ISOMORPHISM CLASSES OF ELLIPTIC CURVES
OVER FINITE FIELDS WITH CHARACTERISTIC 3

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Abstract. We count the isomorphism classes of elliptic curves over finite fields $F_{3^n}$ and list a representative of each isomorphism class. Also we give the number of rational points for each supersingular elliptic curve over $F_{3^n}$.

1. Introduction

Elliptic curves have been intensively studied in algebraic geometry and number theory. Starting in about 1985, the theory of elliptic curves over finite fields has been applied to various problems; factoring integers, primality proving and construction of public key cryptosystems. One of the advantages using elliptic curve cryptosystems is the greater flexibility in choosing the group. That is for each prime power $q$ there is only one multiplicative group $F_q^*$, but there are many elliptic curve groups.

It may be useful to classify the isomorphism classes of elliptic curves over finite fields, in order to know how many essentially different choices of curves are. And this classification is used to produce nonisomorphic elliptic curves, which may be useful for a cryptographic purpose. In [4] isomorphism classes of elliptic curves over $F_{p^n}, p \neq 2, 3$ and $F_{2^n}$ were studied.

In this paper we count the isomorphism classes of elliptic curves defined over finite fields $F_{3^n}$ and list a representative of each isomorphism class. Moreover, we give the order and the group structure of supersingular elliptic curves over $F_{3^n}$. This fields are getting more interest in a cryptographic purpose (see, for instance, [1] and [2]).

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This paper is organized as follows:

In section 2 the necessary definitions and notations are introduced. In section 3 the exact number and a set of representatives of the isomorphism classes of elliptic curves over $\mathbb{F}_{3^n}$ are produced. In section 4 we give the number of points and the group type of supersingular elliptic curves over finite fields with characteristic 3.

2. Elliptic curves

In this section, we recall the basic definitions and properties about the elliptic curves. We follow notations given in [4].

If $K$ is a field, let $\overline{K}$ denote its algebraic closure. A Weierstrass equation is a homogeneous equation of degree 3 of the form

$$Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3,$$

where $a_i \in K$. The Weierstrass equation is said to be smooth or non-singular if for all projective points $P(X, Y, Z) \in \mathbb{P}^2(\overline{K})$ satisfying

$$F(X, Y, Z) = Y^2Z + a_1XYZ + a_3YZ^2 - X^3 - a_2X^2Z - a_4XZ^2 - a_6Z^3 = 0,$$

at least one of the three partial derivatives $\frac{\partial F}{\partial X}, \frac{\partial F}{\partial Y}, \frac{\partial F}{\partial Z}$ is non-zero at $P$. An elliptic curve $E$ is the set of all solutions in $\mathbb{P}^2(\overline{K})$ of a smooth Weierstrass equation. There is exactly one point in $E$ with $Z$-coordinate equal to 0, namely $(0, 1, 0)$. We call this point the point at infinity and denote it by $O$.

For convenience, we will write the Weierstrass equation for an elliptic curve using non-homogeneous (affine) coordinates $x = X/Z, y = Y/Z$,

$$(2.1) \quad y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

An elliptic curve $E$ is then the set of solutions to equation (2.1) in the affine plane $\mathbb{A}^2(\overline{K}) = \overline{K} \times \overline{K}$, together with the extra point at infinity $O$. If $a_1, a_2, a_3, a_4, a_6 \in K$, the $E$ is said to be defined over $K$, and we denote by $E/K$. If $E$ is defined over $K$, then the set of $K$-rational points of $E$, denoted $E(K)$, is the set of points both of whose coordinates lie in $K$, together with the point $O$.

The following lemma will be needed for theorems in the later sections.

**Lemma 2.1.** [3] The trinomial

$$x^p - x - a, \quad a \in \mathbb{F}_q,$$
where $q$ is a prime power $p^n$, has a solution in $F_q$ if and only if $Tr(a) = 0$. Here, $Tr(\alpha) = \alpha + \alpha^p + \alpha^{p^2} + \cdots + \alpha^{p^{n-1}}$.

3. Isomorphism classes of elliptic curves over $F_q, q = 3^n$

Let $E$ be an elliptic curve over a field $\mathbb{K}$. We can write $E$ as the following nonsingular Weierstrass form

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6,$$

where $a_1, a_2, a_3, a_4, a_6 \in \mathbb{K}$.

Two elliptic curves are said to be isomorphic if they are isomorphic as projective varieties. Briefly, two projective varieties $V_1, V_2$ defined over a field $\mathbb{K}$ are isomorphic over $\mathbb{K}$ if there exist morphisms $\phi : V_1 \rightarrow V_2, \psi : V_2 \rightarrow V_1(\phi, \psi$ defined over $\mathbb{K})$, such that $\psi \circ \phi$ and $\phi \circ \psi$ are the identity maps on $V_1, V_2$ respectively. The following result relates the notion of isomorphism of elliptic curves to the coefficients of the Weierstrass equations that define the curves [4].

**Theorem 3.1.** [4] Two elliptic curves $E_1/\mathbb{K}$ and $E_2/\mathbb{K}$ given by the equations

$$E_1 : y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6,$$
$$E_2 : y^2 + \tilde{a}_1 xy + \tilde{a}_3 y = x^3 + \tilde{a}_2 x^2 + \tilde{a}_4 x + \tilde{a}_6$$

are isomorphic over $\mathbb{K}$ if and only if there exists an admissible change of variables $u, r, s, t \in \mathbb{K}, u \neq 0$, such that the change of variables

$$(x, y) \rightarrow (u^2 x + r, u^3 y + u^2 sx + t)$$

transforms equation $E_1$ to equation $E_2$. The relationship of isomorphism is an equivalence relation.

**Theorem 3.2.** [4] Two elliptic curves $E_1/\mathbb{K}$ and $E_2/\mathbb{K}$ are isomorphic over $\mathbb{K}$ if and only if there exist $u, r, s, t \in \mathbb{K}, u \neq 0$, that satisfy the following equations;

$$\begin{cases}
ua_1 = a_1 + 2s \\
u^2a_2 = a_2 - sa_1 + 3r - s^2 \\
u^3a_3 = a_3 + ra_1 + 2t \\
u^4a_4 = a_4 - sa_3 + 2ra_2 - (t + rs)a_1 + 3r^2 - 2st \\
u^6a_6 = a_6 + ra_4 + r^2a_2 + r^3 - ta_3 - t^2 - rta_1.
\end{cases}$$
Proposition 3.3. [6] Let $E/F_q, q = 3^n$ be a curve given by a Weierstrass equation. Then there is an admissible change of variables

$$(x, y) \mapsto (u^2 x + r, u^3 y + u^2 s x + t)$$

with $u \in F_q^*$ and $r, s, t \in F_q$ such that $E/F_q$ has a Weierstrass equation of the indicated form.

$$\begin{cases}
  y^2 = x^3 + a_2 x^2 + a_6, & \Delta(E) = -a_3^3 a_6, \ j(E) = -a_3^2 / a_6 \text{ if } j(E) \neq 0, \\
  y^2 = x^3 + a_4 x + a_6, & \Delta(E) = -a_4^3 \text{ if } j(E) = 0,
\end{cases}$$

where $\Delta(E)$ denotes the discriminant of $E$.

The elliptic curve $E$ is said to be supersingular if $p$ divides $t$, where $\#E(F_q) = q + 1 - t, q = p^n$. Then it is well-known that $E$ is supersingular if and only if $j$-invariant $j(E) = 0$ when $q = 3^n$.

Theorem 3.4. (Nonsupersingular case) There are $2(q - 1)$ isomorphism classes of non-supersingular elliptic curves over $F_q, q = 3^n$.

Proof. Let $E_1, E_2$ be non-supersingular elliptic curves defined over $F_{3^n}$ and given by the equations

$$E_1 : y^2 = x^3 + a_2 x^2 + a_6, \quad (a_2 \neq 0, a_6 \neq 0),$$

$$E_2 : y^2 = x^3 + \bar{a}_2 x^2 + \bar{a}_6, \quad (\bar{a}_2 \neq 0, \bar{a}_6 \neq 0).$$

Then the only admissible change of variables which transforms $E_1$ into $E_2$ is

$$(x, y) \mapsto (u^2 x, u^3 y), \quad u \in F_q^*,$$

such that $u^2 \bar{a}_2 = a_2$ and $u^6 \bar{a}_6 = a_6$. When $u^2 = 1$, the above transformation gives the automorphism. So there are $(q - 1)^2 / ((q - 1)/2) = 2(q - 1)$ isomorphism classes of non-supersingular elliptic curves over $F_q, q = 3^n$.

Theorem 3.5. (Supersingular case) There are 6 isomorphism classes of supersingular elliptic curves over $F_{3^n}$, when $n$ is even. There are 4 isomorphism classes of supersingular elliptic curves, when $n$ is odd.

Proof. Let $E$ be a supersingular elliptic curve over $F_{3^n}$ given by the following equation.

$$E : y^2 = x^3 + a_4 x + a_6, \quad a_4 \neq 0.$$ 

The admissible change of variables in the equation $E$ which transforms into itself is

$$(x, y) \mapsto (u^2 x + r, u^3 y),$$

where

$$u^4 = 1 \text{ and } u^6 a_6 = a_6 + a_4 r + r^3.$$
Substitute $u^4 = 1$ into the second equation above, then we obtain
\[(3.1) \quad r^3 + a_4 r + a_6 (1 - u^2) = 0.\]
We split this into the following cases;
If $\sqrt{-a_4} \notin \mathbb{F}_q^*$, then the map $r \mapsto r^3 + a_4 r$ is bijective. Hence the equation \[(3.1)\] has unique solution, say $r_u$, and the automorphism group is
\[(x, y) \mapsto (u^2 x + r_u, u^3 y).\]
If $\sqrt{-a_4} \in \mathbb{F}_q^*$, then there exists $r_0 \in \mathbb{F}_q^*$ such that $r_0^2 = -a_4$, and the equation \[(3.1)\] has a solution in $\mathbb{F}_q$ iff the equation
\[(3.2) \quad r^3 - r = a_6 (u^2 - 1)/r_0^3\]
has a solution in $\mathbb{F}_q$. If $\text{Tr}(a_6 (u^2 - 1)/r_0^3) \neq 0$, then the equation \[(3.2)\] has no solutions by Lemma 2.2. If $\text{Tr}(a_6 (u^2 - 1)/r_0^3) = 0$, then the equation \[(3.2)\] has three solutions; If $r_u$ is one of them, the other solutions are $r_u \pm 1$. Note that since $u^2 = \pm 1 \in \mathbb{F}_q$, the condition $\text{Tr}(a_6 (u^2 - 1)/r_0^3) = 0$ iff $u^2 \text{Tr}(a_6/r_0^3) = 0$.

We further split this into two cases according to $n$ is even or odd.
When $n$ is even, then the solutions of the equation $u^4 = 1$ are $\{\pm 1, \pm \zeta \mid \zeta^2 = -1, \zeta \in \mathbb{F}_q\}$. If $\text{Tr}(a_6/r_0^3) = 0$, then the automorphism group is
\[(x, y) \mapsto (u^2 x + r, u^3 y), u^4 = 1, r \in \{r_0 r_u, r_0 (r_u \pm 1)\}.\]
If $\text{Tr}(a_6/r_0^3) \neq 0$, then the automorphism group is
\[(x, y) \mapsto (x + r, uy), u^2 = 1, r \in \{0, \pm r_0\}.\]
When $n$ is odd, the solutions to the equation $u^4 = 1$ are $\{\pm 1\}$. So the automorphism group is
\[(x, y) \mapsto (x + r, uy), u^2 = 1, r \in \{0, r_0, -r_0\}.\]
In conclusion, if $n$ is even, the number of supersingular elliptic curves is
\[
\frac{q(q - 1)/2}{q(q - 1)/4} + \frac{q(q - 1)/6}{q(q - 1)/12} + \frac{q(q - 1)/3}{q(q - 1)/6} = 6,
\]
and if $n$ is odd, the number of supersingular elliptic curves is
\[
\frac{q(q - 1)/2}{q(q - 1)/2} + \frac{q(q - 1)/2}{q(q - 1)/6} = 4.
\]

\[\square\]

**Theorem 3.6.** 1. (Nonsupersingular case) A set of representatives of the isomorphism classes of non-supersingular elliptic curves over $\mathbb{F}_{3^n}$ is
\[
\{y^2 = x^3 + a_2 x^2 + a_6 \mid a_2 \in \{1, \alpha\}, a_6 \in \mathbb{F}_{3^n}^*\},
\]
where $\alpha$ is a quadratic nonresidue in $\mathbb{F}_3^*$.

2. (Supersingular case) A representative from each isomorphism class of supersingular elliptic curves over $\mathbb{F}_3$ is

\[
\begin{aligned}
&\begin{cases}
  y^2 = x^3 - \beta x \\
  y^2 = x^3 - \beta^3 x \\
  y^2 = x^3 - x \\
  y^2 = x^3 - \gamma x \\
  y^2 = x^3 - x + \delta \\
  y^2 = x^3 - \gamma x + \delta
\end{cases} & \text{if } n \text{ is even},
\end{aligned}
\]

where $\sqrt{\beta} \notin \mathbb{F}_3^*$, $\sqrt{\gamma} \notin \mathbb{F}_3^*$, $\sqrt{\gamma} \notin \mathbb{F}_3^*$ and $\text{Tr}(\delta) \neq 0$.

And

\[
\begin{aligned}
&\begin{cases}
  y^2 = x^3 + x \\
  y^2 = x^3 - x \\
  y^2 = x^3 - x + \lambda \\
  y^2 = x^3 - x + \mu
\end{cases} & \text{if } n \text{ is odd},
\end{aligned}
\]

where $\text{Tr}(\lambda) = 1$, $\text{Tr}(\mu) = -1$.

4. Number of points

We determine the number of rational points $\sharp E(\mathbb{F}_3^n)$, where $E$ is a supersingular curve over $\mathbb{F}_3$. We summarize the needed Theorems to count the order of supersingular elliptic curves and to find the group structure.

**Theorem 4.1.** (Weil Theorem) Let $E$ be an elliptic curve defined over $\mathbb{F}_q$, and let $t = q + 1 - \sharp E(\mathbb{F}_q)$. Then $\sharp E(\mathbb{F}_q^k) = q^k + 1 - \alpha^k - \beta^k$, where $\alpha, \beta$ are complex numbers determined from the factorization of $1 - tT + qT^2 = (1 - \alpha T)(1 - \beta T)$.

**Theorem 4.2.** [7] Let $p$ be a prime and $q = p^m$. Let $t$ be an integer with $|t| \leq 2\sqrt{q}$ and $N_q(t)$ be the number of isomorphism classes of elliptic curves over $\mathbb{F}_q$ such that $\sharp E(\mathbb{F}_q) = q + 1 - t$. Then
Isomorphism classes of elliptic curves over $\mathbb{F}_{3^{m}}$  

If $q$ is cyclic $\mathbb{Z}$, if $q$ is cyclic $\mathbb{Z}$, even $\mathbb{Q}$  

The number of points and the group structure of $\mathbb{Z}$, even $\mathbb{Q}$  

The curve $E$ can be also viewed as an elliptic curve over any extension field $\mathbb{F}_{q^{m}}$. One can compute $\sharp E(\mathbb{F}_{q^{m}})$, for $m \geq 2$, from $\sharp E(\mathbb{F}_{q})$ using the Weil theorem. The group type of these curves may be determined by using Theorem 4.3.

**Theorem 4.4.** The number of points and the group structure of supersingular elliptic curves in Theorem 3.6 are given following tables:

<table>
<thead>
<tr>
<th>No</th>
<th>Curve $E$</th>
<th>$n$</th>
<th>$\sharp E(\mathbb{F}_{3^{m}})$</th>
<th>Group Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$y^2 = x^3 - \beta x$</td>
<td>even</td>
<td>$q + 1$</td>
<td>cyclic</td>
</tr>
<tr>
<td>2</td>
<td>$y^2 = x^3 - \beta^2 x$</td>
<td>even</td>
<td>$q + 1$</td>
<td>cyclic</td>
</tr>
<tr>
<td>3</td>
<td>$y^2 = x^3 - x$</td>
<td>$n \equiv 0 \pmod{4}$</td>
<td>$q + 1 - 2\sqrt{q}$</td>
<td>$\mathbb{Z}<em>{\sqrt{q} - 1} \oplus \mathbb{Z}</em>{\sqrt{q} + 1}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$n \equiv 2 \pmod{4}$</td>
<td>$q + 1 + 2\sqrt{q}$</td>
<td>$\mathbb{Z}<em>{\sqrt{q} + 1} \oplus \mathbb{Z}</em>{\sqrt{q} - 1}$</td>
</tr>
<tr>
<td>4</td>
<td>$y^2 = x^3 - \gamma x$</td>
<td>$n \equiv 0 \pmod{4}$</td>
<td>$q + 1 + 2\sqrt{q}$</td>
<td>$\mathbb{Z}<em>{\sqrt{q} + 1} \oplus \mathbb{Z}</em>{\sqrt{q} + 1}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$n \equiv 2 \pmod{4}$</td>
<td>$q + 1 - 2\sqrt{q}$</td>
<td>$\mathbb{Z}<em>{\sqrt{q} - 1} \oplus \mathbb{Z}</em>{\sqrt{q} - 1}$</td>
</tr>
<tr>
<td>5</td>
<td>$y^2 = x^3 - x + \delta$</td>
<td>even</td>
<td>$q + 1 \pm \sqrt{q}$</td>
<td>cyclic</td>
</tr>
<tr>
<td>6</td>
<td>$y^2 = x^3 - \gamma x + \delta$</td>
<td>even</td>
<td>$q + 1 \pm \sqrt{q}$</td>
<td>cyclic</td>
</tr>
</tbody>
</table>

Table 1. Orders of supersingular elliptic curves over $\mathbb{F}_{3^{m}}$, where $n$ is even
When \( q \equiv \pm 1 \pmod{3} \) respectively in even). We first observe that where \( q \) is a quadratic nonresidue in \( \mathbb{F}_q \). Hence \( t_1 = t_2 = 0 \). Since the coefficients of the equation \( E_3 \) are in \( \mathbb{F}_3 \), we can apply the Weil Theorem to determine \( E_3 \) and we get \( t_3 = 2\sqrt{q} \) or \(-2\sqrt{q} \) according to whether \( n \equiv 2 \) or 0 (mod 4) respectively. By Theorem 4.2, we obtain that the 6 values of \( t_i \) are 0, 0, \( \pm \sqrt{q} \) and \( \pm \sqrt{2q} \) (not necessarily in that order). We find \( E_4 \) has four 2-torsion points since \( \gamma \) is a quadratic residue in \( \mathbb{F}_q \). Hence \( E_4(\mathbb{F}_q) \) cannot be cyclic, so \( t_4 = \pm 2\sqrt{q} \).

Therefore \( t_5 = \pm \sqrt{q} \) and \( t_6 = -t_5 \).

When \( n \) is odd, there are 4 isomorphism classes of supersingular curves over \( \mathbb{F}_{3^n} \). Let \( \sharp E_i = \sharp E_i(\mathbb{F}_q) = q + 1 - t_i \), for \( 1 \leq i \leq 4 \), where \( q = 3^n \) and the curves \( E_i \) are those of Theorem 3.6 (when \( n \) is odd). We determine the order of curves \( E_i(= q + 1), i = 1, 2 \) over \( \mathbb{F}_{3^n} \) from \( \sharp E_i(\mathbb{F}_q)(=4) \) using the Weil theorem. Since there are two(four) 2-torsion points in the curve \( E_1(\mathbb{F}_2) \), the group structure is cyclic \((\mathbb{Z}_{(q+1)/2} \oplus \mathbb{Z}_2) \).

By Theorem 4.2, we obtain that the 4 values of \( t_i \) are 0, 0, \( \sqrt{3q} \) and \(-\sqrt{3q} \) (not necessarily in that order). Since \( t_1 = t_2 = 0 \), we get \( t_3 = \pm \sqrt{3q} \) and \( t_4 = -t_3 \).

If \( n \not\equiv 0 \pmod{3} \), then \( Tr(1) \not\equiv 0 \) and \( Tr(-1) \not\equiv 0 \). So we can take \( \lambda = 1, \mu = -1 \) or \( \lambda = -1, \mu = 1 \) depending on \( n \equiv 1 \pmod{3} \) or \( n \equiv 2 \pmod{3} \) respectively in \( E_3, E_4 \) when \( n \) is odd. The order of the curve \( y^2 = x^3 - x + 1 \) is \( q + 1 + \sqrt{3q} \) if \( n \equiv 1 \pmod{12} \) and \( q + 1 - \sqrt{3q} \) if \( n \equiv 5 \pmod{12} \). The order of the curve \( y^2 = x^3 - x - 1 \) is \( q + 1 - \sqrt{3q} \) if \( n \equiv 1 \pmod{12} \) and \( q + 1 + \sqrt{3q} \) if \( n \equiv 5 \pmod{12} \).
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References


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