + \frac{r-1}{4} (6r^2 + 8r - 24) - \frac{r-1}{4} t. \) \( \square \)

Lemma 12. Let \( r \) be odd, \( r \geq 5 \) and \( n \geq 7r^2 + 9r - 26 \). Then
\[
\sigma(K_{r,r} - e, n) \leq \left( \frac{5}{2} r - \frac{5}{2} \right) n - \frac{11}{8} r^2 + \frac{5}{8} r + \frac{15}{8} - (n-r+1).
\]

Proof. It is enough to prove that \( (*) \): if \( \pi = (d_1, d_2, \ldots, d_n) \in GS_n \) with \( \sigma(\pi) \geq \left( \frac{5}{2} r - \frac{5}{2} \right) n - \frac{11}{8} r^2 + \frac{5}{8} r + \frac{15}{8} - (n-r+1) \), then \( \pi \) is potentially \( K_{r,r} - e \)-graphic. Apply induction on \( n \). By Lemma 11, \( (*) \) holds for \( n = 7r^2 + 9r - 26 \). Now suppose that \( (*) \) holds for \( n-1 \geq 7r^2 + 9r - 26 \). We will prove that \( (*) \) holds for \( n \). If \( d_n \geq r \), then by Lemma 10, \( \pi \) is potentially \( K_{r,r} - e \)-graphic. If \( d_n \leq r - 1 \), then residual sequence \( \pi' \) satisfies \( \sigma(\pi') = \sigma(\pi) - 2d_n \geq \left( \frac{5}{2} r - \frac{5}{2} \right) (n-1) - \frac{11}{8} r^2 + \frac{5}{8} r + \frac{15}{8} - ((n-1) - r + 1). \) By the induction hypothesis, \( \pi' \) and \( \pi \) are potentially \( K_{r,r} - e \)-graphic. \( \square \)

Theorem 7. Let \( r \geq 5 \) be odd, \( 1 \leq k \leq r \) and \( n \geq 7r^2 + 9r - 26 \). Then
\[
\sigma(K_{r,r} - ke, n) = \begin{cases} \left( \frac{5}{2} r - \frac{5}{2} \right) n - \frac{11}{8} r^2 + \frac{5}{8} r + \frac{15}{8} - (n-r+1), & \text{if } (r,n) \in E_3, \\ \left( \frac{5}{2} r - \frac{5}{2} \right) n - \frac{11}{8} r^2 + \frac{5}{8} r + \frac{15}{8} - (n-r+1), & \text{if } (r,n) \in E_4. \end{cases}
\]

Proof. By Lemmas 8 and 12, \( \sigma(K_{r,r} - ke, n) = \left( \frac{5}{2} r - \frac{5}{2} \right) n - \frac{11}{8} r^2 + \frac{5}{8} r + \frac{15}{8} - (n-r+1) \) for \( (r,n) \in E_3 \) and \( \left( \frac{5}{2} r - \frac{5}{2} \right) n - \frac{11}{8} r^2 + \frac{5}{8} r + \frac{15}{8} - (n-r+1) \leq \sigma(K_{r,r} - ke, n) \leq \left( \frac{5}{2} r - \frac{5}{2} \right) n - \frac{11}{8} r^2 + \frac{5}{8} r + \frac{15}{8} - (n-r+1) \) for \( (r,n) \in E_4 \). Since \( \sigma(K_{r,r} - ke, n) \) is even, we have \( \sigma(K_{r,r} - ke, n) = \left( \frac{5}{2} r - \frac{5}{2} \right) n - \frac{11}{8} r^2 + \frac{5}{8} r + \frac{15}{8} - (n-r+1) \) for \( (r,n) \in E_4. \) \( \square \)
REFERENCES


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