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Multiparameter CUSUM charts with variable sampling intervals

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Abstract

We consider the problem of using control charts to monitor more than one parameter with emphasis on simultaneously monitoring the mean and variance. The fixed sampling interval (FSI) control charts are modified to use variable sampling interval (VSI) control charts depending on what is being observed from the data. In general, approaches of monitoring the mean and variance simultaneously is to use separate charts for each parameter and a combined chart. In this paper, we use three basic strategies which are separate Shewhart charts for each parameter, a combined Shewhart chart and a combined CUSUM chart. We showed that a combined VSI CUSUM chart is comparatively more efficient than any other chart if the shifts in both mean and variance are small.

Keywords: Combined chart, FSI control chart, multiparameter CUSUM chart, separate chart, VSI control chart.

1. Introduction

Control charts are widely used to monitor measured quality variables from a process. It will be assumed that any deterioration in quality is reflected by changes in parameters. This change must be detected quickly so that corrective action can be taken.

The properties of a control chart are determined by the length of time it takes the chart to produce a signal. If the process is in control, then this time should be long so that the rate of false alarms is low, and if the process is out of control, then the time from the shift to the signal should be short so that detection is quick. The time to signal (TS) is the time from the start of the process to the time when the chart signals and the average time to signal (ATS) is the expected value of the time to signal. If the process is in control, the ATS should be large, and if process is out of control, the ATS should be small.

One of the most widely used control charts is the Shewhart control chart. The procedure involves taking sampling of fixed size at fixed sampling intervals and computing the relevant statistics. If the observed sample statistic deviates too far from target value, the chart

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signals and corrective action taken. Otherwise, the process is allowed to continue to the next sampling. When the parameter of interest is the process mean μ, the Shewhart $\bar{X}$ -chart usually has control limits set at $\mu_0 \pm 3$ standard errors, where μ_0 is the target value for the mean. This chart is easy to implement and is good at detecting large shifts from the target value. When the parameter in question is the process variance σ^2, R -chart or S^2 -chart can be used. R -chart uses the sampling range as the control statistic and is relatively simple to use. S^2 -chart uses the sample variance as the control statistic. Both charts are usually designed under the assumption that the process quality characteristic has a normal distribution. In this case the S^2 -chart is more efficient than the R -chart.

The cumulative sum (CUSUM) control chart was introduced by Page (1954) and has been widely used for monitoring the mean of a quality characteristic of a production process. The CUSUM chart has been shown to be more efficient than the standard Shewhart $\bar{X}$ -chart in detecting small and moderate shifts in the process mean.

A standard control chart takes samples from the process at fixed-length sampling intervals. Rather than sampling at fixed time intervals it seems more reasonable to allow the time between samples to vary depending on what has been observed by the sample just taken. That is, the sampling interval should be long if the sample statistic is close to target but the sampling interval should be short if the sample is not close to target.

Reynolds et al. (1988) investigated the properties of VSI control chart. They showed that VSI $\bar{X}$ -charts more efficient than FSI $\bar{X}$ -chart.

Chengalur et al. (1989) considered the Shewhart charts to simultaneously monitor more than one parameter with emphasis on simultaneously monitoring the mean and variance. They proposed a combined statistic which is sensitive to shifts in both mean and variance and showed that the VSI procedure is substantially more efficient than fixed interval procedure.

Reynolds (1988) developed the ATS for VSI CUSUM charts using a Markov-chain approximation. For detecting shifts in process mean, Reynolds et al. (1990) showed that the VSI CUSUM chart is considerably more efficient than the FSI CUSUM chart.

We consider the problem of using control charts to monitor more than one parameter with emphasis on simultaneously monitoring the mean and variance. FSI control charts are modified to use VSI depending on what is being observed from the data. Throughout this paper we use a sampling interval length of unity for the FSI charts and use two intervals $d_1,\ d_2\ (d_1 < d_2)$ for the VSI charts.

Three basic strategies are investigated. One strategy uses separate Shewhart charts for each parameter. Another strategy uses a combined Shewhart statistic which is sensitive to shifts in both the mean and variance. The third strategy uses a combined CUSUM statistic. For CUSUM charts, a Markov-chain approach is used to evaluate the ATS. This paper shows that a combined VSI CUSUM chart is comparatively more efficient than any other chart if the shifts in both mean and variance are small.

2. Shewhart charts

2.1. FSI Shewhart chart

The typical approach of monitoring the mean and variance simultaneously is to use separate charts for each parameter and then to signal if either chart signals. This procedure uses separate $\bar{X}$ and S^2 -charts. Hence the overall signal probability is

$$\alpha = 1 - (1 - \alpha_x)(1 - \alpha_s),$$

where α_x and α_s are the signal probabilities of the $\bar{X}$ and S^2 -charts, respectively. Shifts in the mean are measured in terms of standard errors and are denoted by $\delta = \sqrt{n}(\mu - \mu_0)/\sigma_0$, shifts in variance are measured in terms of σ/σ_0. Here μ_0 and σ_0 denote target values for μ and σ respectively. Then the control limits for $\bar{X}$ and S^2 -charts are given by

$$\left\{\mu_0 - Z_{1-\alpha_x/2}\frac{\sigma_0}{\sqrt{n}},\ \mu_0 + Z_{1-\alpha_x/2}\frac{\sigma_0}{\sqrt{n}}\right\},$$

$$\left\{0, \chi^2_{(n-1),1-\alpha_s}\frac{\sigma_0^2}{(n-1)}\right\},$$

respectively, and the ATS for shifts in the parameters from the target value of $\underline{\theta}_0 = (\mu_0,\ {\sigma^2}_0)$ is given by

$$\frac{1}{1 - [1 - P_{\bar{X}}(\mathrm{signal}|\delta, \sigma/\sigma_0))(1 - P_{S^2}(\mathrm{signal}|\delta, \sigma/\sigma_0))]},$$

where $P_{\bar{X}}(\mathrm{signal}|\delta, \sigma/\sigma_0)$ is a probability that $\bar{X}$ -chart signals and $P_{S^2}(\mathrm{signal}|\delta, \sigma/\sigma_0)$ is a probability that S^2 -chart signals.

As an alternative to using two separate charts, Reynolds and Ghosh (1981) proposed using a single statistic based on the squared standardized deviations of the observations from the target value μ_0. The statistic is given by

$$C_i = \sum_{j=1}^{n}\left(\frac{(X_{ij} - \mu_0)}{\sigma_0}\right)^2,$$

where X_{ij} is the j th observation at the i th sampling. The statistic C_i can be interpreted as the sum of statistics used by the $\bar{X}$ -chart and S^2 -chart, that is, $C_i = U_i + V_i$, where $U_i = n(\bar{X}_i - \mu_0)^2/\sigma_0^2,\ V_i = (n-1)S_i^2/\sigma_0^2$. This combined statistic thus includes informations about both μ and σ^2. If the process is in control, then the combined statistic has a chi-square distribution with n degrees of freedom and if the process is out of control, then the combined statistic has a noncetral chi-square distribution with n degrees of freedom and noncentrality parameter $(\delta\sigma_0/\sigma)^2$. Hence the control limits are set at $\left\{0,\ {\chi^2}_n, 1-\alpha\right\}$ and the ATS is

$$\frac{1}{P(\mathrm{signal}|\delta, \sigma/\sigma_0)},$$

where $P(\mathrm{signal}|\delta, \sigma/\sigma_0) = 1 - P\left\{0 \le C \le \frac{\chi^2_{n,1-\alpha}}{(\sigma/\sigma_0)^2}\right\},\ C \sim {\chi^2}_n\left(\left(\frac{\delta}{\sigma/\sigma_0}\right)^2\right)$.

2.2. VSI Shewhart charts

2.2.1. Separate VSI Shewhart charts

For $i = 1, 2$, let D_{ix} and D_{is} denote the respective marginal regions to which the sampling interval d_i is assigned for the separate $\bar{X}$ and S^2 -charts. That is, $D_{ix} = \left\{\bar{x}|d_x(\bar{X}) = d_i\right\}$ an $D_{is} = \left\{s^2|d_s(S^2) = d_i\right\}$. Let

$$S_x = \left\{\bar{x}|\bar{X} - \text{chart signals}\right\},$$
$$S_s = \left\{s^2|S^2 - \text{chart signals}\right\}.$$

The sampling interval is minimum of the sampling intervals assigned by the $\bar{X}$ and S^2 -charts, respectively. That is,

$$D_1 = (D_{1x} \cap S_s^c) \cup (S_x^c \cap D_{1s}),$$
$$D_2 = (D_{2x} \cap D_{2s}),$$

where D_1 and D_2 denote the regions specifying the intervals d_1 and d_2, respectively. Then the ATS is given by

$$\frac{d_1 P\left(D_1|\underline{\theta}\right) + d_2 P\left(D_2|\underline{\theta}\right)}{P\left(S|\underline{\theta}\right)\left((1 - P(S|\underline{\theta})\right)}, \tag{2.1}$$

where $P(S|\underline{\theta})$ denotes the total signal probability.

2.2.2. Combined VSI Shewhart charts

The sampling interval d_1 is used when $b < C_i \le {\chi^2}_n, 1-\alpha$ and d_2 is used when $0 \le C_i \le b$ where b is a boundary point that depends on the values of d_1 and d_2. The ATS is the same as (2.1).

3. Combined CUSUM charts

The CUSUM control statistic that is used at the i th sample is

$$U_i = \max\{U_i - 1, 0\} + (C_i - n - k),$$

where k is a constant and n is a sample size. The CUSUM chart signals that the process mean and variance have been changed whenever $U_i > h$, where $h \ge 0$ is called the decision interval. With VSI CUSUM chart, sampling interval d_1 is used when $U_j \in (g, h)$ and d_2 is used when $U_j \in (-\infty, g]$, where $-\infty < g \le h$, that is, g is the boundary between the regions specifying d_1 and d_2. One method for numerically evaluating the ATS of the CUSUM chart is based on a Markov-chain approximation. Suppose that the continuation region $C = (-\infty, h)$ for a CUSUM chart is partitioned into r regions $E_1, E_2, \cdots, E_r$, where each region corresponds to a state of Markov-chain. The transition matrix P for the Markov-chain can be written as

$$P = \begin{bmatrix} Q & (I-Q)\underline{1} \\ 0' & 1 \end{bmatrix},$$

where Q is the submatrix of P corresponding to the r transient states, $\underline{0}$ is a vector of 0s of dimension $r \times 1$ and $\underline{1}$ is an $r \times 1$ vector of 1 s.

Suppose that states $1, 2, \cdots, m$ use d_2 and states $m+1, m+2, \cdots, r$ use d_1. State 1 corresponds to $U_i \le 0$. Let $w = g/(m-1)$ and $v = (h-g)/(r-m)$. For $j = 2, 3, \cdots, m$, state j corresponds to $(j-2)w < U_i \le (j-1)w$. For $j = m+1, m+2, \cdots, r$, state j corresponds to $g + (j-m-1)v < U_i \le g + (j-m)v$.

Then transition probabilities p_{ij} for Q are as follows. Let $Y = C_i - n - k$. For $i = 1$,

$$p_{ij} = \begin{cases} P[Y \le 0|\delta, \dfrac{\sigma}{\sigma_0}] & \text{if } j = 1, \\ P[(j-2)w < Y \le (j-1)w|\delta, \dfrac{\sigma}{\sigma_0}] & \text{if } j = 2, \cdots, m, \\ P[g + (j-m-1)v < Y \le g + (j-m)v|\delta, \dfrac{\sigma}{\sigma_0}] & \text{if } j = m+1, \cdots, r. \end{cases}$$

For $i = 2, 3, \cdots, m$,

$$p_{ij} = \begin{cases} P[Y \le -(i - \dfrac{3}{2})w|\delta, \dfrac{\sigma}{\sigma_0}] & \text{if } j = 1, \\ P[(j - i - \dfrac{1}{2})w < Y \le (j - i + \dfrac{1}{2})w|\delta, \dfrac{\sigma}{\sigma_0}] & \text{if } j = 2, \cdots, m, \\ P[(m - i + \dfrac{1}{2})w + (j-m-1)v < Y \le (m - i + \dfrac{1}{2})w & \\ \quad + (j-m)v|\delta, \dfrac{\sigma}{\sigma_0}] & \text{if } j = m+1, \cdots, r. \end{cases}$$

For $i = m+1, m+2, \cdots, r$,

$$p_{ij} = \begin{cases} P[Y \le -g - (i - m - \dfrac{1}{2})v|\delta, \dfrac{\sigma}{\sigma_0}] & \text{if } j = 1, \\ P[-(m-j+1)w - (i - m - \dfrac{1}{2})v < Y \le -(m-j)w & \\ \quad - (i - m - \dfrac{1}{2})v|\delta, \dfrac{\sigma}{\sigma_0}] & \text{if } j = 2, \cdots, m, \\ P[(j - i - \dfrac{1}{2})v < Y \le (j - i + \dfrac{1}{2})v|\delta, \dfrac{\sigma}{\sigma_0}] & \text{if } j = m+1, \cdots, r. \end{cases}$$

If we define the fundamental matrix M as $M = [m_{ij}] = (I - Q)^{-1}$, then

$$ATS_i = d_2 \sum_{j=1}^{m} m_{ij} + d_1 \sum_{j=m+1}^{r} m_{ij},$$

where ATS_i is the ATS when the starting state is i.

4. Numerical results and conclusions

The primary purpose of this paper is to determine which type of chart performs best for various types of process changes. The distribution of the quality variable is $N(0,1)$ and a sample of size 5 is used for each observation and table is constructed for the following shift values in the mean and variance.

$$\delta = \sqrt{n}(\mu - \mu_0)/\sigma_0 : 0, 0.25, 0.5, 1, 2, 4$$

$$\frac{\sigma}{\sigma_0} : 1, 1.1, 1.25, 1.5, 2, 3, 4.$$

Table 4.1 ATS for various control charts

		σ/σ_0						
δ		1	1.1	1.25	1.5	2	3	4
0	ShFSI/s	200.00	64.55	18.83	5.42	1.86	1.12	1.03
	ShVSI/s	200.00	52.45	11.27	2.13	0.45	0.19	0.16
	ShFSI/c	200.00	56.01	15.40	4.53	1.69	1.09	1.02
	ShVSI/c	200.00	44.38	8.78	1.68	0.39	0.19	0.16
	CuFSI/c	200.00	30.72	9.48	4.17	2.03	1.21	1.06
	CuVSI/c	200.00	19.66	5.47	2.97	2.16	1.94	1.91
0.25	ShFSI/s	172.50	52.52	18.14	5.35	1.86	1.12	1.03
	ShVSI/s	169.31	47.62	10.74	2.08	0.45	0.19	0.16
	ShFSI/c	185.21	53.27	14.97	4.48	1.69	1.09	1.02
	ShVSI/c	182.99	41.74	8.46	1.65	0.39	0.19	0.16
	CuFSI/c	174.89	29.00	9.28	4.13	2.02	1.21	1.06
	CuVSI/c	170.18	18.32	5.38	2.96	2.16	1.94	1.91
0.5	ShFSI/s	117.51	47.56	16.29	5.14	1.84	1.12	1.03
	ShVSI/s	109.14	36.34	9.32	1.96	0.44	0.19	0.16
	ShFSI/c	149.43	46.16	13.80	4.32	1.67	1.09	1.02
	ShVSI/c	142.39	35.01	7.60	1.57	0.39	0.19	0.16
	CuFSI/c	120.70	24.72	8.75	4.04	2.01	1.21	1.06
	CuVSI/c	108.33	15.09	5.09	2.92	2.16	1.94	1.91
1	ShFSI/s	41.68	23.64	11.17	4.43	1.78	1.12	1.03
	ShVSI/s	31.21	15.13	5.62	1.57	0.41	0.19	0.16
	ShFSI/c	74.30	28.42	10.35	3.80	1.62	1.09	1.02
	ShVSI/c	61.22	18.94	5.16	1.31	0.37	0.19	0.16
	CuFSI/c	40.72	15.15	7.13	3.71	1.95	1.20	1.06
	CuVSI/c	27.55	8.61	4.29	2.78	2.14	1.94	1.91
2	ShFSI/s	6.44	5.44	4.22	2.79	1.58	1.10	1.03
	ShVSI/s	2.28	1.89	1.36	0.77	0.34	0.19	0.16
	ShFSI/c	14.12	8.35	4.71	2.58	1.46	1.08	1.02
	ShVSI/c	6.61	3.44	1.64	0.72	0.31	0.18	0.16
	CuFSI/c	7.81	5.83	4.19	2.84	1.77	1.18	1.05
	CuVSI/c	4.39	3.57	2.93	2.43	2.08	1.93	1.91
4	ShFSI/s	1.20	1.23	1.26	1.28	1.20	1.06	1.02
	ShVSI/s	0.13	0.15	0.17	0.19	0.20	0.17	0.16
	ShFSI/c	1.62	1.53	1.43	1.31	1.16	1.05	1.01
	ShVSI/c	0.21	0.21	0.21	0.21	0.20	0.17	0.16
	CuFSI/c	2.03	1.93	1.80	1.61	1.35	1.11	1.04
	CuVSI/c	2.05	2.05	2.03	201	1.97	1.92	1.91

ShFSI/s: Separate FSI Shewhart control chart
ShVSI/s: Separate VSI Shewhart control chart
ShFSI/c: Combined FSI Shewhart control chart
ShVSI/c: Combined VSI Shewhart control chart
CuFSI/c: Combined FSI CUSUM control chart
CuVSI/c: Combined VSI CUSUM control chart

We construct Table 4.1 to compare the ATS for various control charts and consider the given shift values. Except for small shifts in σ, the ATS for the combined Shewhart chart with variable sampling interval is seen to be smaller than the ATS for any other chart.

For small shifts in both μ and σ, the ATS for the combined CUSUM chart with variable sampling interval is smaller than the ATS for any other chart.

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