AN ITERATIVE METHOD FOR EQUILIBRIUM PROBLEMS, VARIATIONAL INEQUALITY PROBLEMS AND FIXED POINT PROBLEMS

MEIJUAN SHANG* AND YONGFU SU

ABSTRACT. In this paper, we introduce an iterative scheme for finding a common element of the set of fixed points of a nonexpansive mapping, the set of solutions of the variational inequality for an inverse-strongly monotone mapping and the set of solutions of an equilibrium problem in a Hilbert space. We show that the iterative sequence converges strongly to a common element of the three sets. The results of this paper extend and improve the corresponding results announced by many others.

AMS Mathematics Subject Classification : 47H05; 4709.
Key words and phrases : Nonexpansive mapping; inverse-strongly monotone mapping; variational inequality; equilibrium problem

1. Introduction

Let $H$ be a real Hilbert space with inner product $\langle \cdot , \cdot \rangle$ and norm $\| \cdot \|$, respectively. Let $C$ a nonempty closed convex subset of $H$ and let $P_C$ be the metric projection of $H$ onto $C$. Recall that a mapping $S : C \to C$ is called nonexpansive if

$$\| Sx - Sy \| \leq \| x - y \|$$

for all $x, y \in C$. We denote by $F(S)$ the set of fixed points of $S$. Recall that a mapping $A : C \to H$ is called monotone if for all $x, y \in C$, $\langle x - y, Ax - Ay \rangle \geq 0$.

Recall that a mapping $A : C \to H$ is called inverse-strongly monotone if there exists a positive real number $\alpha$ such that

$$\langle x - y, Ax - Ay \rangle \geq \alpha \| Ax - Ay \|^2$$
for all \( x, y \in C \); see [4,8,14]. For such a case, \( A \) is called \( \alpha \)-inverse-strongly monotone. If \( A \) is an \( \alpha \)-inverse-strongly monotone mapping of \( C \) into \( H \), then it is obvious that \( A \) is \( 1/\alpha \)-Lipschitz continuous.

The classical variational inequality problem is to find a \( u \in C \) such that \( \langle v - u, Au \rangle \geq 0 \) for all \( v \in C \). We denoted by \( VI(A, C) \) the set of solutions of the variational inequality problem. The variational inequality has been extensively studied in the literature. See [2,3,12,13,15] and the references therein.

Let \( F \) be a bifunction of \( C \times C \) into \( \mathbb{R} \), where \( \mathbb{R} \) is the set of real numbers. The equilibrium problem for \( F : C \times C \rightarrow \mathbb{R} \) is to find \( x \in C \) such that

\[
F(x, y) \geq 0 \quad \text{for all } y \in C. \tag{1.1}
\]

The set of solutions of (1.1) is denoted by \( EP(F) \). Given a mapping \( T : C \rightarrow H \), let \( F(x, y) = \langle Tx, y - x \rangle \) for all \( x, y \in C \). Then, \( z \in EP(F) \) if and only if \( \langle Tz, y - z \rangle \geq 0 \) for all \( y \in C \), i.e., \( z \) is a solution of the variational inequality. Numerous problems in physics, optimization and economics reduce to find a solution of (1.1). Some methods have been proposed to solve the equilibrium problem; see, for instance, [5,6,12,13].

For finding an element of \( F(S) \cap VI(C, A) \), Takahashi and Toyoda [16] introduced the following iterative scheme: \( x_1 \in C \) and

\[
x_{n+1} = \alpha_n x_n + (1 - \alpha_n)SP_C(x_n - \lambda_n Ax_n), \quad n \geq 1 \tag{1.2}
\]

and obtain a weak convergence theorem in a Hilbert space. Recently, Iiduka and Takahashi [7] proposed a new iterative scheme: \( x_1 = x \in C \) and

\[
x_{n+1} = \alpha_n x + (1 - \alpha_n)SP_C(x_n - \lambda_n Ax_n), \quad n \geq 1 \tag{1.3}
\]

and obtained a strong convergence theorem in a Hilbert space.

On the other hand, for finding an element of \( EP(F) \cap F(S) \), Takahashi and Takahashi [13] introduced the following iterative scheme by the viscosity approximation method in a Hilbert space: \( x_1 \in H \) and

\[
\begin{cases}
F(u_n, y) + \frac{1}{r_n}(y - u_n, u_n - x_n) \geq 0, \quad \forall y \in C, \\
x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Su_n, \quad n \geq 1,
\end{cases} \tag{1.4}
\]

for all \( n \in \mathbb{N} \), where \( \{\alpha_n\} \subset [0, 1] \) and \( \{r_n\} \subset (0, \infty) \) satisfy some appropriate conditions. Further, They proved \( \{x_n\} \) and \( \{u_n\} \) converge strongly to \( z \in F(S) \cap EP(F) \), where \( z = P_{F(S) \cap EP(F)} f(z) \).

Very recently, Su et al. [12] introduced a new iterative scheme for finding a common element of \( F(S) \cap VI(C, A) \cap EP(F) \) given as follows: \( x_1 \in H \) and

\[
\begin{cases}
F(u_n, y) + \frac{1}{r_n}(y - u_n, u_n - x_n) \geq 0, \quad \forall y \in C, \\
x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)SP_C(u_n - \lambda_n Au_n), \quad n \geq 1.
\end{cases}
\]

\[
\begin{cases}
F(u_n, y) + \frac{1}{r_n}(y - u_n, u_n - x_n) \geq 0, \quad \forall y \in C, \\
x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)SP_C(u_n - \lambda_n Au_n), \quad n \geq 1.
\end{cases}
\]
In this paper, motivated and inspired by the above results, we introduce a new following iterative scheme: \( x_1 \in H \) and
\[
\begin{aligned}
F(u_n, y) + \frac{1}{\lambda_n}(y - u_n, u_n - x_n) &\geq 0, \ \forall y \in C, \\
x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n SP_C(u_n - \lambda_n Au_n), \ n \geq 1,
\end{aligned}
\tag{1.5}
\]
for finding a common element of the set of fixed points of a nonexpansive mapping, the set of solutions of a variational inequality for an \( \alpha \)-inverse-strongly monotone mapping and the set of solutions of an equilibrium problem in a real Hilbert space. Furthermore, we show the iterative sequences \( \{x_n\} \) and \( \{u_n\} \) converge strongly to \( z \in F(S) \cap VI(C, A) \cap EP(F) \), where \( z = P_{F(S) \cap VI(C, A) \cap EP(F)} f(z) \). The results of this paper extended and improved the results of Iiduka and Takahashi [7], Takahashi and Takahashi [13] and Su et al. [12].

2. Preliminaries

Let \( H \) be a real Hilbert space with inner product \( \langle \cdot, \cdot \rangle \) and norm \( \| \cdot \| \), respectively. It is well known that for all \( x, y \in H \) and \( \lambda \in [0,1] \) there holds
\[
\| \lambda x + (1 - \lambda)y \| = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2.
\]
Let \( C \) be a nonempty closed convex subset of \( H \). We denote by "→" strong convergence and "→w" weak convergence. For every point \( x \in H \), there exists a unique nearest point in \( C \), denoted by \( P_C x \), such that
\[
\|x - P_C x\| \leq \|x - y\| \quad \text{for all } y \in C.
\]
\( P_C \) is called the metric projection of \( H \) onto \( C \). We know that \( P_C \) is a nonexpansive mapping of \( H \) onto \( C \). It is also known that \( P_C \) satisfies
\[
\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y \rangle,
\tag{2.1}
\]
for every \( x, y \in H \). Moreover, \( P_C x \) is characterized by the following properties:
\( P_C x \in C \) and \( \langle x - P_C x, P_C x - y \rangle \geq 0 \), for all \( y \in C \).

In the context of the variational inequality problem, this implies that
\[
u \in VI(C, A) \iff u = P_C(u - \lambda Au), \text{ for all } \lambda > 0.
\tag{2.2}
\]

A space \( X \) is said to satisfy Opial's condition [9] if for each sequence \( \{x_n\}_{n=1}^{\infty} \) in \( X \) which converges weakly to point \( x \in X \), we have
\[
\lim_{n \to \infty} \|x_n - x\| < \lim_{n \to \infty} \|x_n - y\|, \ \forall y \in X, \ y \neq x.
\]

A set-valued mapping \( T : H \to \mathcal{P}^H \) is called monotone if for all \( x, y \in H \), \( f \in Tx \) and \( g \in Ty \) imply \( \langle x - y, f - g \rangle \geq 0 \). A monotone mapping \( T : H \to \mathcal{P}^H \) is maximal if the graph \( G(T) \) of \( T \) is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping \( T \) is maximal if and only if for \( (x, f) \in H \times H, \langle x - y, f - g \rangle \geq 0 \) for every \( (y, g) \in G(T) \) implies \( f \in Tx \). Let \( A \) be an inverse-strongly monotone mapping of \( C \) into \( H \) and let \( N_C v \) be the normal cone to \( C \) at \( v \in C \), i.e.,
\[
N_C v = \{w \in H : \langle v - u, w \rangle \geq 0, \forall u \in C\}.
\]
and define

\[ T_v = \begin{cases} 
  Av + N_cv, & v \in C, \\
  \emptyset, & v \notin C.
\end{cases} \]

Then \( T \) is maximal monotone and \( 0 \in T_v \) if and only if \( v \in VI(C, A) \); see [10,11].

For solving the equilibrium problem for a bifunction \( F : C \times C \rightarrow \mathbb{R} \), let us assume that \( F \) satisfies the following conditions:

(A1) \( F(x, x) = 0 \) for all \( x \in C \);

(A2) \( F \) is monotone, i.e., \( F(x, y) + F(y, x) \leq 0 \) for all \( x, y \in C \);

(A3) for each \( x, y, z \in C \),

\[ \lim_{t \downarrow 0} F(tz + (1 - t)x, y) \leq F(x, y); \]

(A4) for each \( x \in C \), \( y \mapsto F(x, y) \) is convex and lower semicontinuous.

We need the following lemmas for the proof of our main results.

**Lemma 2.1** [18]. Let \( \{x_n\} \) and \( \{y_n\} \) be bounded sequences in a Banach space \( X \) and let \( \{\beta_n\} \) be a sequence in \( [0,1] \) with \( 0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1 \). Suppose \( x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n \) for all integers \( n \geq 0 \) and \( \limsup_{n \rightarrow \infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0 \). Then, \( \lim_{n \rightarrow \infty} \|y_n - x_n\| = 0 \).

**Lemma 2.2** [17]. Assume \( \{\alpha_n\} \) is a sequence of nonnegative real numbers such that

\[ \alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \delta_n, \quad n \geq 0, \]

where \( \{\gamma_n\} \) is a sequence in \( (0,1) \) and \( \{\delta_n\} \) is a sequence in \( \mathbb{R} \) such that

(i) \( \sum_{n=1}^{\infty} \gamma_n = \infty \);

(ii) \( \limsup_{n \rightarrow \infty} \delta_n/\gamma_n \leq 0 \) or \( \sum_{n=1}^{\infty} |\delta_n| < \infty \).

Then \( \lim_{n \rightarrow \infty} \alpha_n = 0 \).

**Lemma 2.3** [1]. Let \( C \) be a nonempty closed convex subset of \( H \) and let \( F \) be a bifunction of \( C \times C \) into \( \mathbb{R} \) satisfying (A1) – (A4). Let \( r > 0 \) and \( x \in H \). Then, there exists \( z \in C \) such that

\[ F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \quad \forall y \in C. \]

**Lemma 2.4** [6]. Assume that \( F : C \times C \rightarrow \mathbb{R} \) satisfies (A1) – (A4). For \( r > 0 \) and \( x \in H \), define a mapping \( T_r : H \rightarrow C \) as follows:

\[ T_r(x) = \{ z \in C : F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \quad \forall y \in C \} \]

for all \( x \in H \). Then, the following hold:

(1) \( T_r \) is single-valued;

(2) \( T_r \) is firmly nonexpansive, i.e., for any \( x, y \in H \),

\[ \|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y \rangle; \]

(3) \( F(T_r) = EP(F) \);
(4) \( EP(F) \) is closed and convex.

### 3. Main results

In this section, we introduce an iterative method by the viscosity approximation method for finding a common element of the set of fixed points of a nonexpansive mapping, the set of solutions of the variational inequality for an inverse-strongly monotone mapping and the set of solutions of an equilibrium problem in a Hilbert space. We show that the iterative sequence converges strongly to a common element of the three sets.

**Theorem 3.1.** Let \( C \) be a nonempty closed convex subset of a real Hilbert space \( H \). Let \( F \) be a bifunction from \( C \times C \) to \( \mathbb{R} \) satisfying (A1) – (A4) and \( S \) be a nonexpansive mapping of \( C \) into \( H \). Let \( A \) be an \( \alpha \)-inverse-strongly monotone mapping of \( C \) into \( H \) such that \( F(S) \cap VI(C, A) \cap EP(F) \neq \emptyset \). Let \( f \) be a contraction of \( H \) into itself and let \( \{x_n\} \) and \( \{u_n\} \) be sequences generated by (1.5), where \( \{\alpha_n\}, \{\beta_n\} \) and \( \{\gamma_n\} \) are three sequences in \([0, 1]\), \( \{\lambda_n\} \subset [0, 2\alpha) \) and \( \{r_n\} \subset (0, \infty) \) satisfy the following conditions:

1. \( \alpha_n + \beta_n + \gamma_n = 1 \),
2. \( \lim_{n \to \infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty \),
3. \( 0 < \lim \inf_{n \to \infty} \beta_n \leq \lim \sup_{n \to \infty} \beta_n < 1 \),
4. \( \{\lambda_n\} \subset [a, b] \) for some \( a, b \in (0, 2\alpha) \) and \( \lim_{n \to \infty} |\lambda_{n+1} - \lambda_n| = 0 \),
5. \( \lim \inf_{n \to \infty} r_n > 0 \) and \( \lim_{n \to \infty} r_{n+1} - r_n = 0 \).

Then \( \{x_n\} \) and \( \{u_n\} \) converge strongly to \( z \in F(S) \cap VI(C, A) \cap EP(F) \), where \( z = P_{F(S) \cap VI(C, A) \cap EP(F)} f(z) \).

**Proof.** Let \( Q = P_{F(S) \cap EP(F) \cap VI(C, A)} \). Then \( Qf \) is a contraction of \( H \) into \( C \). In fact, there exists \( a \in [0, 1) \) such that \( \|f(x) - f(y)\| \leq a\|x - y\| \) for all \( x, y \in H \). So, we have that

\[
\|Qf(x) - Qf(y)\| \leq \|f(x) - f(y)\| \leq a\|x - y\|
\]

for all \( x, y \in H \). So, \( Qf \) is a contraction of \( H \) into \( C \). Since \( H \) is complete, there exists a unique element of \( C \), such that \( z = Qf(z) \). Such a \( z \in H \) is an element of \( C \). For all \( x, y \in C \) and \( \lambda_n \in [0, 2\alpha] \),

\[
\|(I - \lambda_n A)x - (I - \lambda_n A)y\|^2 = \|(x - y) - \lambda_n (Ax - Ay)\|^2 \\
= \|x - y\|^2 - 2\lambda_n (x - y, Ax - Ay) + \lambda_n^2 \|Ax - Ay\|^2 \\
\leq \|x - y\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Ax - Ay\|^2
\]

(3.1)

which implies that \( I - \lambda_n A \) is nonexpansive. Let \( v \in F(S) \cap EP(F) \cap VI(C, A) \). Then \( v = P_C(v - \lambda_n Av) \), for all \( \lambda > 0 \). Put \( w_n = P_C(u_n - \lambda_n Au_n) \), for all \( n \geq 1 \). We compute that

\[
\|w_n - v\| = \|P_C(u_n - \lambda_n Au_n) - P_C(v - \lambda_n Av)\| \\
\leq \|(u_n - \lambda_n Au_n) - (v - \lambda_n Av)\| \\
\leq \|u_n - v\|.
\]

(3.2)
From $u_n = T_n x_n$, we have
\[ \|u_n - v\| = \|T_n x_n - T_n v\| \leq \|x_n - v\|, \text{ for all } n \geq 1. \] \tag{3.3}
Then we compute that
\[
\|x_{n+1} - v\| = \|\alpha_n(f(x_n) - v) + \beta_n(x_n - v) + \gamma_n(Sw_n - v)\|
\leq \alpha_n\|f(x_n) - v\| + \beta_n\|x_n - v\| + \gamma_n\|w_n - v\|
\leq \alpha_n a \|x_n - v\| + \alpha_n \|f(v) - v\| + (1 - \alpha_n)\|x_n - v\|
\leq \max\{\|x_n - v\|, \frac{1}{1 - a} \|f(v) - v\|\}.
\]
By induction, we get
\[ \|x_n - v\| \leq \max\{\|x_1 - v\|, \frac{1}{1 - a} \|f(v) - v\|\}, \ n \geq 1. \]
Therefore, $\{x_n\}$ is bounded. $\{u_n\}$, $\{Sw_n\}$ and $\{f(x_n)\}$ are also bounded. Next, we show that $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$. Since $I - \lambda_n A$ is nonexpansive, we have
\[
\|u_{n+1} - w_n\| = \|PC[(I - \lambda_{n+1} A)u_{n+1}] - PC[(I - \lambda_n A)u_n]\|
\leq \|(I - \lambda_{n+1} A)u_{n+1} - (I - \lambda_n A)u_n\|
\leq \|(I - \lambda_{n+1} A)u_{n+1} - (I - \lambda_{n+1} A)u_n\| + \lambda_n - \lambda_{n+1}\|Au_n\|
\leq \|u_{n+1} - u_n\| + \lambda_n - \lambda_{n+1}\|Au_n\|.
\tag{3.4}
\]
On the other hand, from $u_n = T_n x_n$ and $u_{n+1} = T_{n+1} x_{n+1}$, we have
\[ F(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0 \quad \text{for all } y \in C \] \tag{3.5}
and
\[ F(u_{n+1}, y) + \frac{1}{r_{n+1}} \langle y - u_{n+1}, u_{n+1} - x_{n+1} \rangle \geq 0 \quad \text{for all } y \in C \] \tag{3.6}
Putting $y = u_{n+1}$ in (3.5) and $y = u_n$ in (3.6), we have
\[ F(u_n, u_{n+1}) + \frac{1}{r_n} \langle u_{n+1} - u_n, u_n - x_n \rangle \geq 0 \]
and
\[ F(u_{n+1}, u_n) + \frac{1}{r_{n+1}} \langle u_n - u_{n+1}, u_{n+1} - x_{n+1} \rangle \geq 0. \]
So, from (A2) we have
\[ \langle u_{n+1} - u_n, \frac{u_n - x_n}{r_n} - \frac{u_{n+1} - x_{n+1}}{r_{n+1}} \rangle \geq 0 \]
and hence
\[ \langle u_{n+1} - u_n, u_n - u_{n+1} + u_{n+1} - x_n - \frac{r_n}{r_{n+1}}(u_{n+1} - x_{n+1}) \rangle \geq 0. \]
Without loss of generality, let us assume that there exists a real number $b$ such that $r_n > b > 0$ for all $n \in \mathbb{N}$. Then, we have
\[
\|u_{n+1} - u_n\|^2 \leq \langle u_{n+1} - u_n, x_{n+1} - x_n + (1 - \frac{r_n}{r_{n+1}})(u_{n+1} - x_{n+1}) \rangle
\leq \|u_{n+1} - u_n\| \left\{ \|x_{n+1} - x_n\| + |1 - \frac{r_n}{r_{n+1}}|\|u_{n+1} - x_{n+1}\| \right\}.
and hence
\[ \|u_{n+1} - u_n\| \leq \|x_{n+1} - x_n\| + \frac{1}{r_{n+1}}|r_{n+1} - r_n|\|u_{n+1} - x_{n+1}\| \]
\[ \leq \|x_{n+1} - x_n\| + \frac{1}{b}|r_{n+1} - r_n|L, \]
\[ (3.7) \]
where \( L = \sup\{|u_n - x_n| : n \in \mathbb{N}\} \). From (3.4) we have
\[ \|w_{n+1} - w_n\| \leq \|x_{n+1} - x_n\| + \frac{1}{b}|r_{n+1} - r_n|L + |\lambda_n - \lambda_{n+1}|\|Au_n\|. \]
\[ (3.8) \]
Define \( x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n, \ n \geq 0 \). Observe that from the definition of \( y_n \), we obtain
\[ y_{n+1} - y_n = \frac{x_{n+2} - \beta_{n+1}x_{n+1}}{1 - \beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n} \]
\[ = \frac{\alpha_{n+1}f(x_{n+1}) + \gamma_n Sw_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_nf(x_n) + \gamma_n Sw_n}{1 - \beta_n} \]
\[ = \frac{\alpha_{n+1}f(x_{n+1})}{1 - \beta_{n+1}} - \frac{\alpha_nf(x_n)}{1 - \beta_n} + \frac{\gamma_{n+1}Sw_{n+1} - \gamma_n Sw_n}{1 - \beta_{n+1}} \]
\[ = \frac{\alpha_{n+1}f(x_{n+1})}{1 - \beta_{n+1}} - \frac{\alpha_nf(x_n)}{1 - \beta_n} + (1 - \frac{\alpha_{n+1}}{1 - \beta_{n+1}})(Sw_{n+1} - Sw_n) \]
\[ + \frac{(\alpha_n - \alpha_{n+1})}{1 - \beta_{n+1}}Sw_n. \]
\[ (3.9) \]
Combining (3.8) and (3.9), we have
\[ \|y_{n+1} - y_n\| - \|x_{n+1} - x_n\| \]
\[ \leq \frac{\alpha_{n+1}}{1 - \beta_{n+1}}\|f(x_{n+1})\| + \frac{\alpha_n}{1 - \beta_n}\|f(x_n)\| + (1 - \frac{\alpha_{n+1}}{1 - \beta_{n+1}})\|w_{n+1} - w_n\| \]
\[ + |\frac{\alpha_n}{1 - \beta_n} - \frac{\alpha_{n+1}}{1 - \beta_{n+1}}|\|Sw_n\| - \|x_{n+1} - x_n\| \]
\[ \leq \frac{\alpha_{n+1}}{1 - \beta_{n+1}}\|f(x_{n+1})\| + \frac{\alpha_n}{1 - \beta_n}\|f(x_n)\| - \frac{\alpha_{n+1}}{1 - \beta_{n+1}}\|x_{n+1} - x_n\| \]
\[ + \frac{1}{b}|r_{n+1} - r_n|L + |\lambda_n - \lambda_{n+1}|\|Au_n\| + |\frac{\alpha_n}{1 - \beta_n} - \frac{\alpha_{n+1}}{1 - \beta_{n+1}}|\|Sw_n\|. \]
This together with the conditions (2)-(5) implies that
\[ \limsup_{n \to \infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0. \]
Hence using Lemma 2.1, we have \( \lim_{n \to \infty}\|y_n - x_n\| = 0 \). Consequently, we can obtain
\[ \lim_{n \to \infty}\|x_{n+1} - x_n\| = \lim_{n \to \infty}(1 - \beta_n)\|y_n - x_n\| = 0. \]
\[ (3.10) \]
From (3.7) and \( \sum_{n=1}^{\infty}|r_{n+1} - r_n| < \infty \), we have \( \lim_{n \to \infty}\|u_{n+1} - u_n\| = 0 \). From (3.4) and \( \sum_{n=1}^{\infty}|\lambda_{n+1} - \lambda_n| < \infty \), we also have
\[ \lim_{n \to \infty}\|w_{n+1} - w_n\| = 0. \]
\[ (3.11) \]
Next we show \(\|Sw_n - w_n\| \to 0\), as \(n \to \infty\). First, we show \(\|x_n - Sw_n\| \to 0\), as \(n \to \infty\). Notice that

\[
\|x_n - Sw_n\| \leq \|x_n - Sw_{n-1}\| + \|Sw_{n-1} - Sw_n\|
\leq \alpha_{n-1}\|f(x_{n-1}) - Sw_{n-1}\| + \|w_n - w_{n-1}\|.
\]

From \(\alpha_n \to 0\) and (3.11), we get

\[
\lim_{n \to \infty} \|x_n - Sw_n\| = 0. \tag{3.12}
\]

In sequence, we show \(\|x_n - u_n\| \to 0\), as \(n \to \infty\). For \(v \in F(S) \cap FP(F) \cap VI(C, A)\), we have

\[
\|u_n - v\|^2 = \|T_{r_n} x_n - T_{r_n} v\|^2 \leq \langle T_{r_n} x_n - T_{r_n} v, x_n - v \rangle
= \langle u_n - v, x_n - v \rangle
= \frac{1}{2}(\|u_n - v\|^2 + \|x_n - v\|^2 - \|u_n - x_n\|^2)
\]

and hence \(\|u_n - v\|^2 \leq \|x_n - v\|^2 - \|x_n - u_n\|^2\). Therefore, from the convexity of \(\|\cdot\|^2\), we have

\[
\|x_{n+1} - v\|^2 = \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n Sw_n - v\|^2
\leq \alpha_n\|f(x_n) - v\|^2 + \beta_n\|x_n - v\|^2 + \gamma_n\|u_n - v\|^2
\leq \alpha_n\|f(x_n) - v\|^2 + \beta_n\|x_n - v\|^2 + \gamma_n(\|x_n - v\|^2 - \|x_n - u_n\|^2)
= \alpha_n\|f(x_n) - v\|^2 + (1 - \alpha_n)\|x_n - v\|^2 - \gamma_n\|x_n - u_n\|^2
\]

and hence

\[
\gamma_n\|x_n - u_n\|^2
\leq \alpha_n\|f(x_n) - v\|^2 + \|x_n - v\|^2 - \|x_{n+1} - v\|^2
\leq \alpha_n\|f(x_n) - v\|^2 + \|x_n - x_{n+1}\|(\|x_n - v\| + \|x_{n+1} - v\|).
\]

So, from (3.10) and condition (1)-(3), we have

\[
\|x_n - u_n\| \to 0, \quad \text{as } n \to \infty. \tag{3.13}
\]

Next we show \(\|u_n - w_n\| \to 0\). For \(v \in F(S) \cap FP(F) \cap VI(C, A)\), we compute that

\[
\|x_{n+1} - v\|^2
\leq \alpha_n\|f(x_n) - v\|^2 + \beta_n\|x_n - v\|^2 + \gamma_n\|w_n - v\|^2
\leq \alpha_n\|f(x_n) - v\|^2 + \beta_n\|x_n - v\|^2 + \gamma_n\|(I - \lambda_n A)u_n - (I - \lambda_n A)v\|^2
\leq \alpha_n\|f(x_n) - v\|^2 + \beta_n\|x_n - v\|^2 + \gamma_n(\|u_n - v\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Au_n - Av\|^2)
\leq \alpha_n\|f(x_n) - v\|^2 + (1 - \alpha_n)\|x_n - v\|^2 + a(b - 2\alpha)\|Au_n - Av\|^2
\]
Therefore, we have
\[
-a(b - 2\alpha)\left\| Au_n - Av \right\|^2 
\leq \alpha_n \left\| f(x_n) - v \right\|^2 + \left\| x_n - v \right\|^2 - \left\| x_{n+1} - v \right\|^2 
\leq \alpha_n \left\| f(x_n) - v \right\|^2 + (\left\| x_n - v \right\| + \left\| x_{n+1} - v \right\|) \left\| x_{n+1} - x_n \right\|.
\]
Since \( \alpha_n \to 0 \), \( a, b \in (0, 2\alpha) \) and \( \left\| x_{n+1} - x_n \right\| \to 0 \), we obtain
\[
\lim_{n \to \infty} \left\| Au_n - Av \right\| = 0. \tag{3.14}
\]

From (2.1), we have
\[
\left\| w_n - v \right\|^2 = \left\| P_C(u_n - \lambda_n Au_n) - P_C(v - \lambda_n Av) \right\|^2 
\leq \left\langle (u_n - \lambda_n Au_n) - (v - \lambda_n Av), w_n - v \right\rangle 
= \frac{1}{2} \left\{ \left\| (u_n - \lambda_n Au_n) - (v - \lambda_n Av) \right\|^2 + \left\| w_n - v \right\|^2 
- \left\| (u_n - \lambda_n Au_n) - (v - \lambda_n Av) - (w_n - v) \right\|^2 \right\} 
\leq \frac{1}{2} \left\{ \left\| u_n - v \right\|^2 + \left\| w_n - v \right\|^2 - \left\| (u_n - w_n) - \lambda_n (Au_n - Av) \right\|^2 \right\} 
= \frac{1}{2} \left\{ \left\| u_n - v \right\|^2 + \left\| w_n - v \right\|^2 - \left\| u_n - w_n \right\|^2 
+ 2\lambda_n \left\langle u_n - w_n, Au_n - Av \right\rangle - \lambda_n^2 \left\| Au_n - Av \right\|^2 \right\}.
\]

So, we obtain
\[
\left\| w_n - v \right\|^2 
\leq \left\| u_n - v \right\|^2 - \left\| u_n - w_n \right\|^2 + 2\lambda_n \left\langle u_n - w_n, Au_n - Av \right\rangle - \lambda_n^2 \left\| Au_n - Av \right\|^2.
\]

Hence, we have
\[
\left\| x_{n+1} - v \right\|^2 = \left\| \alpha_n f(x_n) + \beta_n x_n + \gamma_n Sw_n - v \right\|^2 
\leq \alpha_n \left\| f(x_n) - v \right\|^2 + \beta_n \left\| x_n - v \right\|^2 + \gamma_n \left\| w_n - v \right\|^2 
\leq \alpha_n \left\| f(x_n) - v \right\|^2 + \beta_n \left\| x_n - v \right\|^2 + \gamma_n \left( \left\| u_n - v \right\|^2 - \left\| u_n - w_n \right\|^2 
+ 2\lambda_n \left\langle u_n - w_n, Au_n - Av \right\rangle - \lambda_n^2 \left\| Au_n - Av \right\|^2 \right) 
\leq \alpha_n \left\| f(x_n) - v \right\|^2 + \left\| x_n - v \right\|^2 - \gamma_n \left\| u_n - w_n \right\|^2 
+ 2\gamma_n \lambda_n \left\langle u_n - w_n, Au_n - Av \right\rangle - \lambda_n^2 \lambda_n \left\| Au_n - Av \right\|^2.
\]

From \( \alpha_n \to 0 \), (3.10) and (3.14), we obtain
\[
\lim_{n \to \infty} \left\| u_n - w_n \right\| = 0. \tag{3.15}
\]

From \( \left\| Sw_n - w_n \right\| \leq \left\| Sw_n - x_n \right\| + \left\| x_n - u_n \right\| + \left\| u_n - w_n \right\| \), we obtain
\[
\lim_{n \to \infty} \left\| Sw_n - w_n \right\| = 0. \tag{3.16}
\]

Next we show that
\[
\limsup_{n \to \infty} \left\langle f(z) - z, x_n - z \right\rangle \leq 0, \tag{3.17}
\]
where \( z = P_{F(S)} \cap VI(C, A) \cap EP(F) f(z) \). To show it, choose a subsequence \( \{w_{n_i}\} \) of \( \{w_n\} \) such that
\[
\limsup_{n \to \infty} (f(z) - z, Sw_n - z) = \lim_{i \to \infty} (f(z) - z, Sw_{n_i} - z).
\]
As \( \{w_{n_i}\} \) is bounded, we have that a subsequence \( \{w_{n_{i_j}}\} \) of \( \{w_{n_i}\} \) converges weakly to \( w \). We may assume without loss of generality that \( w_{n_i} \to w \). From (3.16), we obtain \( Sw_{n_i} \to w \). Then we can obtain \( w \in F(S) \cap VI(C, A) \cap EP(F) \).

In fact, let us first show that \( w \in VI(C, A) \). Define
\[
T_v = \begin{cases} 
Av + NCv, & v \in C, \\
\emptyset, & v \notin C.
\end{cases}
\]
Then \( T \) is maximal monotone. Let \( (v, u) \in G(T) \). Since \( u - Av \in NCv \) and \( w_n \in C \), we have \( (v - w_n, u - Av) \geq 0 \).

On the other hand, from \( w_n = P_C(u_n - \lambda_n Au_n) \), we have \( \langle v - w_n, w_n - (u_n - \lambda_n Au_n) \rangle \geq 0 \) and hence \( \langle v - w_n, (\frac{w_n - w_n}{\lambda_n} + Au_n) \rangle \geq 0 \). Therefore, we have
\[
\langle v - w_n, u \rangle \geq \langle v - w_n, Av \rangle \\
\geq \langle v - w_n, Av \rangle - \langle v - w_n, \frac{w_n - u_n}{\lambda_n} + Au_n \rangle \\
= \langle v - w_n, Av - Au_n - \frac{w_n - u_n}{\lambda_n} \rangle \\
= \langle v - w_n, Av - Au_n \rangle + \langle v - w_n, Au_n - Au_n \rangle - \langle v - w_n, \frac{w_n - u_n}{\lambda_n} \rangle \\
\geq \langle v - w_n, Au_n - Au_n \rangle - \langle v - w_n, \frac{w_n - u_n}{\lambda_n} \rangle,
\]
which together with \( \|w_n - u_n\| \to 0 \) and \( A \) is Lipschitz continuous implies that \( \langle v - w, u \rangle \geq 0 \). Since \( T \) is maximal monotone, we have \( w \in T^{-1}0 \) and hence \( w \in VI(C, A) \). Next, let us show that \( w \in F(S) \). Assume \( w \notin F(S) \). Since \( w_{n_i} \to w \) and \( Sw \neq w \), from Opial’s condition, we have
\[
\liminf_{i \to \infty} \|w_{n_i} - w\| < \liminf_{i \to \infty} \|w_{n_i} - Sw\| = \liminf_{i \to \infty} \|w_{n_i} - Sw_{n_i} + Sw_{n_i} - Sw\| \\
= \liminf_{i \to \infty} \|Sw_{n_i} - Sw\| \leq \liminf_{i \to \infty} \|w_{n_i} - w\|.
\]
This is a contradiction. Thus, we obtain \( w \in F(S) \). Finally, we show that \( w \in EP(F) \). By \( u_n = T_{r_n} x_n \), we have
\[
F(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \forall y \in C.
\]
From (A2), we also have \( \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq F(y, u_n) \) and hence \( \langle y - u_n, \frac{u_n - x_n}{r_n} \rangle \geq F(y, u_n) \). Since \( \|u_n - w_n\| \to 0 \) and \( w_{n_i} \to w \), we have \( u_{n_i} \to w \).
It together with \( \|x_n - u_n\| \to 0 \) and (A4) implies that \( 0 \geq F(y, w) \) for all \( y \in C \).
For $t$ with $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1 - t)w$. Since $y \in C$ and $w \in C$, we have $y_t \in C$ and hence $F(y_t, w) \leq 0$. So, from (A1) and (A4), we have

$$0 = F(y_t, y_t) \leq tF(y_t, y) + (1 - t)F(y_t, w) \leq tF(y_t, y)$$

and hence $0 \leq F(y_t, y)$. From (A3), we have $0 \leq F(w, y)$ for all $y \in C$ and hence $w \in EP(F)$. Therefore $w \in F(S) \cap EP(F) \cap VI(C, A)$. Since $z = P_{F(S) \cap EP(F) \cap VI(C, A)}f(z)$, we have

$$\limsup_{n \to \infty} (f(z) - z, x_n - z) = \limsup_{n \to \infty} (f(z) - z, Sw_n - z) = \lim_{i \to \infty} (f(z) - z, Sw_{ni} - z) = (f(z) - z, w - z) \leq 0.$$

For all $n \geq m$, we have

$$\|x_{n+1} - z\|^2 = \langle \alpha_n f(x_n) + \beta_n x_n + \gamma_n Sw_n - z, x_{n+1} - z \rangle$$
$$= \alpha_n \langle f(x_n) - z, x_{n+1} - z \rangle + \beta_n \langle x_n - z, x_{n+1} - z \rangle$$
$$+ \gamma_n \langle Sw_n - z, x_{n+1} - z \rangle$$
$$\leq \frac{1}{2} \alpha_n \alpha_a \|x_n - z\|^2 + \|x_{n+1} - z\|^2 + \alpha_n \langle f(z) - z, x_{n+1} - z \rangle$$
$$+ \frac{1}{2} (1 - \alpha_n) \left( \|x_n - z\|^2 + \|x_{n+1} - z\|^2 \right)$$
$$\leq \frac{1}{2} \left[ 1 - (1 - a) \alpha_n \right] \|x_n - z\|^2 + \frac{1}{2} \|x_{n+1} - z\|^2 + \alpha_n \langle f(z) - z, x_{n+1} - z \rangle,$$

which implies that

$$\|x_{n+1} - z\|^2 \leq \left[ 1 - (1 - a) \alpha_n \right] \|x_n - z\|^2 + 2 \alpha_n \langle f(z) - z, x_{n+1} - z \rangle.$$

Finally by using condition (2), (3.17) and Lemma 2.1, we can obtain $x_n \to z = P_{F(S) \cap EP(F) \cap VI(C, A)}f(z)$, as $n \to \infty$. This completes the proof. \hfill \Box

4. Applications

From Theorem 3.1, we have the following results immediately.

Theorem 4.1. Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $A$ be an $\alpha$-inverse-strongly monotone mapping of $C$ into $H$ and let $S$ be a nonexpansive mapping of $C$ into $H$ such that $F(S) \cap VI(C, A) \neq \emptyset$. Let $f$ be a contraction of $H$ into itself. Suppose $x_1 \in H$ and $\{x_n\}$ is given by

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)SP_C[(I - \lambda_n A)P_C x_n], \; n \geq 1,$$

where $\{\alpha_n\} \subset [0, 1)$ and $\{\lambda_n\} \subset [0, 2\alpha]$ satisfy the following conditions

(1) $\lim_{n \to \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, 
(2) $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, 2\alpha)$ and $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$.

Then $\{x_n\}$ converges strongly to $z \in F(S) \cap VI(C, A)$, where $z = P_{F(S) \cap VI(C, A)}f(z)$. 

Proof. Put $F(x, y) = 0$ for all $x, y \in C$, $\beta_n = 1$ and $r_n = 1$ for all $n \in \mathbb{N}$ in Theorem 3.1. Then, we have $u_n = P_Cx_n$. So, by Theorem 3.1, we can conclude the desired conclusion easily.

Remark 4.2. Putting $f(y) = x \in C$ for all $y \in H$ and $S$ be a nonexpansive mapping of $C$ in Corollary 4.1, we have $P_Cx_n = x_n$. So, Theorem 3.1 in Iiduka and Takahashi's [7] can be directly obtained.

REFERENCES


Meijuan Shang received her BS from Hebei Normal University and M.D at Tianjin Polytechnic University. She works at Shijiazhuang University from 2002. Her research interests focus on Nonlinear Analysis
Equilibrium problems, variational inequality problems and fixed point problems

Department of Mathematics, Tianjin Polytechnic University, Tianjin 300160, China
Department of Mathematics, Shijiazhuang University, Shijiazhuang 050035, China
e-mail: meijuanshang@yahoo.com.cn

Yongfu Su works at Tianjin Polytechnic University as a professor. He research interests focus on Nonlinear Analysis
Department of Mathematics, Tianjin Polytechnic University, Tianjin 300160, China
e-mail: suyongfu@tjpu.edu.cn