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# An objective Bayesian analysis for multiple step stress accelerated life tests

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**Abstract**

This paper derives noninformative priors for scale parameter of exponential distribution when the data are collected in multiple step stress accelerated life tests. We find the objective priors for this model and show that the reference prior satisfies first order matching criterion. Also, we show that there exists no second order matching prior. Some simulation results are given and using artificial data, we perform Bayesian analysis for proposed priors.

*Keywords*: Bayesian analysis, Jeffreys' prior, multiple step accelerated life test, probability matching prior, reference prior.

## 1. Introduction

In many reliability studies, the life tests were made under various environmental conditions. But for extremely reliable units it is in general impossible to make life tests under the usual conditions because the life times of units under the usual conditions may tend to be large and then the testing time may be very long. As a common approach to overcome this problem, the accelerated life tests (ALTs) are widely used, in which samples of units are subjected to conditions of greater stress than the usual conditions. For example, accelerated test conditions involve higher than usual temperature, voltage, pressure, vibration, cycling rate, load, etc., or some combination of them.

The step stress ALT is commonly used in engineering practice. We interest the step stress ALT wherein the stress on unfailed units is allowed to change at preassigned times until they fail.

The existing literature on analysis of step-stress ALT centered around three types of models.

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DeGroot and Goel (1979) proposed tampered random variables (TRV) model which the effect of changing the stress from $s_1$ to $s_2$ ( $s_1 < s_2$) is to multiply the remaining life of the unit at changing time $\tau$ (which they called it as tampering point) by some unknown factor, called tampering coefficient, $\alpha$ $(0 < \alpha < 1)$. The proposed model is

$$Y = \begin{cases} X & X \le \tau, \\ \tau + \alpha(X - \tau) & X > \tau. \end{cases} \tag{1.1}$$

There is an another model for analyzing the accelerated life test data, which was proposed by Bhattacharyya and Soejoeti (1989). They assumed that the effect of changing the stress is to multiply the initial failure rate function $\lambda_1(y)$ by an unknown factor $\alpha$ subsequent to the change point $\tau$. Denoting the failure rate function of the step stress life length by $\lambda^*(y)$, the tampered failure rate (TFR) model is defined as

$$\lambda^*(y) = \begin{cases} \lambda_1(y) & y \le \tau, \\ \alpha\lambda_1(y) & y > \tau. \end{cases} \tag{1.2}$$

Nelson (1980) proposed cumulative exposure (CE) model as follows: Let $F^*$ be the cumulative distribution function of the step stress data which can be specified by $F_i(y) = F(y|s_i), i = 1, 2,$ where $F(y|s_i)$ is the cumulative distribution function of life length under the constant stress setting $s_i$. The CE model is defined by

$$F^*(y) = \begin{cases} F_1(y) & y \le \tau, \\ F_2(\nu_1 + y - \tau) & y > \tau, \end{cases} \tag{1.3}$$

where $\nu_1$ is defined to be the solution of the equation $F_2(\nu_1) = F_1(\tau)$.

Bhattacharyya and Soejoeti (1989) indicated that the TRV, CE and TFR models are identical in the sense that TRV can be expressed by the other models through reparameterization, when distribution under use stress is exponential.

For applications of ALT model to the real data, multiple ( a model with more than two tampering points) step stress ALT model will be applicable to the extremely reliable items. There have been several works extending two step stress (or simple step stress) ALT model to multiple step stress model. Typical examples are Shaked and Singpurwalla (1983) and Madi (1993).

The papers mentioned above except Degroot and Geol (1976) solved the estimation problem in step stress ALT models using the classical or nonparametric methods. From a Bayesian point of view, DeGroot and Goel (1979) studied the Bayesian estimation of parameters and optimal design of the model (1.1) when the lifetime under use stress is exponential distribution. They considered two independent gamma priors for parameter estimation.

Owing to the lack of prior knowledge about parameters or lack of time to accumulate the information about the model, there may be an inevitable situation to use noninformative priors. The most commonly used noninformative prior is Jeffreys' (1961) prior, which is proportional to the positive square root of the determinant of the Fisher information matrix. Jeffreys' prior plays a major role in many one parameter models, but Jeffreys' prior frequently runs into serious difficulties in the presence of nuisance parameters. Jeffreys' prior does not hold invariant property under reparameterization and does not match frequentist coverage probability.

In recent years, many efforts have been done for finding noninformative priors such as reference or probability matching prior in Bayesian analysis. There has been a great deal of studies for finding noninformative priors.

Welch and Peers (1963), Peers (1965) and Stein (1985) found a prior which requires the frequentist coverage probability of the posterior region of a real-valued parametric function to match the normal level with a remainder of $o(n^{-\frac{1}{2}})$, where $n$ is the sample size. Tibshirani (1989) reconsidered the case when the real valued parameter of interest is orthogonal to the nuisance parameter vector. These priors, as usually referred to as 'first order' matching priors, were further studied in Datta and Ghosh (1995a, 1995b, 1996).

Recently, Ghosh and Mukerjee (1997) developed a 'second order', that is, $o(n^{-1})$, matching prior. They extend the finding in Mukerjee and Dey (1993) to the case of multiple nuisance parameters based on quantiles, and also develop a second order matching prior based on distribution function.

On the other hand, Ghosh and Mukerjee (1992), and Berger and Bernardo (1989, 1992) extended Bernardo's (1979) reference prior approach, giving a general algorithm to derive a reference prior by splitting the parameters into several groups according to their order of inferential importance. This approach is very successful in various practical problems. Quite often reference priors satisfy the matching criterion described earlier.

In this paper, we will generalize two step stress TRV model (1.1) to multiple step stress TRV model.

For the Bayesian analysis, we derive the reference prior and matching prior for the scale parameter when the lifetime distribution under normal stress is exponential. Through the orthogonal transformation in the sense of Cox and Reid (1987), we first find the orthogonal reparameterization for scale parameter and then find reference prior and matching prior. We show that the proposed matching prior is the first order matching prior and that there exists no second order matching for multiple step stress ALT model. We show that, under the proposed noninformative priors, the joint posterior for the parameters is proper. And some simulation results and example are given.

## 2. Multiple step stress accelerated life test model

Consider the realistic situation of accelerated life testing where we continue increasing the stress level on the unfailed items over a preassigned number $k(\geq 1)$ of times. And we assume that the lifetime distribution under normal stress follows exponential distribution with parameter $\theta$ of which the probability density function (pdf) is given by

$$f_1(x|\theta) = \theta \exp\{-\theta x\}, 0 < x < \infty, 0 < \theta < \infty. \tag{2.1}$$

When $k \geq 2$, we call it as a multiple step stress ALT model. Now, we generalize the simple step stress TRV model (1.1) to a multiple step stress TRV model. Let

$$0 < \tau_1 < \tau_2 < \cdots < \tau_k < \infty$$

be the $k$'s tampering points. Starting to the normal stress level $s_0$, at the tampering point $\tau_1$, we raise stress level to $s_1(> s_0)$, and so on. According to the stress level $s_i$, $i = 1, 2, \cdots, k$, there is tampering coefficient $\alpha_i$ which represents the effect of stress change. Let $Y$ be the lifetime under multiple step stress pattern, then, the multiple step stress TRV model can be described as follows.

$$Y = \begin{cases} X, & \text{if } X \le \tau_1, \\ \dfrac{X - \frac{\sum_{i=1}^{l-1}(\tau_i - \tau_{i-1})}{\prod_{j=0}^{i-1}\alpha_j}}{\prod_{j=0}^{l-1}\alpha_j^{-1}} + \tau_{l-1}, & \text{if } \frac{\sum_{i=1}^{l-1}(\tau_i-\tau_{i-1})}{\prod_{j=0}^{i-1}\alpha_j} < X \le \frac{\sum_{i=1}^{l}(\tau_i-\tau_{i-1})}{\prod_{j=0}^{i-1}\alpha_j}, \\ & l = 2, \cdots, k, \\ \dfrac{X - \sum_{i=1}^{k}(\tau_i - \tau_{i-1})\prod_{j=0}^{i-1}\alpha_j^{-1}}{\prod_{j=0}^{k}\alpha_j^{-1}} + \tau_k, & \text{if } X > \frac{\sum_{i=1}^{k}(\tau_i-\tau_{i-1})}{\prod_{j=0}^{i-1}\alpha_j}, \end{cases} \tag{2.2}$$

where $\alpha_0 \equiv 1$, $\tau_0 \equiv 0$, $0 < \alpha_i < 1, i = 1, \cdots, k$ and $X$ is exponentially distributed random variable with density (2.1).

If we denote $F_1(y|\theta)$ as the distribution function of $X$, then the distribution function $F$ of $Y$ is given by

$$F(y|\theta, \underline{\alpha}) = \begin{cases} F_1(y|\theta), & y \le \tau_1, \\ F_1\left(\frac{\sum_{i=1}^{l-1}(\tau_i-\tau_{i-1})}{\prod_{j=0}^{i-1}\alpha_j} + \frac{(y-\tau_{l-1})}{\prod_{j=0}^{l-1}\alpha_j}|\theta\right), & \tau_{l-1} < y \le \tau_l, \quad l = 2, \cdots, k, \\ F_1\left(\frac{\sum_{i=1}^{k}(\tau_i-\tau_{i-1})}{\prod_{j=0}^{i-1}\alpha_j} + \frac{(y-\tau_k)}{\prod_{j=0}^{k}\alpha_j}|\theta\right), & y > \tau_k, \end{cases} \tag{2.3}$$

where $\underline{\alpha} = (\alpha_0, \alpha_1, \cdots, \alpha_k)$.

The probability density function $f(y|\theta, \underline{\alpha})$ of $Y$ is given by

$$f(y|\theta, \underline{\alpha}) = \begin{cases} \theta \exp\{-\theta y\}, & y \le \tau_1, \\ \frac{\theta}{\prod_{j=0}^{l-1}\alpha_j} e^{-\theta\left[\frac{\sum_{i=1}^{l-1}(\tau_i-\tau_{i-1})}{\prod_{j=0}^{i-1}\alpha_j} + \frac{(y-\tau_{l-1})}{\prod_{j=0}^{l-1}\alpha_j}\right]}, & \tau_{l-1} < y \le \tau_l, l = 2, \cdots, k, \\ \frac{\theta}{\prod_{j=0}^{k}\alpha_j} e^{-\theta\left[\frac{\sum_{i=1}^{k}(\tau_i-\tau_{i-1})}{\prod_{j=0}^{i-1}\alpha_j} + \frac{(y-\tau_k)}{\prod_{j=0}^{k}\alpha_j}\right]}, & y > \tau_k, \end{cases} \tag{2.4}$$

where $0 < \theta < \infty$ and $0 < \alpha_i < 1$, $i = 1, 2, \cdots, k$.

Let $\delta_1 = I(y \le \tau_1)$, $\delta_l = I(\tau_{l-1} < y \le \tau_l)$, $l = 2, \cdots, k$, and $\delta_{k+1} = I(y > \tau_k)$, where $I$ is indicator function. Then $\sum_{i=1}^{k+1}\delta_i = 1$. And the likelihood function per one observation $y$ is given by,

$$L(\theta, \underline{\alpha}|y) = \frac{\theta e^{-\theta y \delta_1}}{\prod_{l=1}^{k}\alpha_l^{\sum_{i=l+1}^{k+1}\delta_i}} \exp\left\{-\theta \sum_{l=2}^{k+1}\delta_l \left[\frac{\sum_{i=1}^{l-1}(\tau_i - \tau_{i-1})}{\prod_{j=0}^{i-1}\alpha_j} + \frac{(y - \tau_{l-1})}{\prod_{j=0}^{l-1}\alpha_j}\right]\right\}. \tag{2.5}$$

The log-likelihood function for one observation is

$$\mathcal{L}(\theta, \underline{\alpha}|y) = \log\theta - \sum_{l=2}^{k}\log\alpha_l \sum_{i=l+1}^{k+1}\delta_i - \theta y \delta_1 - \theta\sum_{l=2}^{k+1}\delta_l\left[\frac{\sum_{i=1}^{l-1}(\tau_i - \tau_{i-1})}{\prod_{j=0}^{i-1}\alpha_j} + \frac{(y - \tau_{l-1})}{\prod_{j=0}^{l-1}\alpha_j}\right]. \tag{2.6}$$

Usually, one purpose of the ALTs is the information about the parameter under the normal stress level. In our multiple step stress TRV model, $\theta$, which is the failure rate at the normal stress, is more important parameter than the others.

Now, we consider the reparameterization of the original parameters to accomplish the parameter orthogonality in the sense of Cox and Reid (1987). To do this, let $\omega_1 = \theta$ and $\omega_l = \theta / \prod_{j=1}^{l-1} \alpha_j$, $l = 2, \cdots, k+1$. Then the log-likelihood function under reparameterization is given by

$$\mathcal{L}(\underline{\omega}|y) = \sum_{l=1}^{k+1} \delta_l \log \omega_l - \omega_1 y \delta_1 - \sum_{l=2}^{k+1} \delta_l \left[ \sum_{i=1}^{l-1} (\tau_i - \tau_{i-1}) \omega_i + (y - \tau_{l-1}) \omega_l \right], \tag{2.7}$$

where $\underline{\omega} = (\omega_1, \cdots, \omega_{k+1}) \in \Theta_\omega$, $\Theta_\omega = \{\underline{\omega} | 0 < \omega_1 < \omega_2 < \omega_3 < \cdots < \omega_{k+1} < \infty\}$.

From the above reparameterized log-likelihood function (2.7), one can find the Fisher information matrix for the $\underline{\omega}$.

Let $I_\omega$ be the Fisher information matrix of $\underline{\omega}$. Then the information matrix $I_\omega$ is a diagonal matrix with elements $I_i, i = 1, 2, \cdots, k+1$. The elements are given by

$$I_1 = \frac{1}{\omega_1^2} \left(1 - \exp\{-\omega_1 \tau_1\}\right),$$

$$I_l = \frac{1}{\omega_l^2} \exp\left( - \sum_{i=1}^{l-1} (\tau_i - \tau_{i-1}) \omega_i \right) \left(1 - \exp\{-(\tau_l - \tau_{l-1}) \omega_l\}\right), l = 2, 3, \cdots, k,$$

and

$$I_{k+1} = \frac{1}{\omega_{k+1}^2} \exp\left( - \sum_{i=1}^{k} (\tau_i - \tau_{i-1}) \omega_i \right).$$

## 3. Noninformative priors

In this section, we will derive the noninformative priors in multiple step stress TRV model. From the information matrix $I_\omega$, one can find Jeffrey's prior for $\underline{\omega}$ as follows.

$$\pi^J(\underline{\omega}) \propto \left( \prod_{l=1}^{k+1} \omega_l^{-1} \right) \times \left[ \left( \prod_{l=1}^{k} (1 - \exp\{-(\tau_l - \tau_{l-1}) \omega_l\}) \right) \left( \prod_{l=2}^{k+1} \exp\{ - \sum_{i=1}^{l-1} (\tau_i - \tau_{i-1}) \omega_i \} \right) \right]^{1/2}, \tag{3.1}$$

where $\underline{\omega} \in \Theta_\omega$. Using the identity

$$\sum_{l=2}^{k+1} \sum_{i=1}^{l-1} (\tau_i - \tau_{i-1}) \omega_i = \sum_{l=1}^{k} (k - l + 1)(\tau_l - \tau_{l-1}) \omega_l,$$

Jeffrey's prior can be rewritten as

$$\pi^J(\underline{\omega}) \propto \frac{\prod_{l=1}^{k} (1 - \exp\{-(\tau_l - \tau_{l-1}) \omega_l\})^{1/2} \exp\left\{ -\frac{1}{2} \left( \sum_{l=1}^{k} (k - l + 1)(\tau_l - \tau_{l-1}) \omega_l \right) \right\}}{\prod_{l=1}^{k+1} \omega_l}.$$

It is well known that the prior (3.1) does not meet the nominal level coverage probability in case of the presence of nuisance parameters. Also, the prior does not satisfy the invariance property under transformation. To remedy these problems, we will find a reference prior and a matching prior when $\omega_1$ is a parameter of interest.

We introduce the brief concept of a matching prior. For a prior $\pi$, let $\theta_1^{1-\alpha}(\pi;\mathbf{Y})$ be a percentile of the posterior distribution of $\theta_1$, that is,

$$P^{\pi}\{\theta_1 \le \theta_1^{1-\alpha}(\pi;\mathbf{Y})|\mathbf{Y}\} = 1-\alpha. \tag{3.2}$$

We want to find priors which satisfy

$$P^{\pi}\{\theta_1 \le \theta_1^{1-\alpha}(\pi;\mathbf{Y})|\theta,\alpha\} = 1-\alpha + o(n^{-\frac{u}{2}}) \tag{3.3}$$

for some $u > 0$, as $n$ goes to $\infty$. Priors $\pi$ satisfying (3.3) are called matching priors. If $u = 1$, then $\pi$ is called a first order matching prior, if $u = 2$, $\pi$ is called a second order matching prior.

Now, we want to find a matching prior when the parameter of interest is $\omega_1$. Denote $\omega^{(2)} = (\omega_2, \cdots, \omega_{k+1})$ as a nuisance parameters. Based on the work of Tibshirani (1989), the first order probability priors, when the parameter of interest is $\omega_1$, is given by,

$$\pi^M(\underline{\omega}) \propto \omega_1^{-1}\left(1-\exp\{-\omega_1\tau_1\}\right)^{1/2} d(\omega^{(2)}), \tag{3.4}$$

where $d(\cdot)$ is an arbitrary differentiable function in its arguments. Clearly, the Jeffrey's prior (3.1) is not the first order matching prior.

The class of priors given in (3.4) is large, and it may be necessary to narrow down this class of priors. Mukerjee and Ghosh (1997) suggested the second order matching prior which give more accurate frequentist coverage probability than first order matching prior. They showed that a second order probability matching prior is of the form (3.4), and $d(\omega^{(2)})$ must satisfy the following differential equation,

$$\frac{1}{6}d(\omega_{(2)})\frac{\partial}{\partial\omega_1}\left[(I_1^{-\frac{3}{2}})L_{1,1,1}\right] + \sum_{\nu=2}^{k+1}\sum_{s=2}^{k+1}\frac{\partial}{\partial\omega_\nu}\left\{I_1^{-\frac{1}{2}}L_{11s}(I^{s\nu})d(\omega^{(2)})\right\} = 0, \tag{3.5}$$

where

$$L_{1,1,1} = E\left[\left(\frac{\partial\mathcal{L}(\underline{\omega}|y)}{\partial\omega_1}\right)^3\right],$$

$$L_{11s} = E\left[\frac{\partial^3\mathcal{L}(\underline{\omega}|y)}{\partial\omega_1^2\partial\omega_s}\right],$$

and $I^{s\nu}$ is the $(s,\nu)$-th element of the inverse of Fisher information matrix.

It can be easily verified that , for $s = 2, 3, \cdots, k+1$,

$$\frac{\partial^3\mathcal{L}(\underline{\omega}|y)}{\partial\omega_1^2\partial\omega_s} = 0,$$

and $I^{s\nu} = 0, s \ne \nu$. Hence the second term in equation (3.5) is 0.

The only way which the prior (3.4) satisfies the second order matching condition (3.5) is that $(I_1^{-\frac{3}{2}})L_{1,1,1}$ is a function of $\omega^{(2)}$ or a constant. To verify that whether the second order matching criterion is satisfied or not, we compute $L_{1,1,1}$ in equation (3.5). From the log-likelihood function (2.7),

$$\frac{\partial \mathcal{L}(\underline{\omega}|y)}{\partial \omega_1} = \frac{\delta_1}{\omega_1} - y\delta_1 - \tau_1(1-\delta_1).$$

Then,

$$\begin{aligned} L_{1,1,1} &= E\left[\left(\delta_1(\omega_1^{-1} - y) - \tau_1(1-\delta_1)\right)^3\right] \\ &= E\left[\left(\delta_1(\omega_1^{-1} - y)\right)^3\right] - \tau_1^3 E\left[1-\delta_1\right]. \end{aligned}$$

The second term in the above equation is

$$\tau_1^3 E\left[1-\delta_1\right] = \tau_1^3 \exp\{-\tau_1\omega_1\},$$

and the first term can be expanded as the following equation :

$$E\left[\left(\delta_1(\omega_1^{-1} - y)\right)^3\right] = \omega_1^{-3}E[\delta_1] - E[\delta_1 Y^3] - 3\omega_1^{-2}E[\delta_1 Y] + 3\omega_1^{-1}E[\delta_1 Y^2].$$

The expectations in the above equation are calculated to be

$$\begin{aligned} E[\delta_1 Y] &= \int_0^{\tau_1} y\omega_1 \exp\{-\omega_1 y\} dy \\ &= -\tau_1 \exp\{-\omega_1\tau_1\} + \omega_1^{-1}(1-\exp\{-\omega_1\tau_1\}), \end{aligned}$$

$$\begin{aligned} E[\delta_1 Y^2] &= \int_0^{\tau_1} y^2\omega_1 \exp\{-\omega_1 y\} dy \\ &= -\tau_1^2 \exp\{-\omega_1\tau_1\} + 2\omega_1^{-2}(1-\exp\{-\omega_1\tau_1\}) - 2\omega_1^{-1}\tau_1 \exp\{-\omega_1\tau_1\} \end{aligned}$$

and

$$\begin{aligned} E[\delta_1 Y^3] &= \int_0^{\tau_1} y^2\omega_1 \exp\{-\omega_1 y\} dy \\ &= -\tau_1^3 \exp\{-\omega_1\tau_1\} + 6\omega_1^{-3}(1-\exp\{-\omega_1\tau_1\}) - 6\omega_1^{-2}\tau_1 \exp\{-\omega_1\tau_1\} \\ &\quad -3\omega_1^{-1}\tau_1^2 \exp\{-\omega_1\tau_1\}. \end{aligned}$$

Hence,

$$L_{1,1,1} = 3\omega_1^{-2}\tau_1 \exp\{-\omega_1\tau_1\} - 2\omega_1^{-3}(1-\exp\{-\omega_1\tau_1\}).$$

So,

$$I_1^{-\frac{3}{2}} \times L_{1,1,1} = 3\omega_1\tau_1 \exp\{-\omega_1\tau_1\}(1-\exp\{-\omega_1\tau_1\})^{-\frac{3}{2}} - 2(1-\exp\{-\omega_1\tau_1\})^{\frac{1}{2}}. \qquad (3.6)$$

The equation (3.6) is not a function of $\omega^{(2)}$ or a constant. The second order matching prior does not exist in this case.

Berger and Bernardo (1992a) developed the algorithm to find a reference prior. And Datta and Ghosh (1995) proposed the method of developing reference priors when the orthogonality of the parameters is satisfied. In our case, $\omega_1$ is of more inferential importance than $\omega^{(2)}$, the one at a time reference prior is given by

$$\pi^R(\underline{\omega}) \propto \left(\prod_{i=1}^{k+1}\omega_i\right)^{-1}\left[\prod_{i=1}^{k}(1-\exp\{-\omega_i(\tau_i-\tau_{i-1})\})\right]^{1/2}, \tag{3.7}$$

where $\underline{\omega} \in \Theta_\omega$. One can find the fact that this prior is also the first order matching prior.

## 4. Posterior analysis

Suppose that $y_1, y_2, \cdots, y_n$ is random sample from pdf (2.4). Then the likelihood function of $(\theta, \underline{\alpha})$ is given by

$$L(\theta,\underline{\alpha}|\underline{y}) = \prod_{v=1}^{n} f(y_v|\theta,\underline{\alpha}) = \theta^n \prod_{l=1}^{k}\alpha_l^{-\sum_{v=1}^{n}\sum_{i=l+1}^{k+1}\delta_i^v}$$
$$\times \exp\left[-\theta\sum_{v=1}^{n}\delta_1^v y_v - \theta\sum_{v=1}^{n}\sum_{l=2}^{k+1}\delta_l^v\{\sum_{i=1}^{l-1}(\tau_i-\tau_{i-1})\prod_{j=0}^{i-1}\alpha_j^{-1} + (y_v-\tau_{l-1})\prod_{j=0}^{l-1}\alpha_j^{-1}\}\right],$$

where $\delta_1^v = I(y_v \le \tau_1)$, $\delta_l^v = I(\tau_{l-1} < y_v \le \tau_l)$, $l = 2, 3, \cdots, k$ and $\delta_{k+1}^v = I(y_v > \tau_k)$. Let $m_1 = \sum_{v=1}^n \delta_1^v$ be the number of observations failed until $\tau_1$, let $m_l = \sum_{v=1}^n \delta_l^v$ be the number of observations failed between $\tau_{l-1}$ and $\tau_l$, $l = 2, \cdots, k$, and let $m_{k+1} = \sum_{v=1}^n \delta_{k+1}^v$ be the number of observations failed beyond $\tau_k$. Then,

$$\sum_{v=1}^{n}\sum_{i=l+1}^{k+1}\delta_i^v = \sum_{i=l+1}^{k+1} m_i = n - \sum_{i=1}^{l} m_i.$$

Let

$$V^{(1)} = \sum_{v=1}^{n}\delta_1^v y_v,$$

and, for $l = 2, 3, \cdots, k+1$,

$$V^{(l)} = \sum_{v=1}^{n}\delta_l^v(y_v - \tau_{l-1}).$$

Under the reparametrization, the above likelihood function is

$$L(\underline{\omega}|\underline{y}) = \prod_{l=1}^{k+1}\omega_l^{m_l}\exp\{-\omega_1 V^{(1)} - \sum_{l=2}^{k+1} m_l\sum_{i=1}^{l-1}(\tau_i-\tau_{i-1})\omega_i - \sum_{l=2}^{k+1}\omega_l V^{(l)}\}. \tag{4.1}$$

Combining the above likelihood (4.1) and the Jeffrey's prior (3.1), one can obtain the joint posterior pdf of $\underline{\omega}$ as follows:

$$\pi^J(\,\underline{\omega}\,|\underline{y}) \propto \left(\prod_{l=1}^{k+1}\omega_l^{m_l-1}\right)\left(\prod_{l=1}^{k}(1-\exp\{-(\tau_l-\tau_{l-1})\omega_l\})^{1/2}\right)$$
$$\times \exp\left\{-\sum_{l=1}^{k}\omega_l\left(V^{(l)} + (\tau_l-\tau_{l-1})(\sum_{i=l+1}^{k+1} m_i + \frac{(k-l+1)}{2})\right) - \omega_{k+1}V^{(k+1)}\right\}. \tag{4.2}$$

The propriety of the above joint posterior is proved in the next theorem.

**Theorem 4.1** The joint posterior (4.2) is proper, if $m_l > 0$, $l = 1, 2, \cdots, k+1$.

**Proof**:

$$
\begin{aligned}
&\int_{\Theta_\omega} \pi^J(\underline{\omega}|\underline{y})d\underline{\omega} \\
&= \int_0^\infty \omega_1^{m_1-1}(1-e^{-\omega_1\tau_1})^{\frac{1}{2}} e^{-\omega_1(\frac{1}{2}k\tau_1+V^{(1)}+\tau_1\sum_{i=2}^{k+1}m_i)} \\
&\times \int_{\omega_1}^\infty \omega_2^{m_2-1}(1-e^{-\omega_2(\tau_2-\tau_1)})^{\frac{1}{2}} e^{-\omega_2\left(\frac{1}{2}(k-1)(\tau_2-\tau_1)+V^{(2)}+(\tau_2-\tau_1)\sum_{i=3}^{k+1}m_i\right)} \\
&\times \int_{\omega_2}^\infty \omega_3^{m_3-1}(1-e^{-\omega_3(\tau_3-\tau_2)})^{\frac{1}{2}} e^{-\omega_3\left(\frac{1}{2}(k-2)(\tau_3-\tau_2)+V^{(3)}+(\tau_3-\tau_2)\sum_{i=4}^{k+1}m_i\right)} \\
&\times \cdots \\
&\times \int_{\omega_{k-2}}^\infty \omega_{k-1}^{m_{k-1}-1}(1-e^{-\omega_{k-1}(\tau_{k-1}-\tau_{k-2})})^{\frac{1}{2}} e^{-\omega_{k-1}\left((\tau_{k-1}-\tau_{k-2})+V^{(k-1)}+(\tau_{k-1}-\tau_{k-2})\sum_{i=k}^{k+1}m_i\right)} \\
&\times \int_{\omega_{k-1}}^\infty \omega_k^{m_k-1}(1-e^{-\omega_k(\tau_k-\tau_{k-1})})^{\frac{1}{2}} e^{-\omega_k\left(\frac{1}{2}(\tau_k-\tau_{k-1})+V^{(k)}+(\tau_k-\tau_{k-1})m_{k+1}\right)} \\
&\times \int_{\omega_k}^\infty \omega_{k+1}^{m_{k+1}-1} e^{-\omega_{k+1}V^{(k+1)}} d\omega_{k+1}d\omega_k\cdots d\omega_3 d\omega_2 d\omega_1.
\end{aligned}
$$

Since, for $l = 1, \cdots, k$, $(1-e^{-\omega_l(\tau_l-\tau_{l-1})})^{\frac{1}{2}} \leq 1$,

$$
\begin{aligned}
\int_{\Theta_\omega} \pi^J(\underline{\omega}|\underline{y})d\underline{\omega} \leq & \int_0^\infty \omega_1^{m_1-1} e^{-\omega_1(\frac{1}{2}k\tau_1+V^{(1)}+\tau_1\sum_{i=2}^{k+1}m_i)} \\
&\times \int_0^\infty \omega_2^{m_2-1} e^{-\omega_2\left(\frac{1}{2}(k-1)(\tau_2-\tau_1)+V^{(2)}+(\tau_2-\tau_1)\sum_{i=3}^{k+1}m_i\right)} \\
&\times \int_0^\infty \omega_3^{m_3-1} e^{-\omega_3\left(\frac{1}{2}(k-2)(\tau_3-\tau_2)+V^{(3)}+(\tau_3-\tau_2)\sum_{i=4}^{k+1}m_i\right)} \\
&\times \cdots \\
&\times \int_0^\infty \omega_{k-1}^{m_{k-1}-1} e^{-\omega_{k-1}\left((\tau_{k-1}-\tau_{k-2})+V^{(k-1)}+(\tau_{k-1}-\tau_{k-2})\sum_{i=k}^{k+1}m_i\right)} \\
&\times \int_0^\infty \omega_k^{m_k-1} e^{-\omega_k\left(\frac{1}{2}(\tau_k-\tau_{k-1})+V^{(k)}+(\tau_k-\tau_{k-1})m_{k+1}\right)} \\
&\times \int_0^\infty \omega_{k+1}^{m_{k+1}-1} e^{-\omega_{k+1}V^{(k+1)}} d\omega_{k+1}d\omega_k\cdots d\omega_3 d\omega_2 d\omega_1 \\
&< \infty,
\end{aligned}
$$

if $m_l > 0$, $l = 1, 2, \cdots, k+1$. This completes the proof. □

Using the first order matching prior given in (3.4), the joint posterior is given by

$$
\pi^M(\underline{\omega}|\underline{y}) \propto \frac{\omega_1^{m_1-1}\exp\left\{-\sum_{l=1}^k \omega_l\left[(\tau_l-\tau_{l-1})\sum_{i=l+1}^{k+1}m_i + V^{(l)}\right] - \omega_{k+1}V^{(k+1)}\right\}}{(1-e^{-\omega_1\tau_1})^{-\frac{1}{2}}\left(\prod_{l=2}^{k+1}\omega_l^{-m_l}\right)}. \quad (4.3)
$$

The propriety of (4.3) can be proved similarly.

**Theorem 4.2** The joint posterior (4.3) is proper, if $m_1 > 0$ and $m_l \geq 0$, $l = 2, 3, \cdots, k+1$. The joint posterior under reference prior (3.7) is given by

$$\pi^R(\underline{\omega}|\underline{y}) \propto (\prod_{l=1}^{k+1} \omega_l^{m_l-1})(\prod_{l=1}^{k}(1-e^{-\omega_l \tau_l})^{\frac{1}{2}}) \\ \times \exp\left\{-\sum_{l=1}^{k}\omega_l\left[(\tau_l-\tau_{l-1})\sum_{i=l+1}^{k+1} m_i + V^{(l)}\right] - \omega_{k+1}V^{(k+1)}\right\}. \qquad (4.4)$$

**Theorem 4.3** The joint posterior (4.4) is proper, if $m_l > 0$, $l = 1, 2, \cdots, k+1$.

Now, we find the marginal posterior pdf's of the interest parameter $\omega_1$, under noninformative priors.

**Theorem 4.4**

1. The marginal posterior of $\omega_1$ under Jeffrey's prior is given by

$$\pi^J(\omega_1|\underline{y}) = \frac{\omega_1^{m_1-1}(1-e^{-\omega_1\tau_1})^{\frac{1}{2}} e^{-\omega_1(\frac{1}{2}k\tau_1 + V^{(1)} + \tau_1 \sum_{i=2}^{k+1} m_i)}}{N_J^{(k+1)}} \\ \times \int_{\omega_1}^{\infty}\int_{\omega_2}^{\infty}\cdots\int_{\omega_{k-2}}^{\infty}\int_{\omega_{k-1}}^{\infty}[\prod_{l=2}^{k}\omega_l^{m_l-1}(1-e^{-\omega_l(\tau_l-\tau_{l-1})})^{\frac{1}{2}} \\ \times e^{-\omega_l(\frac{1}{2}(k-l+1)(\tau_l-\tau_{l-1}) + V^{(l)} + (\tau_l-\tau_{l-1})\sum_{i=l+1}^{k+1} m_i)}] \\ \times [1 - IG(\omega_k|m_{k+1}, V^{(k+1)})]d\omega_k d\omega_{k-1}\cdots d\omega_3 d\omega_2, \qquad (4.5)$$

where $N_J^{(k+1)}$ is the normalizing constant given by

$$N_J^{(k+1)} = \int_0^{\infty}\int_{\omega_1}^{\infty}\cdots\int_{\omega_{k-2}}^{\infty}\int_{\omega_{k-1}}^{\infty}[\prod_{l=1}^{k}\omega_l^{m_l-1}(1-e^{-\omega_l(\tau_l-\tau_{l-1})})^{\frac{1}{2}} \\ \times e^{-\omega_l(\frac{1}{2}(k-l+1)(\tau_l-\tau_{l-1}) + V^{(l)} + (\tau_l-\tau_{l-1})\sum_{i=l+1}^{k+1} m_i)}] \\ \times [1 - IG(\omega_k|m_{k+1}, V^{(k+1)})]d\omega_k d\omega_{k-1}\cdots d\omega_2 d\omega_1$$

and $IG(x|\gamma_1, \gamma_2)$ is an incomplete gamma function with parameters $\gamma_1$ and $\gamma_2$.

2. The marginal posterior of $\omega_1$ under matching prior is given by

$$\pi^M(\omega_1|\underline{y}) = \frac{\omega_1^{m_1-1}(1-e^{-\omega_1\tau_1})^{\frac{1}{2}} e^{-\omega_1(V^{(1)} + \tau_1\sum_{i=2}^{k+1} m_i)}}{N_M^{(k+1)}} \\ \times \int_{\omega_1}^{\infty}\int_{\omega_2}^{\infty}\cdots\int_{\omega_{k-2}}^{\infty}\int_{\omega_{k-1}}^{\infty}[\prod_{l=2}^{k}\omega_l^{m_l} e^{-\omega_l(V^{(l)} + (\tau_l-\tau_{l-1})\sum_{i=l+1}^{k+1} m_i)}] \\ \times [1 - IG(\omega_k|m_{k+1}+1, V^{(k+1)})]d\omega_k d\omega_{k-1}\cdots d\omega_3 d\omega_2, \qquad (4.6)$$

where $N_M^{(k+1)}$ is the normalizing constant given by

$$N_M^{(k+1)} = \int_0^\infty \int_{\omega_1}^\infty \cdots \int_{\omega_{k-2}}^\infty \int_{\omega_{k-1}}^\infty \omega_1^{m_1-1}(1-e^{-\omega_1\tau_1})^{\frac{1}{2}} e^{-\omega_1(V^{(1)}+\tau_1\sum_{i=2}^{k+1} m_i)}$$
$$\times [\prod_{l=2}^{k} \omega_l^{m_l} e^{-\omega_l(V^{(l)}+(\tau_l-\tau_{l-1})\sum_{i=l+1}^{k+1} m_i)}]$$
$$\times [1 - IG(\omega_k|m_{k+1}+1, V^{(k+1)})] d\omega_k d\omega_{k-1}\cdots d\omega_2 d\omega_1.$$

3. The marginal posterior of $\omega_1$ under reference prior is given by

$$\pi^R(\omega_1|\underline{y}) = \frac{\omega_1^{m_1-1}(1-e^{-\omega_1\tau_1})^{\frac{1}{2}} e^{-\omega_1(V^{(1)}+\tau_1\sum_{i=2}^{k+1} m_i)}}{N_R^{(k+1)}}$$
$$\times \int_{\omega_1}^\infty \int_{\omega_2}^\infty \cdots \int_{\omega_{k-2}}^\infty \int_{\omega_{k-1}}^\infty [\prod_{l=2}^{k} \omega_l^{m_l-1}(1-e^{-\omega_l(\tau_l-\tau_{l-1})})^{\frac{1}{2}}$$
$$\times e^{-\omega_l(V^{(l)}+(\tau_l-\tau_{l-1})\sum_{i=l+1}^{k+1} m_i)}]$$
$$\times [1 - IG(\omega_k|m_{k+1}, V^{(k+1)})] d\omega_k d\omega_{k-1}\cdots d\omega_3 d\omega_2, \qquad (4.7)$$

where $N_R^{(k+1)}$ is the normalizing constant given by

$$N_R^{(k+1)} = \int_0^\infty \int_{\omega_1}^\infty \cdots \int_{\omega_{k-2}}^\infty \int_{\omega_{k-1}}^\infty \left[\prod_{l=1}^{k} \frac{\omega_l^{m_l-1}(1-e^{-\omega_l(\tau_l-\tau_{l-1})})^{\frac{1}{2}}}{e^{\omega_l(V^{(l)}+(\tau_l-\tau_{l-1})\sum_{i=l+1}^{k+1} m_i)}}\right]$$
$$\times [1 - IG(\omega_k|m_{k+1}, V^{(k+1)})] d\omega_k d\omega_{k-1}\cdots d\omega_2 d\omega_1.$$

## 5. Numerical examples

We will show some simulation results and example based on artificial data set. We will compare the coverage probability of the priors when $n$ is small and moderate.

Let $\theta^\gamma(\pi;\mathbf{Y})$ be the posterior $\gamma$-quantile of $\theta$ given $\mathbf{Y}$ under the prior $\pi$. So, $(0, \theta^\gamma(\pi;\mathbf{Y}))$ is the one-sided $\gamma$ posterior confidence interval. Let $Q_{(\theta,\alpha)}(\gamma;\theta)$ be a frequentist coverage probability of this posterior confidence interval.

$$Q_{(\theta,\alpha)}(\gamma;\theta) = P\{0 < \theta \le \theta^\gamma(\pi;\mathbf{Y})\} = \gamma.$$

Similarly, we can define $\alpha^\gamma(\pi;\mathbf{Y})$ and $Q_{(\theta,\alpha)}(\gamma;\alpha)$ to be the posterior $\gamma$ quantile of $\alpha$ and the corresponding frequentist coverage probability, respectively. In Table 5.1, the estimated $Q_{(\theta,\alpha)}(\gamma;\theta)$ and $Q_{(\theta,\alpha)}(\gamma;\alpha)$ are shown, when $\gamma = 0.05(0.95)$. To obtain the table, we generate 10,000 independent random samples for fixed $\theta$, $\tau$ and $\alpha$ from simple step stress ALT model. Note that under the prior $\pi$ and given $\mathbf{Y}$, the event $\theta \le \theta^\gamma(\pi;\mathbf{Y})$ is equivalent to the event $F_\theta(\theta^\gamma(\pi;\mathbf{Y})|\mathbf{Y}) \le \gamma$. So, we calculate the relative frequency of $F_\theta(\theta^\gamma(\pi;\mathbf{Y})|\mathbf{Y}) \le \gamma$.

Table 5.1 shows the frequentist coverage probability of $\omega_1$. For sample sizes 5 to 60, the simulation is repeated 10000 times. In this simulation, we assume that $\theta = 2$, $\alpha_1 = 0.5$ and $\alpha_2 = 0.5$. The tampering points $\tau_1$ and $\tau_2$ are determined to satisfy $P\{Y \le \tau_1\} = \frac{1}{3}$,

**Table 5.1** Coverage probability for $\theta$ in three-step model

| | | sample size | $\pi^J$ | | $\pi^M$ | | $\pi^R$ | |
|---|---|---|---|---|---|---|---|---|
| $\tau_1$ | $\tau_2$ | $n$ | 0.05 | 0.95 | 0.05 | 0.95 | 0.05 | 0.95 |
| | | 5 | 0.000 | 0.990 | 0.000 | 1.000 | 0.000 | 1.000 |
| | | 10 | 0.006 | 0.888 | 0.022 | 0.920 | 0.015 | 0.910 |
| | | 15 | 0.013 | 0.909 | 0.029 | 0.925 | 0.022 | 0.921 |
| | | 20 | 0.014 | 0.917 | 0.029 | 0.934 | 0.024 | 0.930 |
| | | 25 | 0.019 | 0.924 | 0.032 | 0.948 | 0.028 | 0.946 |
| | | 30 | 0.021 | 0.920 | 0.034 | 0.943 | 0.031 | 0.941 |
| 0.2027 | 0.3760 | 35 | 0.024 | 0.929 | 0.037 | 0.939 | 0.034 | 0.938 |
| | | 40 | 0.026 | 0.933 | 0.040 | 0.948 | 0.036 | 0.946 |
| | | 45 | 0.026 | 0.928 | 0.037 | 0.944 | 0.036 | 0.943 |
| | | 50 | 0.031 | 0.939 | 0.042 | 0.948 | 0.040 | 0.947 |
| | | 55 | 0.033 | 0.936 | 0.044 | 0.950 | 0.041 | 0.950 |
| | | 60 | 0.032 | 0.935 | 0.044 | 0.946 | 0.042 | 0.945 |
| | | 5 | 0.000 | 0.756 | 0.000 | 0.990 | 0.000 | 0.935 |
| | | 10 | 0.001 | 0.862 | 0.017 | 0.929 | 0.012 | 0.916 |
| | | 15 | 0.009 | 0.890 | 0.028 | 0.932 | 0.022 | 0.925 |
| | | 20 | 0.016 | 0.901 | 0.035 | 0.936 | 0.030 | 0.931 |
| | | 25 | 0.016 | 0.906 | 0.034 | 0.938 | 0.030 | 0.933 |
| | | 30 | 0.022 | 0.919 | 0.038 | 0.944 | 0.035 | 0.941 |
| 0.3466 | 0.4479 | 35 | 0.020 | 0.920 | 0.037 | 0.944 | 0.034 | 0.941 |
| | | 40 | 0.028 | 0.921 | 0.043 | 0.941 | 0.040 | 0.939 |
| | | 45 | 0.025 | 0.917 | 0.040 | 0.941 | 0.037 | 0.939 |
| | | 50 | 0.024 | 0.928 | 0.037 | 0.949 | 0.035 | 0.947 |
| | | 55 | 0.029 | 0.932 | 0.044 | 0.948 | 0.041 | 0.947 |
| | | 60 | 0.030 | 0.931 | 0.043 | 0.948 | 0.041 | 0.947 |

**Table 5.2** Estimates of parameters

| | MLE | $\pi^J$ | $\pi^M$ | $\pi^R$ |
|---|---|---|---|---|
| $\theta$ | $1.182 \times 10^{-2}$ | $8.010 \times 10^{-3}$ | $8.014 \times 10^{-3}$ | $8.013 \times 10^{-3}$ |
| $\alpha_1$ | $2.159 \times 10^{-1}$ | $1.917 \times 10^{-1}$ | $1.454 \times 10^{-1}$ | $1.693 \times 10^{-1}$ |
| $\alpha_2$ | $5.732 \times 10^{-2}$ | $7.921 \times 10^{-2}$ | $6.793 \times 10^{-2}$ | $8.877 \times 10^{-2}$ |

$P\{Y \geq \tau_2\} = \frac{1}{3}$ and $P\{Y \leq \tau_1\} = \frac{1}{2}$, $P\{Y \geq \tau_2\} = \frac{1}{3}$. The points are given by $\tau_1 = 0.2027$, $\tau_2 = 0.3760$ and $\tau_1 = 0.3466$, $\tau_2 = 0.4479$, respectively.

From this simulation, we can find the fact that matching prior and reference prior achieve the frequentist coverage probability relatively well. And the tampering point does not affect the coverage probability.

**Example.** The following 15 data, given by Proschan (1963), are a part of time intervals of successive failures of the air conditioning system in Boeing jet airplanes.

74 57 48 29 502 12 70 21 29 386 59 27 153 26 326

We apply the data to the model (2.2) with $k = 2$. We assume that the acceleration factors $\alpha_1 = 0.1$ and $\alpha_2 = 0.1$. And the tempering point $\tau_1 = 50$ and $\tau_2 = 75$. Here, we choose $\tau_2$ such that $\tau_1 + (\tau_2 - \tau_1)\alpha_1^{-1} = 300$. The data are given in Table 5.3.

Using the above data, we compute the MLE and various Bayes estimates for the parameters. The estimates are given in Table 5.2. The estimated values are not quite different. All the estimates give overestimated values for $\theta$ and $\alpha_1$, but $\alpha_2$ is underestimated.

**Table 5.3** Accelerated data with two tempering points

| $X$ | $\delta_1$ | $\delta_2$ | $\delta_3$ | $Y$ |
|---|---|---|---|---|
| 74 | 0 | 1 | 0 | 52.40 |
| 57 | 0 | 1 | 0 | 50.70 |
| 48 | 1 | 0 | 0 | 48.00 |
| 29 | 1 | 0 | 0 | 29.00 |
| 502 | 0 | 0 | 1 | 77.02 |
| 12 | 1 | 0 | 0 | 12.00 |
| 70 | 0 | 1 | 0 | 52.00 |
| 21 | 1 | 0 | 0 | 21.00 |
| 29 | 1 | 0 | 0 | 29.00 |
| 386 | 0 | 0 | 1 | 75.86 |
| 59 | 0 | 1 | 0 | 50.90 |
| 27 | 1 | 0 | 0 | 27.00 |
| 153 | 0 | 1 | 0 | 60.30 |
| 26 | 1 | 0 | 0 | 26.00 |
| 326 | 0 | 0 | 1 | 75.26 |

From Table 5.2, we see that Bayes estimates give better estimates than MLE. For parameter $\alpha_1$, the Bayes estimate under $\pi^M$ is the closest to 0.1 whereas for $\alpha_2$, the Bayes estimate under $\pi^R$ is the closest to 0.1. And we can conclude that the overall performance of the Bayes estimates are superior to MLE.

From this example, one can find the fact that three step stress ALT shorten the original life time remarkably. This can save more money and time than two step stress ALT. But the estimate of parameter of interest are close to the estimate based on original life time.

## 6. Concluding remarks

There has been done a little work for Bayesian analysis of ALTs data. Almost all the Bayesian inferences related to ALTs data were based on the conjugate priors.

In this paper, we developed the noninformative priors such as probability matching and reference prior for the parameter of interest in the presence of nuisance parameters. And simulations and examples were given to verify our proposed Bayesian analysis performed well.

Finally, we recommend the use of probability matching prior or reference prior to analyze multiple step stress ALTs data.

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