GENERAL HÖLDER TYPE INEQUALITIES ON THE 
FUNCTIONS OF $G_kG_\phi$-BOUNDED VARIATIONS

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ABSTRACT. For $G\phi$-sequences $\phi_i$ and $\kappa$-functions $\kappa_i (i = 1, 2, 3)$ we obtain the most general Hölder type inequalities on the functions of $G_kG_\phi$-bounded variations.

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In his studies on generalized functions of bounded variation and their application to the theory of harmonic functions, D. S. Cyphert([1]) viewed $\kappa$-function $\kappa$ as a rescaling of lengths of subintervals in $[a, b]$ such that the length of $[a, b]$ is 1 if $\kappa(1) = 1$. We are requiring that $\kappa$ has the following properties on a closed interval $[0, 1]$;

(1) $\kappa$ is continuous with $\kappa(0) = 0$ and $\kappa(1) = 1$,
(2) $\kappa$ is concave and strictly increasing, and
(3) $\lim_{x \to 0^+} (\kappa(x)/x) = \infty$.

We shall say that $\kappa_i (i = 1, 2, 3)$ satisfy the $\Delta_\kappa$-condition (briefly $\kappa_i \in \Delta_\kappa (i = 1, 2, 3)$) if $\kappa$-functions $\kappa_1$, $\kappa_2$ and $\kappa_3$ satisfy $\kappa_1^{-1}(x)\kappa_2^{-1}(x) \geq \kappa_3^{-1}(x)$ for $x > 0$.

Let $\phi = \{\phi_n\}$ be a sequence of monotone nondecreasing convex functions defined on the nonnegative real numbers such that $\phi_n(0) = 0$ and $\phi_n(x) > 0$ for $x > 0$ and $n = 1, 2, \ldots$. We shall say that $\phi$ is a $G\phi$-sequence, in symbol

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\( \phi \in G\phi S \) if \( \phi_n(x) \geq \phi_{n+1}(x) \) for all \( n \) and \( x \) and in addition if \( \sum_n \phi_n(x) \) diverges for \( x > 0 \) [4].

Note that if \( \phi = \{\phi_n\} \) is a \( \phi \)-sequence, in symbol \( \phi \in \phi S \), then \( \phi = \{\phi_n\} \) is a \( G\phi \)-sequence, that is, \( \phi S \in G\phi S \) in the sense of D.S. Cyphert[1].

Let us define \( \psi_n(y) = \sup \{ |y| - \phi_n(x) : x \geq 0 \} = \int_0^y \varphi_n^{-1}(s)ds, y \geq 0 \).

Then \( \psi = \{\psi_n\} \) is called as the complementary function to \( \phi_n \) and \( (\phi_n, \psi_n) \) the complementary pair.

In sequel we denote by \( A \times B \) the collection of all products \( f \times g \) for any functions \( f \in A \) and \( g \in B \), and define \( R_{\phi}(t) = \frac{\phi_i(t)}{t} \) and \( S_{\phi_i}(t) = \frac{\psi_i(t)}{t} \) for the corresponding complementary pair \( (\phi_i, \psi_i) \) of a \( G\phi \)-sequence \( \phi = \{\phi_n\} \).

Note that \( R_{\phi_i}(t) \) and \( S_{\phi_i}(t) \) are substitutes for the useful expressions \( t^{p-1} \) and \( t^{1/(p-1)} \) in the \( L^p \)-spaces, but not mutually complementary inverse, in general.

A \( G\phi \)-sequence \( \phi = \{\phi_n\} \) is said to satisfy the \( \Delta_2 \)-condition, in symbol \( \phi \in \Delta_2 \) if there is a constant \( c > 0 \) and \( u_0 \geq 0 \) such that \( 2R_{\phi_n}(2x) \leq cR_{\phi_n}(x) \), for all \( x \geq u_0 \), and the \( \Delta \)-condition, in symbol \( \phi \in \Delta'(c) \) if there is a constant \( c > 0 \) such that \( R_{\phi_n}(xy) \leq cR_{\phi_n}(x)R_{\phi_n}(y), x, y \geq x_0 \geq 0 \).

For \( \phi_1 = \{\phi_n\} \), \( \phi_2 = \{\phi_n\} \in G\phi S \), \( \phi_1 \) is stronger than \( \phi_2 \), \( \phi_1 \succ \phi_2 \) in symbols, if \( R_{\phi_2}(x) \leq aR_{\phi_1}(ax), x \geq x_0 \geq 0 \) for some \( a > 0 \) and \( x_0 \) (depending on \( a \)). We can define equivalence of pair of \( G\phi \)-sequences: \( \phi_1 \sim \phi_2 \) iff \( \phi_1 \succ \phi_2 \) and \( \phi_2 \succ \phi_1 \): there exist numbers \( 0 < a \leq b < \infty, x_0 \geq 0 \) such that \( aR_{\phi_1}(ax) \leq R_{\phi_2}(x) \leq bR_{\phi_1}(bx), x \geq x_0 \).

Let \( \phi = \{\phi_n\} \in G\phi S \) be defined on \([0, \infty)\) for all \( n \). The average function \( A(\phi_n) \) of \( \phi_n \) is the function given by \( A(\phi_n)(x) = \frac{1}{x} \int_0^x \phi_n(\tau)d\tau \) for all \( x > 0 \) and \( A(\phi_n)(0) = 0 \).

A function \( f \) is of \( \kappa \)-bounded variation on \([a, b]\) if there exists a positive constant \( c \) such that, for every collection \( \{I_n\} \) of nonoverlapping subintervals of \([a, b]\), \( \sum |f(I_n)| \leq c \sum \kappa(|I_n|/(b-a)) \), where \( I_n = [x_n, y_n] \) and \( |I_n| = y_n - x_n \).

M. J. Schramm [4] generalized the above idea by considering a sequence of increasing convex functions \( \phi = \{\phi_n\} \) defined on \([0, \infty)\); \( f \) is of \( \phi \)-bounded variation on \([a, b]\) if \( V_\phi(f; a, b) = V_\phi(f) = \sup \left( \sum \phi_n(|f(I_n)|) \right) \) is finite.

A function \( f \) is said to be of \( \kappa G\phi \)-bounded variation on \([a, b]\) if there exists a positive constant \( c \) such that for any appropriate collection \( \{I_n\} \) of nonoverlapping subintervals of \([a, b]\), \( \sum \phi_n(|f(I_n)|) \leq c \sum \kappa(|I_n|/(b-a)) \), where \([a, b] = \bigcup I_n \) and \( |I_n| \) is the length of \( I_n \).
The total variation of $f$ over $[a, b]$ is defined by

$$
kV_{G\phi}(f) = \kappa V_{G\phi}(f : a, b) = \sup \left( \sum \phi_n(|f(I_n)|) \bigg/ \sum \kappa(|I_n|/(b - a)) \right),$$

where the supremum is taken over all collections $\{I_n\}$ of nonoverlapping subintervals in $[a, b]$. We denote by $\kappa G\phi BV$ the collection of all functions of $\kappa G\phi$-bounded variation on $[a, b]$.

A function $f$ is said to be of generalized $\kappa G\phi$-bounded variation on $[a, b]$, in symbols $f \in \kappa G\phi BV[a, b]$ if there exists a positive constant $c$ such that for any sequences $\{I_n\}$ of intervals in $A$,

$$
\sum \phi_n(|f(I_n)|) \leq c \sum \kappa(|I_n|/(b - a)).
$$

where $[a, b] = \bigcup I_n$ and $|I_n|$ is the length of $I_n$. The total $G\kappa G\phi$-variation of $f$ over $[a, b]$ is defined by

$$
kV_{G\phi}(f, A) = \sup \left( \sum \phi_n(|f(I_n)|) \bigg/ \sum \kappa(|I_n|/(b - a)) \right),$$

where the supremum is taken over all collections $\{I_n\}$ of intervals in $A[2]$.

Let $G\kappa G\phi BV_0 = \left\{ f \in G\kappa G\phi BV[a, b] \mid f(a) = 0 \right\}$. For any $f$ in $G\kappa G\phi BV_0$, let us define the norm as in the Orlicz spaces;

$$
\|f\| = \|f\|_{G\kappa G\phi} = \inf \left\{ c > 0 ; G\kappa V_{G\phi}(f) \leq 1 \right\}
$$

Then $(G\kappa G\phi BV_0, \| \cdot \|)$ is a Bannach space and $G\kappa G\phi BV$ may be a Bannach space with the norm $\|f - f(a)\| + |f(a)|$.

From now on we will consider $G\phi$-sequences $\phi_i = \{\phi_{in}\}$ and $\kappa$-functions $\kappa_i \in \Delta_\kappa$ for $i = 1, 2, 3$.

**Theorem 1.** (a) If $\phi_i = \{\phi_{in}\} \in G\phi S$ $(i = 1, 2, 3)$ satisfy the inequality

$$
R_{R_{\phi_{i-1}^{\phi_{i}}}^{\phi_{i}}} (x) R_{R_{\phi_{i-1}^{\phi_{i}}}^{\phi_{i}}} (x) \leq \alpha R_{R_{\phi_{i-1}^{\phi_{i}}}^{\phi_{i}}} (x)
$$

for all $n$ and $x \geq 0$, then we have, for all $f \in G\kappa_1 G\phi_1 BV_0$ and $g \in G\kappa_2 G\phi_2 BV_0$,

$$
G\kappa_1 G\phi_1 BV_0 \cdot G\kappa_2 G\phi_2 BV_0 \subset G\kappa_3 G\phi_3 BV_0
$$

and

$$
\|fg\|_{G\kappa_3 G\phi_3} \leq 2\alpha \|f\|_{G\kappa_1 G\phi_1} \|g\|_{G\kappa_2 G\phi_2}.
$$
(b) If the inequality (1) in part (a) is replaced by

$$\frac{1}{\alpha} \leq R_{\phi_1}(x)R_{(\phi_3^{-1}0\phi_2)}(y) + R_{\phi_2}(y)R_{(\phi_3^{-1}0\phi_1)}(x)$$

then, for all \( f \in G\kappa_1G\phi_1BV_0 \) and \( g \in G\kappa_2G\phi_2BV_0 \),

$$G\kappa_1G\phi_1BV_0 \cdot G\kappa_2G\phi_2BV_0 \subset G\kappa_3G\phi_3BV_0$$

and

$$\|fg\|_{G\kappa_3G\phi_3} \leq 4\alpha\|f\|_{G\kappa_1G\phi_1}\|g\|_{G\kappa_2G\phi_2}.$$  

Proof. (a) By the convexity of \( \phi_{in} \), since the inequality (1) implies the inequality (2), the part (a) holds.

(b) By the inequality (2), since

$$\frac{\sum \phi_{3n}(f(I_n)g(I_n))/\alpha(1+\varepsilon)^2}{\sum \kappa_3(|I_n|/(b-a))} \leq \frac{1}{2}(V_{G\phi_1}(f) + V_{G\phi_2}(g)) \leq 1,$$

we have \( \kappa_3V_{G\phi_3}(fg/4(1+\varepsilon)^2) \leq 1 \), which implies \( \|fg\|_{G\kappa_3G\phi_3} \leq 4(1+\varepsilon)^2 \), and hence the theorem follows by letting \( \varepsilon \to 0 \). \( \square \)

Corollary 2. For \( \phi_i = \{\phi_{in}\} \in G\phi_S \ (i = 1, 2) \) with \( \int_0^1 R_{\phi_{in}^{-1}}(t)R_{\phi_{in}^{-1}}(t)dt < \infty \), if we let \( \phi_{3n}^{-1}(x) = \alpha \int_0^x R_{\phi_{3n}^{-1}}(t)R_{\phi_{3n}^{-1}}(t)dt \) for some constant \( \alpha \), then \( \phi_3 = \{\phi_{3n}\} \in G\phi_S \), and, for all \( f \in G\kappa_1G\phi_1BV_0 \) and \( g \in G\kappa_2G\phi_2BV_0 \),

$$G\kappa_1G\phi_1BV_0 \cdot G\kappa_2G\phi_2BV_0 \subset G\kappa_3G\phi_3BV_0$$

and

$$\|fg\|_{G\kappa_3G\phi_3} \leq 2\alpha\|f\|_{G\kappa_1G\phi_1}\|g\|_{G\kappa_2G\phi_2}.$$  

Proof. Since \( R_{\phi_{nin}^{-1}}(x) \) (i = 1, 2) are nonincreasing, it follow that \( xR_{\phi_{3n}^{-1}}(x) \) is concave and \( \phi_{3n}^{-1}(0) = 0 \). By the inequality (1), this is proved. \( \square \)

Lemma 3. For \( \phi_i = \{\phi_{in}\} \in G\phi_S(i = 1, 2, 3) \), the followings are equivalent; for nonnegative \( x, y, z \geq 0 \),

(a) \( \alpha_1xyR_{\phi_{3n}}(\alpha_1xy) \leq xR_{\phi_{1n}}(x) + yR_{\phi_{2n}}(y) \) for some \( \alpha > 0 \);

(b) \( \lim_{x \to \infty} \sup xR_{\phi_{1n}^{-1}}(x)R_{\phi_{2n}^{-1}}(x)/R_{\phi_{3n}^{-1}}(x) \) < \( \infty \);

(c) \( \alpha_2xyz \leq xR_{\phi_{1n}}(x) + yR_{\phi_{2n}}(y) + zS_{\phi_{3n}}(z) \);
(d) \( \alpha_3 y z S_{\phi_1n}(\alpha_3 y z) \leq y R_{\phi_2n}(y) + z S_{\phi_3n}(z) \);
(e) \( \alpha_4 x z S_{\phi_2n}(\alpha_4 x z) \leq x R_{\phi_1n}(x) + z S_{\phi_3n}(z) \),

where \( \alpha_i > 0 \) are some constants and \( \phi_{i_n}, \psi_{i_n} \) the corresponding complementary pairs.

Proof. Assume that (a) holds. Since \( x \leq (\phi_{i_n}^{-1} \circ \phi_{i_n})(x) \), if we let \( x = u R_{\phi_1n}(u) \) and \( y = u R_{\phi_2n}(u) \), then (a) implies (b).

Conversely, if (b) holds, then there are \( \alpha_1 > 0 \) and \( u_0 \geq 0 \) such that

\[
R_{\phi_1n}(u) R_{\phi_2n}(u) \leq \frac{u}{\alpha_1} R_{\phi_3n}(u), \quad u \geq u_0,
\]

and letting \( x = u R_{\phi_1n}(u) \) and \( y = u R_{\phi_2n}(u) \) for \( x, y \geq \max(u_0 R_{\phi_1n}(u_0), u_0 R_{\phi_2n}(u_0)) \), this shows that (b) implies (a).

By the property of \( G \phi S \), (a) iff (c).

For \( x, y, z \geq x_2 \geq 0 \), we have the following inequalities:

\[
\alpha_2 y z S_{\phi_1n}(\alpha_2 y z) = \alpha_2 x y z - x R_{\phi_1n}(x) \leq y R_{\phi_2n}(y) + z S_{\phi_3n}(z),
\]

\[
\alpha_2 x y z - x y z R_{\phi_1n}(x) \leq \alpha_2 y z R_{\phi_1n}(\alpha_2 y z) \leq y R_{\phi_2n}(y) + z R_{\phi_3n}(z),
\]

which implies that (c) iff (d). Similarly (c) iff (e). \( \square \)

By Lemma 3, we have the following Theorem 4:

**Theorem 4.** For \( \kappa_i \in \Delta_\kappa \) and \( \phi_i = \{\phi_{i_n}\} \in G \phi S(i = 1, 2, 3) \), suppose that one of (a) \sim (e) in Lemma 3 is satisfied. Then for the corresponding complementary pairs \( \phi_{i_n}, \psi_{i_n} \) there is a constant \( \alpha_i \) such that

\[
G\kappa_1 G\phi_1 BV_0 \cdot G\kappa_2 G\phi_2 BV_0 \subset G\kappa_3 G\phi_3 BV_0
\]

and

\[
G\kappa_3 V_{G\phi_3}(fg) \leq \frac{1}{\alpha_i} G\kappa_1 V_{G\phi_1}(f) G\kappa_2 V_{G\phi_2}(g)
\]

for any \( f \in G\kappa_1 G\phi_1 BV_0 \) and \( g \in G\kappa_2 G\phi_2 BV_0 \).

**Theorem 5.** For \( \kappa_i \in \Delta_\kappa, \phi_i = \{\phi_{i_n}\} \in G \phi S(i = 1, 2, 3) \) and the corresponding complementary pairs \( \phi_{i_n}, \psi_{i_n} \), suppose that one of the following conditions is satisfied:

(i) there is a complementary pair \( \phi_{4n}, \psi_{4n} \) such that

\[
\phi_{1n} \succ \phi_{3n} \circ \phi_{4n} \quad \text{and} \quad \phi_{2n} \succ \phi_{3n} \circ \psi_{4n},
\]
(ii) for the above \((\phi_{4n}, \psi_{4n})\), if \(\phi_{3n} \in \Delta'\), suppose that

\[
\phi_{1n} \succ \phi_{4n} \circ \phi_{3n} \quad \text{and} \quad \phi_{2n} \succ \psi_{4n} \circ \phi_{3n},
\]

Then there is a constant \(\alpha \geq 0\) such that, for any \(f \in G_{K_1}G_{\phi_1}BV_0\) and \(g \in G_{K_2}G_{\phi_2}BV_0\),

\[
G_{K_1}G_{\phi_1}BV_0 \cdot G_{K_2}G_{\phi_2}BV_0 \subset G_{K_3}G_{\phi_3}BV_0
\]

and

\[
G_{K_3}V_{G_{\phi_3}}(fg) \leq \frac{1}{\alpha} G_{K_1}V_{G_{\phi_1}}(f)G_{K_2}V_{G_{\phi_2}}(g)
\]

Proof. By (i) and the definition of \(\succ\), we have, for some \(\beta > 0\),

\[
2\beta x R_{(\phi_{3n} \circ \phi_{4n})}(2\beta x) \leq x R_{\phi_{1n}}(x)
\]

and

\[
2\beta y R_{(\phi_{3n} \circ \phi_{4n})}(2\beta y) \leq y R_{\phi_{2n}}(y), \ x, y \geq x_0 \geq 0.
\]

Hence if we let \(\alpha = \beta^2\), then

\[
\alpha xy R_{\phi_{3n}}(\alpha xy) \leq \beta x R_{(\phi_{3n} \circ \phi_{4n})}(2\beta x) + \beta y R_{(\phi_{3n} \circ \phi_{4n})}(2\beta y)
\]

\[
\leq x R_{\phi_{1n}}(x) + y R_{\phi_{2n}}(y),
\]

which is (a) of Lemma 3.

Next let (ii) be true. Then, for \(x, y \geq x_0 \geq 0\),

\[
cxy R_{\phi_{3n}}(cxy) \leq x R_{\phi_{3n}}(x) y R_{\phi_{3n}}(y)
\]

\[
\leq bx R_{\phi_{1n}}(bx) + by R_{\phi_{2n}}(by), \ x, y \geq x_0 \geq 0,
\]

for some \(b, c > 0\). If we let \(\alpha = \frac{c}{b^2}, u = bx, v = by\) and \(u_0 = bx_0\), then

\[
\alpha uv R_{\phi_{3n}}(\alpha uv) = \alpha b^2 xy R_{\phi_{3n}}(\alpha b^2 xy)
\]

\[
\leq \frac{u}{b} R_{\phi_{3n}}\left(\frac{u}{b}\right) \frac{v}{b} R_{\phi_{3n}}\left(\frac{v}{b}\right)
\]

\[
\leq u R_{\phi_{1n}}(u) + v R_{\phi_{2n}}(v),
\]

which reduces to (a) of Lemma 3. So the result holds. \(\square\)
Lemma 6. For \( \phi_1 = \{ \phi_{1n} \} \in G\phi S \), let \( \phi_{2n}(t) = \int_0^t R_{\phi_{1n}}(u)du \), \( \phi_{3n}(t) = \int_0^t R_{\phi_{2n}}(\tau)d\tau \) and \( \phi_{1n}(0) = 0 \) for \( i = 1, 2, 3 \). Then \( R_{\phi_{1n}} \) and \( S_{\phi_{1n}} \) are strictly increasing continuous functions in \( G\phi S \) with continuous derivatives which map \([0, \infty)\) onto itself and satisfy the followings; for any \( t \geq 0 \),

\[
R_{\phi_{1n}}(t) \leq 2R_{\phi_{1n}}(2t) \tag{3}
\]

\[
S_{\phi_{1n}}(R_{\phi_{1n}}(t)) \leq t \leq 2S_{\phi_{1n}}(2R_{\phi_{1n}}(t)) \tag{4},
\]

\[
\phi_1 \sim \phi_2, \phi_2 \sim \phi_3 \tag{5},
\]

and the complementary version;

\[
S_{\phi_{1n}}(t) \leq 2S_{\phi_{1n}}(2t) \tag{3'},
\]

\[
R_{\phi_{1n}}(S_{\phi_{1n}}(t)) \leq t \leq 2R_{\phi_{1n}}(2S_{\phi_{1n}}(t)) \tag{4'},
\]

\[
\psi_1 \sim \psi_2, \psi_2 \sim \psi_3 \tag{5'}.
\]

Proof. Note that \( 1 \leq tR_{\phi_{1n}}^{-1}(t)S_{\phi_{1n}}^{-1}(t) \leq 2 \), \( t \geq 0 \). Substituting \( t = tR_{\phi_{1n}}(t) \), we get

\[
R_{\phi_{1n}}(t) \leq \psi_{1n}^{-1}(\phi_{1n}(t)) \leq 2R_{\phi_{1n}}(t).
\]

Hence

\[
R_{\phi_{1n}}(t) \leq 2R_{\phi_{1n}}(t) \leq 2R_{\phi_{1n}}(2t)
\]

and

\[
\psi_{1n}(R_{\phi_{1n}}(t)) \leq \phi_{1n}(t) \leq \psi_{1n}(2R_{\phi_{1n}}(t)) \tag{\star}.
\]

Dividing (\star) with \( R_{\phi_{1n}}(t) \) or with \( 2R_{\phi_{1n}}(t) \), we have

\[
S_{\phi_{1n}}(R_{\phi_{1n}}(t)) \leq (\phi_{1n}(t)/R_{\phi_{1n}}(t)) \leq t \leq 2S_{\phi_{1n}}(2R_{\phi_{1n}}(t)).
\]

By the continuity of \( \phi_{1n} \) and definitions of \( G\phi S \)-sequences, we have

\[
\lim_{x \to 0} R_{\phi_{2n}}(t) = \lim_{x \to 0} \phi_{2n}'(t) = \lim_{x \to 0} R_{\phi_{1n}}(t) = 0
\]

and

\[
\lim_{x \to \infty} R_{\phi_{2n}}(t) = \lim_{x \to \infty} R_{\phi_{1n}}(t) = \infty.
\]

Since \( \phi_{2n}'(t) = R_{\phi_{1n}}(t) \geq 0 \), \( \phi_{2n} \) is strictly increasing and \( R_{\phi_{1n}}(t_1) < R_{\phi_{2n}}(t_2) \) if \( t_1 < t_2 \), which implies that \( \phi_{2n} \) is convex on \([0, \infty)\).
Since $R_{\phi_{in}}(t) \leq \psi_{in}(t) = S_{\psi_{in}}(t)$, each statement of the Lemma has its complementary version; Substituting $t \to \psi_{in}(t)$ and dividing with $t$,

$$S_{\phi_{in}}(t) \leq \phi_{in}^{-1}(\psi_{in}(t)) \leq S_{\phi_{in}}(t) \to \phi_{in}(S_{\phi_{in}}(t)) \leq \psi_{in}(t) \leq \phi_{in}(2S_{\phi_{in}}(t))$$

and

$$R_{\phi_{in}}(t) \leq 2R_{\phi_{in}}(2t).$$

Also, $R_{\phi_{in}}(S_{\phi_{in}}(t)) \leq t \leq 2R_{\phi_{in}}(2S_{\phi_{in}}(t))$. By the similar way, other cases are proved \qed

Note that if $\phi = \{\phi_{in}\}$ is a G$\phi$S-function, then $R_{\phi_{in}}$ and $S_{\phi_{in}}$ are increasing continuous functions in G$\phi$S with continuous derivatives which map $[0, \infty)$ onto itself and satisfy for any $t \geq 0$ the above inequalities; (3) $\sim$ (5) and (3') $\sim$ (5').

**Theorem 7.** For $\phi_1 = \{\phi_{1n}\} \in G\phi S$, let $\phi_{2n}(t) = \int_0^t R_{\phi_{1n}}(u)du$, $\phi_{3n}(t) = \int_0^t R_{\phi_{2n}}(u)du$ and $\phi_{in}(0) = 0 (i = 1, 2, 3)$, then we have the followings;

(i) for a $\kappa$-function $\kappa_i (i = 1, 2, 3)$ with $\kappa_1 \leq \kappa_2 \leq \kappa_3$,

$$G_{\kappa_1}G_{\phi_1}BV_0 \subset G_{\kappa_2}G_{\phi_1}BV_0 \subset G_{\kappa_3}G_{\phi_2}BV_0 \subset G_{\kappa_3}G_{\phi_3}BV_0$$

and

$$G_{\kappa_1}G_{\psi_1}BV_0 \subset G_{\kappa_2}G_{\psi_1}BV_0 \subset G_{\kappa_3}G_{\psi_2}BV_0 \subset G_{\kappa_3}G_{\psi_3}BV_0$$

(ii) for $\kappa_i \in \Delta_{\kappa}$,

$$G_{\kappa_1}G_{\phi_1}BV_0 \subset G_{\kappa_3}G_{\phi_1}BV_0 \subset G_{\kappa_3}G_{\phi_2}BV_0$$

and

$$G_{\kappa_2}G_{\psi_1}BV_0 \subset G_{\kappa_3}G_{\psi_1}BV_0 \subset G_{\kappa_3}G_{\psi_2}BV_0$$

**Proof.** By (5) and (5') of Lemma 3, these are proved. \qed
REFERENCES


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