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ON INTUITIONISTIC FUZZY SUBSPACES

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Abstract. We introduce a new concept of intuitionistic fuzzy topological subspace, which coincides with the usual concept of intuitionistic fuzzy topological subspace due to Samanta and Mondal [18] in the case that $\mu = \chi_A$ for $A \subseteq X$. Also, we introduce and study some concepts such as continuity, separation axioms, compactness and connectedness in this sense.

1. Introduction and preliminaries

Šostak [19], introduce the fundamental concept of a fuzzy topological structure as an extension of both crisp topology and Chang's fuzzy topology [4], in the sense that not only the object were fuzzified, but also the axiomatics. In [20, 21] Šostak gave some rules and showed how such an extension can be realized. Chattopadhyay et al. [5, 6] have redefined the similar concept. In [16] Ramadan gave a similar definition namely "Smooth fuzzy topology" for lattice $L = [0, 1]$, it has been developed in many direction [9-11]. As a generalization of fuzzy sets, the notion of intuitionistic fuzzy sets was introduced by Atanassov [1-3]. By using intuitionistic fuzzy sets, Çoker and his colleague [7, 8] introduced the topology of intuitionistic fuzzy sets. Samanta and Mondal [17, 18] introduced the notion of intuitionistic fuzzy topology which is a generalization of the concepts of fuzzy topology and the topology of intuitionistic fuzzy sets. Recently, much work has been done with this concept [12-14].

Throughout this paper, let X be a nonempty set $I = [0, 1]$, $I_0 = (0, 1]$, $I_1 = [0, 1)$ and I^X denote the set of all fuzzy subsets of X. For $\mu \in I^X$, we call $\mathcal{A}_\mu = \{\nu \in I^X : \nu \leq \mu\}$. IF stand for intuitionistic fuzzy. For $\alpha \in I$, $\underline{\alpha}(x) = \alpha$ for all $x \in X$. A fuzzy point x_t for $t \in I_0$ is an element of I^X such that, for $y \in X$,

$$x_t(y) = \begin{cases} t & \text{if } y = x, \\ 0 & \text{if } y \neq x. \end{cases}$$

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The set of all fuzzy points in X is denoted by $Pt(X)$. A fuzzy set λ is quasi-coincident with a fuzzy set μ, denoted by $\lambda q \mu$, if there exists $x \in X$ such that $\lambda(x) + \mu(x) > 1$. Otherwise $\lambda \not{q} \mu$ [15].

Lemma 1.1 ([22])**.** *If $f(x_t) q \lambda[f(\mu)]$, then $x_t q f^{-1}(\lambda)[\mu]$, where $x_t q \lambda[\mu]$ means $t + \lambda(x) > \mu(x)$.*

Definition 1.1 ([18])**.** An intuitionistic gradation of openness (IGO, for short) on X is an ordered pair (τ, τ^*) of mappings $\tau, \tau^* : I^X \to I$ satisfies the following conditions:

(IGO1) $\tau(\lambda) + \tau^*(\lambda) \leq 1$ for each $\lambda \in I^X$;

(IGO2) $\tau(\underline{0}) = \tau(\underline{1}) = 1$, $\tau^*(\underline{0}) = \tau^*(\underline{1}) = 0$;

(IGO3) $\tau(\lambda_1 \wedge \lambda_2) \geq \tau(\lambda_1) \wedge \tau(\lambda_2)$ and $\tau^*(\lambda_1 \wedge \lambda_2) \leq \tau^*(\lambda_1) \vee \tau^*(\lambda_2)$ for any $\lambda_1, \lambda_2 \in I^X$;

(IGO4) $\tau(\bigvee_{i\in\Gamma} \lambda_i) \geq \bigwedge_{i\in\Gamma} \tau(\lambda_i)$ and $\tau^*(\bigvee_{i\in\Gamma} \lambda_i) \leq \bigvee_{i\in\Gamma} \tau^*(\lambda_i)$ for any $\{\lambda_i : i \in \Gamma\} \subseteq I^X$.

The triplet (X, τ, τ^*) is called an intuitionistic fuzzy topological space (briefly, IFTS). τ and τ^* may be interpreted as gradation of openness and gradation of nonopenness, respectively.

Definition 1.2 ([18])**.** Let $f : (X, \tau, \tau^*) \to (Y, \sigma, \sigma^*)$ be a mapping from an IFTS (X, τ, τ^*) to another IFTS (Y, σ, σ^*). Then f is said to be IF-continuous if for each $\nu \in I^Y$,

$$\sigma(\nu) \leq \tau(f^{-1}(\nu)) \quad \text{and} \quad \sigma^*(\nu) \geq \tau^*(f^{-1}(\nu)).$$

2. Intuitionistic fuzzy subspaces

Definition 2.1. Let (X, τ, τ^*) be an IFTS and $\mu \in I^X$. The pair of mappings $(\tau_\mu, \tau^*_\mu) : \mathcal{A}_\mu \to I$ defined by:

$$\tau_\mu(\nu) = \bigvee \{\tau(\lambda) : \lambda \in I^X, \lambda \wedge \mu = \nu\}$$

$$\tau^*_\mu(\nu) = \bigwedge \{\tau^*(\lambda) : \lambda \in I^X, \lambda \wedge \mu = \nu\}$$

is an intuitionistic fuzzy μ-topology induced over μ by (τ, τ^*). For any $\nu \in \mathcal{A}_\mu$, the number $\tau_\mu(\nu)$ is called the μ-openness degree of ν, while $\tau^*_\mu(\nu)$ is called μ-nonopenness degree of ν.

Remark 2.1. If $A \subseteq X$ and $\mu = \chi_A$, we have just the usual concept of intuitionistic fuzzy subspace due to Samanta and Mondal [18]. Given (τ_μ, τ^*_μ) and $\nu \in \mathcal{A}_\mu$ we can define $((\tau_\mu)_\nu, (\tau^*_\mu)_\nu)$, the intuitionistic fuzzy ν-topology induced over ν by (τ_μ, τ^*_μ). We have trivially $\tau_\nu = (\tau_\mu)_\nu$ and $\tau^*_\nu = (\tau^*_\mu)_\nu$, that is, an intuitionistic fuzzy subspace of an intuitionistic fuzzy subspace is also an intuitionistic fuzzy subspace.

Theorem 2.1. *Let (X, τ, τ^*) be an IFTS and $\mu \in I^X$. Then (τ_μ, τ^*_μ) verifies the following properties*:

(μIGO1) $\tau_\mu(\nu) + \tau^*_\mu(\nu) \leq 1$ *for each* $\nu \in \mathcal{A}_\mu$.

(μIGO2) $\tau_\mu(\underline{0}) = \tau_\mu(\mu) = 1$, $\tau^*_\mu(\mu) = \tau^*_\mu(\underline{0}) = 0$.

(μIGO3) $\tau_\mu(\nu_1 \wedge \nu_2) \geq \tau_\mu(\nu_1) \wedge \tau_\mu(\nu_2)$ *and* $\tau^*_\mu(\nu_1 \wedge \nu_2) \leq \tau^*_\mu(\nu_1) \vee \tau^*_\mu(\nu_2)$ *for each* $\nu_1, \nu_2 \in \mathcal{A}_\mu$.

(μIGO4) $\tau_\mu(\bigvee_{i\in J} \nu_i) \geq \bigwedge_{i\in J} \tau_\mu(\nu_i)$ *and* $\tau^*_\mu(\bigvee_{i\in J} \nu_i) \leq \bigvee_{i\in J} \tau^*_\mu(\nu_i)$ *for each* $\{\nu_i : i \in J\} \subseteq \mathcal{A}_\mu$.

Proof. (μIGO1) and (μIGO2) are clear.

(μIGO3) Suppose that there exist $\nu_1, \nu_2 \in \mathcal{A}_\mu$ such that

$$\tau^*_\mu(\nu_1 \wedge \nu_2) \not\leq \tau^*_\mu(\nu_1) \vee \tau^*_\mu(\nu_2).$$

Then there exists $s \in (0, 1)$ such that

$$\tau^*_\mu(\nu_1 \wedge \nu_2) > s \geq \tau^*_\mu(\nu_1) \vee \tau^*_\mu(\nu_2).$$

Since $\tau^*_\mu(\nu_1) \leq s$ and $\tau^*_\mu(\nu_2) \leq s$, there exist $\lambda_1, \lambda_2 \in I^X$ with $\tau^*(\lambda_1) \leq s$ and $\tau^*(\lambda_2) \leq s$ such that $\nu_1 = \lambda_1 \wedge \mu$ and $\nu_2 = \lambda_2 \wedge \mu$ and hence $\nu_1 \wedge \nu_2 = (\lambda_1 \wedge \lambda_2) \wedge \mu$. Since $\tau^*(\lambda_1 \wedge \lambda_2) \leq \tau^*(\lambda_1) \vee \tau^*(\lambda_2) \leq s$, $\tau^*_\mu(\nu_1 \wedge \nu_2) \leq s$. It is a contradiction. Hence, $\tau^*_\mu(\nu_1 \wedge \nu_2) \leq \tau^*_\mu(\nu_1) \vee \tau^*_\mu(\nu_2)$ for each $\nu_1, \nu_2 \in \mathcal{A}_\mu$. Similarly, we can show $\tau_\mu(\nu_1 \wedge \nu_2) \geq \tau_\mu(\nu_1) \wedge \tau_\mu(\nu_2)$ for each $\nu_1, \nu_2 \in \mathcal{A}_\mu$.

(μIGO4) Suppose that there exist a family $\{\nu_i : i \in J\} \subseteq \mathcal{A}_\mu$ such that

$$\tau^*_\mu(\bigvee_{i\in J} \nu_i) \not\leq \bigvee_{i\in J} \tau^*_\mu(\nu_i).$$

Then there exists $s \in (0, 1)$ such that

$$\tau^*_\mu(\bigvee_{i\in J} \nu_i) > s \geq \bigvee_{i\in J} \tau^*_\mu(\nu_i).$$

Since $\tau^*_\mu(\nu_i) \leq s$ for all $i \in J$, there exists $\lambda_i \in I^X$ with $\tau^*(\lambda_i) \leq s$ such that $\nu_i = \lambda_i \wedge \mu$. Thus $\bigvee_{i\in J} \nu_i = (\bigvee_{i\in J} \lambda_i) \wedge \mu$.

Since $\tau^*(\bigvee_{i\in J} \lambda_i) \leq \bigvee_{i\in J} \tau^*(\lambda_i) \leq s$, $\tau^*_\mu(\bigvee_{i\in J} \nu_i) \leq s$. It is a contradiction. Hence $\tau^*_\mu(\bigvee_{i\in J} \nu_i) \leq \bigvee_{i\in J} \tau^*_\mu(\nu_i)$ for each $\{\nu_i : i \in J\} \subseteq \mathcal{A}_\mu$. Similarly we can show $\tau_\mu(\bigvee_{i\in J} \nu_i) \geq \bigwedge_{i\in J} \tau_\mu(\nu_i)$ for each $\{\nu_i : i \in J\} \subseteq \mathcal{A}_\mu$. □

Theorem 2.2. *Let (X, τ, τ^*) be an IFTS and $\mu \in I^X$. Define the mappings $\mathcal{F}_{\tau_\mu}, \mathcal{F}^*_{\tau^*_\mu} : \mathcal{A}_\mu \to I$ by: $\mathcal{F}_{\tau_\mu}(\nu) = \tau_\mu(\mu - \nu)$ and $\mathcal{F}^*_{\tau^*_\mu}(\nu) = \tau^*_\mu(\mu - \nu)$ for each $\nu \in \mathcal{A}_\mu$. Then $(\mathcal{F}_{\tau_\mu}, \mathcal{F}^*_{\tau^*_\mu})$ satisfies the following properties*:

(μGIC1) $\mathcal{F}_{\tau_\mu}(\nu) + \mathcal{F}^*_{\tau^*_\mu}(\nu) \leq 1$ *for each* $\nu \in \mathcal{A}_\mu$.

(μGIC2) $\mathcal{F}_{\tau_\mu}(\underline{0}) = \mathcal{F}_{\tau_\mu}(\mu) = 1$, $\mathcal{F}^*_{\tau^*_\mu}(\mu) = \mathcal{F}^*_{\tau^*_\mu}(\underline{0}) = 0$.

(μGIC3) $\mathcal{F}_{\tau_\mu}(\nu_1 \vee \nu_2) \geq \mathcal{F}_{\tau_\mu}(\nu_1) \wedge \mathcal{F}_{\tau_\mu}(\nu_2)$ *and* $\mathcal{F}^*_{\tau^*_\mu}(\nu_1 \vee \nu_2) \leq \mathcal{F}^*_{\tau^*_\mu}(\nu_1) \vee \mathcal{F}^*_{\tau^*_\mu}(\nu_2)$ *for each* $\nu_1, \nu_2 \in \mathcal{A}_\mu$.

(μGIC4) $\mathcal{F}_{\tau_\mu}(\bigwedge_{i\in J} \nu_i) \geq \bigwedge_{i\in J} \mathcal{F}_{\tau_\mu}(\nu_i)$ *and* $\mathcal{F}^*_{\tau^*_\mu}(\bigwedge_{i\in J} \nu_i) \leq \bigvee_{i\in J} \mathcal{F}^*_{\tau^*_\mu}(\nu_i)$ *for each* $\{\nu_i : i \in J\} \subseteq \mathcal{A}_\mu$.

Proof. It is clear. □

Definition 2.2. Let (X,τ,τ^*) be an IFTS, $\mu\in I^X$ and $x_t\in\mu$. Then for $r\in I_0$, $s\in I_1$ with $r+s\le 1$. We say that $\nu\in\mathcal{A}_\mu$ is (r,s)-$IF\mu$-q-neighborhood of x_t, if there is $\eta\in\mathcal{A}_\mu$ with $\tau_\mu(\eta)\ge r$ and $\tau^*_\mu(\eta)\le s$ such that $x_t q\eta[\mu]$ and $\eta\le\nu$. We denote the family of all (r,s)-$IF\mu$-q-neighborhoods of x_t by $Q_{\tau_\mu,\tau^*_\mu}(x_t,r,s)$.

Theorem 2.3. *Let (X,τ,τ^*) be an IFTS, $\mu\in I^X$ and $x_t\in\mu$. Then for $r\in I_0$, $s\in I_1$ with $r+s\le 1$,*

$$Q_{\tau_\mu,\tau^*_\mu}(x_t,r,s)=\{\lambda\wedge\mu:\lambda\in Q_{\tau,\tau^*}(x_t,r,s)\}.$$

Proof. Let $\lambda\in Q_{\tau,\tau^*}(x_t,r,s)$. Then there is $\xi\in I^X$ with $\tau(\xi)\ge r$ and $\tau^*(\xi)\le s$ such that $x_tq\xi$ and $\xi\le\lambda$. Then, $\xi\wedge\mu\le\lambda\wedge\mu$. Put $\eta=\xi\wedge\mu$. Then $\tau_\mu(\eta)\ge r$ and $\tau^*_\mu(\eta)\le s$. Since $x_tq\xi$, $t+\xi(x)>1$ which implies that $t+(\xi\wedge\mu)(x)>\mu(x)$. Then $x_tq\eta[\mu]$. Thus there is $\eta=\xi\wedge\mu\in\mathcal{A}_\mu$ with $\tau_\mu(\eta)\ge r,\tau^*_\mu(\eta)\le s$, $x_tq\eta[\mu]$ and $\eta\le\lambda\wedge\mu$. Hence $\lambda\wedge\mu\in Q_{\tau_\mu,\tau^*_\mu}(x_t,r,s)$. □

Theorem 2.4. *Let (X,τ,τ^*) be an IFTS, $\mu\in I^X$ and $x_t\in\mu$. Then for $r\in I_0$, $s\in I_1$ with $r+s\le 1$, $Q_{\tau_\mu,\tau^*_\mu}(x_t,r,s)$ satisfies the following*:

(μQ1) *If $\nu\in Q_{\tau_\mu,\tau^*_\mu}(x_t,r,s)$, then $x_tq\nu[\mu]$.*

(μQ2) *If $\nu_1,\nu_2\in Q_{\tau_\mu,\tau^*_\mu}(x_t,r,s)$, then $\nu_1\wedge\nu_2\in Q_{\tau_\mu,\tau^*_\mu}(x_t,r,s)$.*

(μQ3) *If $\nu\in Q_{\tau_\mu,\tau^*_\mu}(x_t,r,s)$ and $\nu^*\in\mathcal{A}_\mu$ such that $\nu\le\nu^*$, then $\nu^*\in Q_{\tau_\mu,\tau^*_\mu}(x_t,r,s)$.*

(μQ4) *If $\nu\in Q_{\tau_\mu,\tau^*_\mu}(x_t,r,s)$, there is $\nu^*\in Q_{\tau_\mu,\tau^*_\mu}(x_t,r,s)$ such that $\nu\in Q_{\tau_\mu,\tau^*_\mu}(y_m,r,s)$ for each $y_mq\nu^*[\mu]$.*

Proof. It is clear. □

Theorem 2.5. *Let (X,τ,τ^*) be an IFTS, $\mu\in I^X$. Then for each $\nu\in\mathcal{A}_\mu$ and each $r\in I_0$, $s\in I_1$ with $r+s\le 1$, we define the operator $C_{\tau_\mu,\tau^*_\mu}:\mathcal{A}_\mu\times I_0\times I_1\to\mathcal{A}_\mu$ as follows*:

$$C_{\tau_\mu,\tau^*_\mu}(\nu,r,s)=\bigwedge\{\eta\in\mathcal{A}_\mu:\eta\ge\nu,\tau_\mu(\mu-\eta)\ge r,\tau^*_\mu(\mu-\eta)\le s\}.$$

For each $\nu,\nu_1,\nu_2\in\mathcal{A}_\mu$ and $r\in I_0$, $s\in I_1$ with $r+s\le 1$, the operator C_{τ_μ,τ^*_μ} satisfies the following:

(μC1) $C_{\tau_\mu,\tau^*_\mu}(\underline{0},r,s)=\underline{0}$.

(μC2) $\nu\le C_{\tau_\mu,\tau^*_\mu}(\nu,r,s)$.

(μC3) $C_{\tau_\mu,\tau^*_\mu}(\nu_1\vee\nu_2,r,s)=C_{\tau_\mu,\tau^*_\mu}(\nu_1,r,s)\vee C_{\tau_\mu,\tau^*_\mu}(\nu_2,r,s)$.

(μC4) $C_{\tau_\mu,\tau^*_\mu}(\nu,r,s)\le C_{\tau_\mu,\tau^*_\mu}(\nu,m,n)$ *if $r\le m$, $s\ge n$ and $m+n\le 1$.*

(μC5) $C_{\tau_\mu,\tau^*_\mu}(C_{\tau_\mu,\tau^*_\mu}(\nu,r,s),r,s)=C_{\tau_\mu,\tau^*_\mu}(\nu,r,s)$.

Proof. It is straightforward. □

Theorem 2.6. *Let (X,τ,τ^*) be an IFTS, $\mu\in I^X$. Then for each $\nu\in\mathcal{A}_\mu$ and each $r\in I_0$, $s\in I_1$ with $r+s\le 1$, we define the operator $I_{\tau_\mu,\tau^*_\mu}:\mathcal{A}_\mu\times I_0\times I_1\to$*

$\mathcal{A}_\mu$ as follows:

$$I_{\tau_\mu,\tau^*_\mu}(\nu,r,s)=\bigvee\{\eta\in\mathcal{A}_\mu:\eta\le\nu,\tau_\mu(\eta)\ge r,\tau^*_\mu(\eta)\le s\}.$$

For each $\nu,\nu_1,\nu_2\in\mathcal{A}_\mu$ and $r\in I_0$, $s\in I_1$ with $r+s\le 1$, the operator I_{τ_μ,τ^*_μ} satisfies the following:

(μI1) $I_{\tau_\mu,\tau^*_\mu}(\mu,r,s)=\mu$.
(μI2) $I_{\tau_\mu,\tau^*_\mu}(\nu,r,s)\le\nu$.
(μI3) $I_{\tau_\mu,\tau^*_\mu}(\nu_1\wedge\nu_2,r,s)=I_{\tau_\mu,\tau^*_\mu}(\nu_1,r,s)\wedge I_{\tau_\mu,\tau^*_\mu}(\nu_2,r,s)$.
(μC4) $I_{\tau_\mu,\tau^*_\mu}(\nu,r,s)\ge I_{\tau_\mu,\tau^*_\mu}(\nu,m,n)$ *if* $r\le m$, $s\ge n$ *and* $m+n\le 1$.
(μC5) $I_{\tau_\mu,\tau^*_\mu}(I_{\tau_\mu,\tau^*_\mu}(\nu,r,s),r,s)=I_{\tau_\mu,\tau^*_\mu}(\nu,r,s)$.

Proof. It is straightforward. □

Theorem 2.7. *Let (X,τ,τ^*) be an IFTS and $\mu\in I^X$. For each $\nu\in\mathcal{A}_\mu$ and each $r\in I_0$, $s\in I_1$ with $r+s\le 1$, we have*

(i) $I_{\tau_\mu,\tau^*_\mu}(\mu-\nu,r,s)=\mu-C_{\tau_\mu,\tau^*_\mu}(\nu,r,s)$.
(ii) $C_{\tau_\mu,\tau^*_\mu}(\mu-\nu,r,s)=\mu-I_{\tau_\mu,\tau^*_\mu}(\nu,r,s)$.

Proof. (i) For each $\nu\in\mathcal{A}_\mu$ and each $r\in I_0$, $s\in I_1$ with $r+s\le 1$, we have the following:

$$\begin{aligned}\mu-C_{\tau_\mu,\tau^*_\mu}(\nu,r,s)&=\mu-\bigwedge\{\eta\in\mathcal{A}_\mu:\nu\le\eta,\tau_\mu(\mu-\eta)\ge r,\tau^*_\mu(\mu-\eta)\le s\}\\&=\bigvee\{\mu-\eta:\mu-\eta\le\mu-\nu,\tau_\mu(\mu-\eta)\ge r,\tau^*_\mu(\mu-\eta)\le s\}\\&=I_{\tau_\mu,\tau^*_\mu}(\mu-\nu,r,s).\end{aligned}$$

(ii) It is similar to (i). □

Theorem 2.8. *Let (X,τ,τ^*) be an IFTS, $\mu\in I^X$ and $x_t\in\mu$. For $\nu\in\mathcal{A}_\mu$ and $r\in I_0$, $s\in I_1$ with $r+s\le 1$, $x_t\in C_{\tau_\mu,\tau^*_\mu}(\nu,r,s)$ if and only if for each $\eta\in\mathcal{A}_\mu$ with $\tau_\mu(\eta)\ge r$, $\tau^*_\mu(\eta)\le s$ and $x_tq\eta[\mu]$ we have $\nu q\eta[\mu]$.*

Proof. Let $x_t\in C_{\tau_\mu,\tau^*_\mu}(\nu,r,s)$, $\eta\in\mathcal{A}_\mu$ with $\tau_\mu(\eta)\ge r$, $\tau^*_\mu(\eta)\le s$ and $x_tq\eta[\mu]$. Suppose that $\nu\not q\eta[\mu]$, then $\nu\le\mu-\eta$. Since $x_tq\eta[\mu]$, $t+\eta(x)>\mu(x)$. This implies $x_t\notin\mu-\eta$. Since $\nu\le\mu-\eta$, $\tau_\mu(\mu-(\mu-\eta))=\tau_\mu(\eta)\ge r$ and $\tau^*_\mu(\mu-(\mu-\eta))=\tau^*_\mu(\eta)\le s$, we have $x_t\notin C_{\tau_\mu,\tau^*_\mu}(\nu,r,s)$. It is a contradiction. Conversely, let $\eta\in\mathcal{A}_\mu$ with $\tau_\mu(\eta)\ge r$, $\tau^*_\mu(\eta)\le s$, $x_tq\eta[\mu]$ and $\nu q\eta[\mu]$. Suppose that $x_t\notin C_{\tau_\mu,\tau^*_\mu}(\nu,r,s)$. Then there exists $\xi\in\mathcal{A}_\mu$ with $\tau_\mu(\mu-\xi)\ge r$, $\tau^*_\mu(\mu-\xi)\le s$, $\nu\le\xi$ and $x_t\notin\xi$. Then $\xi(x)<t$ which implies $(\mu-\xi)(x)+t>\mu(x)$. Thus $x_xq(\mu-\xi)[\mu]$. Then from our hypothesis $\nu q(\mu-\xi)[\mu]$. Thus there is $y\in X$ such that $\nu(y)+(\mu-\xi)(y)>\mu(y)$. Thus $\nu(y)>\xi(y)$ which is a contradiction with $\nu\le\xi$. Hence $x_t\in C_{\tau_\mu,\tau^*_\mu}(\nu,r,s)$. □

3. $IF\mu$-continuity

Definition 3.1. Let (X,τ,τ^*) and (Y,σ,σ^*) be two IFTSs and $\mu\in I^X$. Then the mapping $f:(X,\tau,\tau^*)\to(Y,\sigma,\sigma^*)$ is called $IF\mu$-continuous if $\tau_\mu(f^{-1}(\nu)\wedge\mu)\geq\sigma_{f(\mu)}(\nu)$ and $\tau^*_\mu(f^{-1}(\nu)\wedge\mu)\leq\sigma^*_{f(\mu)}(\nu)$ for each $\nu\in\mathcal{A}_{f(\mu)}$.

Remark 3.1. Every IF-continuous mapping is $IF\mu$-continuous for all μ but the converse is not true in general as the following example shows.

Example 3.1. Let $X=I$. We define the $IGO(\tau,\tau^*)$ and $IGO(\sigma,\sigma^*)$ on X as follows: for each $\lambda\in I^X$

$$\tau(\lambda)=\begin{cases}1, & \text{if} \quad \lambda=\underline{0},\underline{1}\\ 0.5, & \text{if} \quad \lambda=\underline{t}, t\leq 0.6\\ 0, & \text{otherwise.}\end{cases} \qquad \tau^*(\lambda)=\begin{cases}0, & \text{if} \quad \lambda=\underline{0},\underline{1}\\ 0.4, & \text{if} \quad \lambda=\underline{t}, t\leq 0.6\\ 1, & \text{otherwise.}\end{cases}$$

$$\sigma(\lambda)=\begin{cases}1, & \text{if} \quad \lambda=\underline{0},\underline{1}\\ 0.2, & \text{if} \quad \lambda=\underline{t}, t\leq 0.6\\ 0.4, & \text{if} \quad \lambda=\underline{0.7}\\ 0, & \text{otherwise.}\end{cases} \qquad \sigma^*(\lambda)=\begin{cases}0, & \text{if} \quad \lambda=\underline{0},\underline{1}\\ 0.7, & \text{if} \quad \lambda=\underline{t}, t\leq 0.6\\ 0.5, & \text{if} \quad \lambda=\underline{0.7}\\ 1, & \text{otherwise.}\end{cases}$$

Let $\mu=\underline{0.5}$, then

$$\tau_\mu(\nu)=\begin{cases}1, & \text{if} \quad \nu=\underline{0},\mu\\ 0.5, & \text{if} \quad \underline{0}<\nu<\underline{0.5}\\ 0, & \text{otherwise.}\end{cases} \qquad \tau^*_\mu(\nu)=\begin{cases}0, & \text{if} \quad \lambda=\underline{0},\mu\\ 0.4, & \text{if} \quad \underline{0}<\nu<\underline{0.5}\\ 1, & \text{otherwise.}\end{cases}$$

$$\sigma_{f(\mu)}(\nu)=\begin{cases}1, & \text{if} \quad \nu=\underline{0},f(\mu)\\ 0.2, & \text{if} \quad \underline{0}<\nu<\underline{0.5}\\ 0, & \text{otherwise.}\end{cases} \qquad \sigma^*_{f(\mu)}(\nu)=\begin{cases}0, & \text{if} \quad \lambda=\underline{0},f(\mu)\\ 0.7, & \text{if} \quad \underline{0}<\nu<\underline{0.5}\\ 1, & \text{otherwise.}\end{cases}$$

Then the identity mapping $id_X:(X,\tau,\tau^*)\to(X,\sigma,\sigma^*)$ is an $IF\mu$-continuous but not IF-continuous.

Theorem 3.1. *Let (X,τ,τ^*) and (Y,σ,σ^*) be two IFTSs, $f:X\to Y$ be a mapping and $\{\mu_i : i\in J\}\subseteq I^X$ such that $\bigvee_{i\in J}\mu_i=\underline{1}$. Then f is $IF\mu_i$-continuous for each $i\in J$ if and only if f is IF-continuous.*

Proof. Due to Remark 3.1, it suffices to show that if f is $IF\mu_i$-continuous for each $i\in J$, then f is IF-continuous. Suppose there exists $\lambda\in I^Y$ such that

$$\tau^*(f^{-1}(\lambda))\not\leq\sigma^*(\lambda).$$

Then there exists $s\in(0,1)$ such that

$$\tau^*(f^{-1}(\lambda))>s\geq\sigma^*(\lambda).$$

Since $\sigma^*(\lambda)\leq s$, $\sigma^*_{f(\mu_i)}(\lambda\wedge f(\mu_i))\leq s$ for each $i\in J$. Since f is an $IF\mu_i$-continuous for each $i\in J$ we have

$$\tau^*_{\mu_i}(f^{-1}(\lambda\wedge f(\mu_i))\wedge\mu_i)\leq\sigma^*_{f(\mu_i)}(\lambda\wedge f(\mu_i))\leq s$$

but

$$f^{-1}(\lambda \wedge f(\mu_i)) \wedge \mu_i = f^{-1}(\lambda) \wedge f^{-1}(f(\mu_i)) \wedge \mu_i = f^{-1}(\lambda) \wedge \mu_i.$$

Then $\tau^*_{\mu_i}(f^{-1}(\lambda) \wedge \mu_i) \leq s$ for each $i \in J$. Then for each $i \in J$ there exists $\nu_i \in I^X$ with $\tau^*(\nu_i) \leq s$ such that $f^{-1}(\lambda) \wedge \mu_i = \nu_i \wedge \mu_i$. This implies that $\bigvee_{i \in J}(f^{-1}(\lambda) \wedge \mu_i) = \bigvee_{i \in J}(\nu_i \wedge \mu_i)$, thus

$$f^{-1}(\lambda) \wedge (\bigvee_{i \in J} \mu_i) = (\bigvee_{i \in J} \nu_i) \wedge (\bigvee_{i \in J} \mu_i).$$

Since $\bigvee_{i \in J} \mu_i = \underline{1}$, $f^{-1}(\lambda) = \bigvee_{i \in J} \nu_i$. Then

$$\tau^*(f^{-1}(\lambda)) = \tau^*(\bigvee_{i \in J} \nu_i) \leq \bigvee_{i \in J} \tau^*(\nu_i) \leq s.$$

It is a contradiction. Thus $\tau^*(f^{-1}(\lambda)) \leq \sigma^*(\lambda)$ for each $\lambda \in I^Y$. Similarly, we can show $\tau(f^{-1}(\lambda)) \geq \sigma(\lambda)$ for each $\lambda \in I^Y$. Thus f is an IF-continuous. □

Theorem 3.2. *Let (X, τ, τ^*) and (Y, σ, σ^*) be two IFTSs, $\mu \in I^X$ and $f : X \to Y$ be an injective mapping. For $r \in I_0$, $s \in I_1$ with $r + s \leq 1$, the following statements are equivalent*:

(i) *f is $IF\mu$-continuous.*
(ii) *$\mathcal{F}_{\tau_\mu}(f^{-1}(\lambda) \wedge \mu) \geq \mathcal{F}_{\sigma_{f(\mu)}}(\lambda)$ and $\mathcal{F}^*_{\tau^*_\mu}(f^{-1}(\lambda) \wedge \mu) \leq \mathcal{F}^*_{\sigma^*_{f(\mu)}}(\lambda)$ for each $\lambda \in \mathcal{A}_{f(\mu)}$.*
(iii) *$f(C_{\tau_\mu, \tau^*_\mu}(\nu, r, s)) \leq C_{\sigma_{f(\mu)}, \sigma^*_{f(\mu)}}(f(\nu), r, s)$ for each $\nu \in \mathcal{A}_\mu$.*
(iv) *$C_{\tau_\mu, \tau^*_\mu}(f^{-1}(\lambda) \wedge \mu, r, s) \leq f^{-1}(C_{\sigma_{f(\mu)}, \sigma^*_{f(\mu)}}(\lambda, r, s)) \wedge \mu$ for each $\lambda \in \mathcal{A}_{f(\mu)}$.*
(v) *$f^{-1}(I_{\sigma_{f(\mu)}, \sigma^*_{f(\mu)}}(\lambda, r, s)) \wedge \mu \leq I_{\tau_\mu, \tau^*_\mu}(f^{-1}(\lambda) \wedge \mu, r, s)$ for each $\lambda \in \mathcal{A}_{f(\mu)}$.*
(vi) *For each $x_t \in \mu$ and $\lambda \in \mathcal{A}_{f(\mu)}$ with $\sigma_{f(\mu)}(\lambda) \geq r$, $\sigma^*_{f(\mu)}(\lambda) \leq s$ and $f(x_t) q \lambda[f(\mu)]$, there is $\nu \in \mathcal{A}_\mu$ with $\tau_\mu(\nu) \geq r$, $\tau^*_\mu(\nu) \leq s$ such that $x_t q \nu[\mu]$ and $f(\nu) \leq \lambda$.*
(vii) *For each $x_t \in \mu$ and $\lambda \in Q_{\sigma_{f(\mu)}, \sigma^*_{f(\mu)}}(f(x_t), r, s)$, there exists $\nu \in Q_{\tau_\mu, \tau^*_\mu}(x_t, r, s)$ such that $f(\nu) \leq \lambda$.*
(viii) *For each $x_t \in \mu$ and each $\lambda \in Q_{\sigma_{f(\mu)}, \sigma^*_{f(\mu)}}(f(x_t), r, s)$,*

$$f^{-1}(\lambda) \in Q_{\tau_\mu, \tau^*_\mu}(x_t, r, s).$$

Proof. (i)⇒ (ii) For each $\lambda \in \mathcal{A}_{f(\mu)}$, we have

$$\begin{aligned}\mathcal{F}^*_{\sigma^*_{f(\mu)}}(\lambda) &= \sigma^*_{f(\mu)}(f(\mu) - \lambda) \geq \tau^*_\mu(f^{-1}(f(\mu) - \lambda) \wedge \mu) \\ &= \tau^*_\mu(f^{-1}(f(\mu)) - f^{-1}(\lambda)) \wedge \mu) \\ &= \tau^*_\mu(\mu - f^{-1}(\lambda) \wedge \mu) = \mathcal{F}^*_{\tau^*_\mu}(f^{-1}(\lambda) \wedge \mu).\end{aligned}$$

Similarly, we can show $\mathcal{F}_{\sigma_{f(\mu)}}(\lambda) \leq \mathcal{F}_{\tau_\mu}(f^{-1}(\lambda) \wedge \mu)$ for each $\lambda \in \mathcal{A}_{f(\mu)}$.

(ii)⇒(iii) For each $\nu \in \mathcal{A}_\mu$ and $r \in I_0$, $s \in I_1$ with $r+s \leq 1$, we have

$$\begin{aligned}
&f^{-1}(C_{\sigma_{f(\mu)},\sigma^*_{f(\mu)}}(f(\nu), r, s))\\
&= f^{-1}(\bigwedge\{\lambda \in \mathcal{A}_{f(\mu)} : f(\nu) \leq \lambda, \sigma_{f(\mu)}(f(\mu)-\lambda) \geq r, \sigma^*_{f(\mu)}(f(\mu)-\lambda) \leq s\})\\
&= \bigwedge\{f^{-1}(\lambda) : \nu \leq f^{-1}(\lambda), \mathcal{F}_{\sigma_{f(\mu)}}(\lambda) \geq r, \mathcal{F}^*_{\sigma^*_{f(\mu)}}(\lambda) \leq s\}\\
&\geq \bigwedge\{f^{-1}(\lambda) \wedge \mu \in \mathcal{A}_\mu : \nu \leq f^{-1}(\lambda) \wedge \mu, \mathcal{F}_{\tau_\mu}(f^{-1}(\lambda) \wedge \mu) \geq r,\\
&\qquad \mathcal{F}^*_{\tau^*_\mu}(f^{-1}(\lambda) \wedge \mu) \leq s\}\\
&\geq \bigwedge\{f^{-1}(\lambda) \wedge \mu \in \mathcal{A}_\mu : \nu \leq f^{-1}(\lambda) \wedge \mu, \tau_\mu(\mu - (f^{-1}(\lambda) \wedge \mu)) \geq r,\\
&\qquad \tau^*_\mu(\mu - (f^{-1}(\lambda) \wedge \mu)) \leq s\}\\
&= C_{\tau_\mu,\tau^*_\mu}(\nu, r, s).
\end{aligned}$$

This implies that $f(C_{\tau_\mu,\tau^*_\mu}(\nu, r, s)) \leq C_{\sigma_{f(\mu)},\sigma^*_{f(\mu)}}(f(\nu), r, s)$.

(iii)⇒(iv) For each $\lambda \in \mathcal{A}_{f(\mu)}$ and $r \in I_0$, $s \in I_1$ with $r+s \leq 1$, we have

$$\begin{aligned}
f(C_{\tau_\mu,\tau^*_\mu}(f^{-1}(\lambda) \wedge \mu, r, s)) &\leq C_{\sigma_{f(\mu)},\sigma^*_{f(\mu)}}(f(f^{-1}(\lambda) \wedge \mu), r, s)\\
&\leq C_{\sigma_{f(\mu)},\sigma^*_{f(\mu)}}(\lambda, r, s).
\end{aligned}$$

Thus

$$C_{\tau_\mu,\tau^*_\mu}(f^{-1}(\lambda) \wedge \mu, r, s) \leq f^{-1}(C_{\sigma_{f(\mu)},\sigma^*_{f(\mu)}}(\lambda, r, s)) \wedge \mu.$$

(iv)⇒(v) It is clear from Theorem 2.7.

(v)⇒(i) Suppose that there exist $\lambda \in \mathcal{A}_{f(\mu)}$ and $r_0 \in I_0$, $s_0 \in I_1$ with $r_0 + s_0 \leq 1$ such that

$$\tau_\mu(f^{-1}(\lambda) \wedge \mu) < r_0 \leq \sigma_{f(\mu)}(\lambda) \quad \text{and} \quad \tau^*_\mu(f^{-1}(\lambda) \wedge \mu) > s_0 \geq \sigma^*_{f(\mu)}(\lambda).$$

Since $\sigma_{f(\mu)}(\lambda) \geq r_0$ and $\sigma^*_{f(\mu)}(\lambda) \leq s_0$, $\lambda = I_{\sigma_{f(\mu)},\sigma^*_{f(\mu)}}(\lambda, r_0, s_0)$. By (v) we have

$$f^{-1}(\lambda) \wedge \mu = f^{-1}(I_{\sigma_{f(\mu)},\sigma^*_{f(\mu)}}(\lambda, r_0, s_0)) \wedge \mu \leq I_{\tau_\mu,\tau^*_\mu}(f^{-1}(\lambda) \wedge \mu, r_0, s_0).$$

Thus

$$f^{-1}(\lambda) \wedge \mu = I_{\tau_\mu,\tau^*_\mu}(f^{-1}(\lambda) \wedge \mu, r_0, s_0).$$

This meaning, $\tau_\mu(f^{-1}(\lambda) \wedge \mu) \geq r_0$ and $\tau^*_\mu(f^{-1}(\lambda) \wedge \mu) \leq s_0$. It is a contradiction. Then $\tau_\mu(f^{-1}(\lambda) \wedge \mu) \geq \sigma_{f(\mu)}(\lambda)$ and $\tau^*_\mu(f^{-1}(\lambda) \wedge \mu) \leq \sigma^*_{f(\mu)}(\lambda)$ for each $\lambda \in \mathcal{A}_{f(\mu)}$. Hence f is $IF\mu$-continuous.

(i)⇒(vi) Let $x_t \in \mu$, $\lambda \in \mathcal{A}_{f(\mu)}$ and $r \in I_0$, $s \in I_1$ with $r+s \leq 1$ such that $\sigma_{f(\mu)}(\lambda) \geq r$, $\sigma^*_{f(\mu)}(\lambda) \leq s$ and $f(x_t) q \lambda [f(\mu)]$. Since f is $IF\mu$-continuous we have

$$\tau_\mu(f^{-1}(\lambda) \wedge \mu) \geq \sigma_{f(\mu)}(\lambda) \geq r$$

and

$$\tau^*_\mu(f^{-1}(\lambda) \wedge \mu) \leq \sigma^*_{f(\mu)}(\lambda) \leq s.$$

Since $f(x_t)q\lambda[f(\mu)]$ and by Lemma 1.1, we have $x_t q f^{-1}(\lambda)[\mu]$. Since f is injective, $f^{-1}(\lambda) \leq \mu$ and hence $x_t q f^{-1}(\lambda) \wedge \mu[\mu]$. Then there exists $\nu = f^{-1}(\lambda) \wedge \mu \in \mathcal{A}_\mu$ with $\tau_\mu(\nu) \geq r$, $\tau^*_\mu(\nu) \leq s$ and $x_t q\nu[\mu]$. Also,

$$f(\nu) = f(f^{-1}(\lambda) \wedge \mu) \leq f(f^{-1}(\lambda)) \wedge f(\mu) \leq \lambda \wedge f(\mu) = \lambda.$$

(vi)⇒(iii) Let $\nu \in \mathcal{A}_\mu$, $x_t \in C_{\tau_\mu,\tau^*_\mu}(\nu, r, s)$ and $\lambda \in \mathcal{A}_{f(\mu)}$ with $\sigma_{f(\mu)}(\lambda) \geq r$ and $\sigma^*_{f(\mu)}(\lambda) \leq s$ such that $f(x_t)q\lambda[f(\mu)]$. By (vi), there exists $\eta \in \mathcal{A}_\mu$ with $\tau_\mu(\eta) \geq r$, $\tau^*_\mu(\eta) \leq s$ such that $x_t q\eta[\mu]$ and $f(\eta) \leq \lambda$. Since $x_t \in C_{\tau_\mu,\tau^*_\mu}(\nu, r, s)$, $\tau_\mu(\eta) \geq r, \tau^*_\mu(\eta) \leq s$ and $x_t q\eta[\mu]$, then by using Theorem 2.8, we have $\nu q\eta[\mu]$ which implies that $f(\nu)qf(\eta)[f(\mu)]$ and hence $f(\nu)q\lambda[f(\mu)]$. Thus $f(x_t) \in C_{\sigma_{f(\mu)},\sigma^*_{f(\mu)}}(f(\nu), r, s)$. This implies that $x_t \in f^{-1}(C_{\sigma_{f(\mu)},\sigma^*_{f(\mu)}}(f(\nu), r, s))$. Then

$$C_{\tau_\mu,\tau^*_\mu}(\nu, r, s) \leq f^{-1}(C_{\sigma_{f(\mu)},\sigma^*_{f(\mu)}}(f(\nu), r, s)).$$

Hence

$$f(C_{\tau_\mu,\tau^*_\mu}(\nu, r, s)) \leq C_{\sigma_{f(\mu)},\sigma^*_{f(\mu)}}(f(\nu), r, s).$$

(vi)⇒(vii) Let $x_t \in \mu$ and $\lambda \in Q_{\sigma_{f(\mu)},\sigma^*_{f(\mu)}}(f(x_t), r, s)$. Then there exists $\eta \in \mathcal{A}_{f(\mu)}$ with $\sigma_{f(\mu)}(\eta) \geq r$, $\sigma^*_{f(\mu)}(\eta) \leq s$ such that $f(x_t)q\eta[f(\mu)]$ and $\eta \leq \lambda$. By (vi), there exists $\nu \in \mathcal{A}_\mu$ with $\tau_\mu(\nu) \geq r$, $\tau^*_\mu(\nu) \leq s$ such that $x_t q\nu[\mu]$ and $f(\nu) \leq \eta \leq \lambda$. Hence $\nu \in Q_{\tau_\mu,\tau^*_\mu}(x_t, r, s)$ and $f(\nu) \leq \lambda$.

(vii)⇒(viii) Let $x_t \in \mu$ and $\lambda \in Q_{\sigma_{f(\mu)},\sigma^*_{f(\mu)}}(f(x_t), r, s)$. By (vii), there exists $\nu \in Q_{\tau_\mu,\tau^*_\mu}(x_t, r, s)$ such that $f(\nu) \leq \lambda$. So, there is $\xi \in \mathcal{A}_\mu$ with $\tau_\mu(\xi) \geq r$, $\tau^*_\mu(\xi) \leq s$ such that $x_t q\xi[\mu]$ and $\xi \leq \nu \leq f^{-1}(\lambda)$. Hence $f^{-1}(\lambda) \in Q_{\tau_\mu,\tau^*_\mu}(x_t, r, s)$.

(viii)⇒(vi) Let $x_t \in \mu$ and $\lambda \in \mathcal{A}_{f(\mu)}$ with $\sigma_{f(\mu)}(\lambda) \geq r$, $\sigma^*_{f(\mu)}(\lambda) \leq s$ and $f(x_t)q\lambda[f(\mu)]$. Then $\lambda \in Q_{\sigma_{f(\mu)},\sigma^*_{f(\mu)}}(f(x_t), r, s)$. By (viii), we have $f^{-1}(\lambda) \in Q_{\tau_\mu,\tau^*_\mu}(x_t, r, s)$ and hence there is $\nu \in \mathcal{A}_\mu$ with $\tau_\mu(\nu) \geq r$, $\tau^*_\mu(\nu) \leq s$ such that $x_t q\nu[\mu]$ and $\nu \leq f^{-1}(\lambda)$, so $f(\nu) \leq \lambda$. □

Theorem 3.3. *Let (X, τ, τ^*), (Y, σ, σ^*) and (Z, δ, δ^*) be IFTSs, $\mu \in I^X$, $f : X \to Y$ and $g : Y \to Z$. If f is $IF\mu$-continuous and g is $IFf(\mu)$-continuous, then $g \circ f$ is $IF\mu$-continuous.*

Proof. For each $\lambda \in \mathcal{A}_{(g\circ f)(\mu)}$ we have

$$\begin{aligned}\delta^*_{g(f(\mu))}(\lambda) &\geq \sigma^*_{f(\mu)}(g^{-1}(\lambda) \wedge f(\mu)) \\ &\geq \tau^*_\mu(f^{-1}(g^{-1}(\lambda) \wedge f(\mu)) \wedge \mu) \\ &= \tau^*_\mu((g \circ f)^{-1}(\lambda) \wedge f^{-1}(f(\mu)) \wedge \mu) \\ &= \tau^*_\mu((g \circ f)^{-1}(\lambda) \wedge \mu).\end{aligned}$$

Similarly, $\delta_{g(f(\mu))}(\lambda) \leq \tau_\mu((g \circ f)^{-1}(\lambda) \wedge \mu)$. Thus $g \circ f$ is $IF\mu$-continuous. □

Definition 3.2. Let (X, τ, τ^*) be an IFTS and $\mu \in I^X$. For $r \in I_0$, $s \in I_1$ with $r + s \leq 1$, $\nu \in \mathcal{A}_\mu$ is called:

(i) (r,s)-$IF\mu$-regular open set if $I_{\tau_\mu,\tau^*_\mu}(C_{\tau_\mu,\tau^*_\mu}(\nu,r,s),r,s)=\nu$.
(ii) (r,s)-$IF\mu$-regular closed set if $C_{\tau_\mu,\tau^*_\mu}(I_{\tau_\mu,\tau^*_\mu}(\nu,r,s),r,s)=\nu$.

Definition 3.3. Let (X,τ,τ^*), (Y,σ,σ^*) be two IFTSs and $\mu\in I^X$. Then the mapping $f:X\to Y$ is called $IF\mu$-almost continuous if $\tau_\mu(f^{-1}(\nu)\wedge\mu)\geq r$ and $\tau^*_\mu(f^{-1}(\nu)\wedge\mu)\leq s$ for each (r,s)-$IF\mu$-regular open set ν in $\mathcal{A}_{f(\mu)}$.

Remark 3.2. Every $IF\mu$-continuous mapping is also $IF\mu$-almost continuous but the converse is not true in general, as the following example shows.

Example 3.2. Consider Example 3.1, and put

$$\sigma(\lambda)=\begin{cases}1, & \text{if } \lambda=\underline{0},\underline{1}\\ 0.5, & \text{if } \lambda=\underline{0.5}\\ 0.6, & \text{if } \lambda=\underline{2.5}\\ 0, & \text{otherwise.}\end{cases}\qquad \sigma^*(\lambda)=\begin{cases}0, & \text{if } \lambda=\underline{0},\underline{1}\\ 0.5, & \text{if } \lambda=\underline{0.5}\\ 0.2, & \text{if } \lambda=\underline{2.5}\\ 1, & \text{otherwise.}\end{cases}$$

Since $\mu=\underline{0.5}$, we have

$$\sigma_{f(\mu)}(\nu)=\begin{cases}1, & \text{if } \nu=\underline{0},f(\mu)\\ 0.6, & \text{if } \nu=\underline{2.5}\\ 0, & \text{otherwise.}\end{cases}\qquad \sigma^*_{f(\mu)}(\nu)=\begin{cases}0, & \text{if } \lambda=\underline{0},f(\mu)\\ 0.2, & \text{if } \underline{\nu}=\underline{2.5}\\ 1, & \text{otherwise.}\end{cases}$$

Then, the identity mapping $id_X:(X,\tau,\tau^*)\to(X,\sigma,\sigma^*)$ is $IF\mu$-almost continuous but not $IF\mu$-continuous.

Theorem 3.4. *Let (X,τ,τ^*) and (Y,σ,σ^*) be two IFTSs, $\mu\in I^X$ and $f: X\to Y$ be an injective mapping. For $r\in I_0$, $s\in I_1$ such that $r+s\leq 1$ the following statements are equivalent*:

(i) *f is $IF\mu$-almost continuous.*
(ii) *$\mathcal{F}_{\tau_\mu}(f^{-1}(\lambda)\wedge\mu)\geq r$ and $\mathcal{F}^*_{\tau^*_\mu}(f^{-1}(\lambda)\wedge\mu)\leq s$ for each (r,s)-$IF\mu$-regular closed set λ in $\mathcal{A}_{f(\mu)}$.*
(iii) *$f^{-1}(\lambda)\wedge\mu\leq I_{\tau_\mu,\tau^*_\mu}(f^{-1}(I_{\sigma_{f(\mu)},\sigma^*_{f(\mu)}}(C_{\sigma_{f(\mu)},\sigma^*_{f(\mu)}}(\lambda,r,s),r,s)\wedge\mu),r,s)$ for each $\lambda\in\mathcal{A}_{f(\mu)}$ with $\sigma_{f(\mu)}(\lambda)\geq r$ and $\sigma^*_{f(\mu)}(\lambda)\leq s$.*
(iv) *$f^{-1}(\lambda)\wedge\mu\geq C_{\tau_\mu,\tau^*_\mu}(f^{-1}(C_{\sigma_{f(\mu)},\sigma^*_{f(\mu)}}(I_{\sigma_{f(\mu)},\sigma^*_{f(\mu)}}(\lambda,r,s),r,s)\wedge\mu),r,s)$ for each $\lambda\in\mathcal{A}_{f(\mu)}$ with $\sigma_{f(\mu)}(f(\mu)-\lambda)\geq r$ and $\sigma^*_{f(\mu)}(f(\mu)-\lambda)\leq s$.*
(v) *For each $x_t\in\mu$ and $\lambda\in\mathcal{A}_{f(\mu)}$ with $\sigma_{f(\mu)}(\lambda)\geq r$, $\sigma^*_{f(\mu)}(\lambda)\leq s$ and $f(x_t)q\lambda[f(\mu)]$, there is $\nu\in\mathcal{A}_\mu$ with $\tau_\mu(\nu)\geq r$, $\tau^*_\mu(\nu)\leq s$ such that $f(\nu)\leq I_{\sigma_{f(\mu)},\sigma^*_{f(\mu)}}(C_{\sigma_{f(\mu)},\sigma^*_{f(\mu)}}(\lambda,r,s),r,s)$.*
(vi) *For each $x_t\in\mu$ and $\lambda\in Q_{\sigma_{f(\mu)},\sigma^*_{f(\mu)}}(f(x_t),r,s)$, there exists $\nu\in Q_{\tau_\mu,\tau^*_\mu}(x_t,r,s)$ such that $f(\nu)\leq I_{\sigma_{f(\mu)},\sigma^*_{f(\mu)}}(C_{\sigma_{f(\mu)},\sigma^*_{f(\mu)}}(\lambda,r,s),r,s)$.*
(vii) *For each $x_t\in\mu$ and each $\lambda\in Q_{\sigma_{f(\mu)},\sigma^*_{f(\mu)}}(f(x_t),r,s)$ we have*

$$f^{-1}(I_{\sigma_{f(\mu)},\sigma^*_{f(\mu)}}(C_{\sigma_{f(\mu)},\sigma^*_{f(\mu)}}(\lambda,r,s),r,s))\in Q_{\tau_\mu,\tau^*_\mu}(x_t,r,s).$$

Proof. (i)⇒ (ii) Let λ be (r,s)-$IF\mu$-regular closed set in $\mathcal{A}_{f(\mu)}$. Then $f(\mu)-\lambda$ is (r,s)-$IF\mu$-regular open set. Since f is $IF\mu$-almost continuous we have

$$\tau_\mu(f^{-1}(f(\mu)-\lambda)\wedge\mu)\geq r \quad \text{and} \quad \tau^*_\mu(f^{-1}(f(\mu)-\lambda)\wedge\mu)\leq s.$$

Since f is injective, $\tau_\mu((\mu-f^{-1}(\lambda))\wedge\mu)\geq r$ and $\tau^*_\mu((\mu-f^{-1}(\lambda))\wedge\mu)\leq s$. Let $\nu=\mu-(f^{-1}(\lambda)\wedge\mu)$. Then

$$\nu(x)=\mu(x)-(f^{-1}(\lambda)\wedge\mu)(x)=\mu(x)-\min\{f^{-1}(\lambda)(x),\mu(x)\}.$$

If $\mu(x)\leq f^{-1}(\lambda)(x)$, then $\nu(x)=\mu(x)-\mu(x)=0$. Then

$$\mathcal{F}_{\tau_\mu}(f^{-1}(\lambda)\wedge\mu)=\tau_\mu(\mu-(f^{-1}(\lambda)\wedge\mu))=\tau_\mu(\nu)=\tau_\mu(\underline{0})=1\geq r.$$

and

$$\mathcal{F}^*_{\tau^*_\mu}(f^{-1}(\lambda)\wedge\mu)=\tau^*_\mu(\mu-(f^{-1}(\lambda)\wedge\mu))=\tau^*_\mu(\nu)=\tau^*_\mu(\underline{0})=0\leq s.$$

If $\mu(x)>f^{-1}(\lambda)(x)$, then

$$\nu=\mu-f^{-1}(\lambda)=(\mu-f^{-1}(\lambda))\wedge\mu.$$

Then

$$\tau_\mu(\nu)=\tau_\mu((\mu-f^{-1}(\lambda))\wedge\mu)\geq r \quad \text{and} \quad \tau^*_\mu(\nu)=\tau^*_\mu((\mu-f^{-1}(\lambda))\wedge\mu)\leq s.$$

Thus

$$\mathcal{F}_{\tau_\mu}(f^{-1}(\lambda)\wedge\mu)=\tau_\mu(\nu)\geq r \quad \text{and} \quad \mathcal{F}^*_{\tau^*_\mu}(f^{-1}(\lambda)\wedge\mu)=\tau^*_\mu(\nu)\leq s.$$

(ii)⇒ (i) It is clear.

(i)⇒ (iii) Let $\lambda\in\mathcal{A}_{f(\mu)}$ and $r\in I_0$, $s\in I_1$ with $r+s\leq 1$ such that $\sigma_{f(\mu)}(\lambda)\geq r$, $\sigma^*_{f(\mu)}(\lambda)\leq s$. Then $\lambda\leq I_{\sigma_{f(\mu)},\sigma^*_{f(\mu)}}(C_{\sigma_{f(\mu)},\sigma^*_{f(\mu)}}(\lambda,r,s),r,s)$. Since $I_{\sigma_{f(\mu)},\sigma^*_{f(\mu)}}(C_{\sigma_{f(\mu)},\sigma^*_{f(\mu)}}(\lambda,r,s),r,s)$ is (r,s)-$IF\mu$-regular open and f is $IF\mu$-almost continuous,

$$\tau_\mu(f^{-1}(I_{\sigma_{f(\mu)},\sigma^*_{f(\mu)}}(C_{\sigma_{f(\mu)},\sigma^*_{f(\mu)}}(\lambda,r,s),r,s))\wedge\mu)\geq r$$

and

$$\tau^*_\mu(f^{-1}(I_{\sigma_{f(\mu)},\sigma^*_{f(\mu)}}(C_{\sigma_{f(\mu)},\sigma^*_{f(\mu)}}(\lambda,r,s),r,s))\wedge\mu)\leq s.$$

So,

$$\begin{aligned}f^{-1}(\lambda)\wedge\mu&\leq f^{-1}(I_{\sigma_{f(\mu)},\sigma^*_{f(\mu)}}(C_{\sigma_{f(\mu)},\sigma^*_{f(\mu)}}(\lambda,r,s),r,s))\wedge\mu\\&=I_{\tau_\mu,\tau^*_\mu}(f^{-1}(I_{\sigma_{f(\mu)},\sigma^*_{f(\mu)}}(C_{\sigma_{f(\mu)},\sigma^*_{f(\mu)}}(\lambda,r,s),r,s))\wedge\mu,r,s).\end{aligned}$$

(iii)⇒ (iv) It follows from Theorem 2.7,

(iv)⇒ (ii) Let λ be (r,s)-$IFf(\mu)$-regular closed. Then $\sigma_{f(\mu)}(f(\mu)-\lambda)\geq r$ $\sigma^*_{f(\mu)}(f(\mu)-\lambda)\leq s$. By (iv) we have,

$$\begin{aligned}f^{-1}(\lambda)\wedge\mu&\geq C_{\tau_\mu,\tau^*_\mu}(f^{-1}(C_{\sigma_{f(\mu)},\sigma^*_{f(\mu)}}(I_{\sigma_{f(\mu)},\sigma^*_{f(\mu)}}(\lambda,r,s),r,s)\wedge\mu),r,s)\\&=C_{\tau_\mu,\tau^*_\mu}(f^{-1}(\lambda)\wedge\mu,r,s).\end{aligned}$$

Thus $f^{-1}(\lambda)\wedge\mu = C_{\tau_\mu,\tau^*_\mu}(f^{-1}(\lambda)\wedge\mu, r, s)$. By Theorem 2.5, we have

$$\mathcal{F}_{\tau_\mu}(f^{-1}(\lambda)\wedge\mu) = \tau_\mu(\mu - f^{-1}(\lambda)\wedge\mu) \geq r$$

and

$$\mathcal{F}^*_{\tau^*_\mu}(f^{-1}(\lambda)\wedge\mu) = \tau^*_\mu(\mu - f^{-1}(\lambda)\wedge\mu) \leq s.$$

(i)⇒ (v)⇒ (iii) and (v)⇒ (vi)⇒ (vii)⇒ (v) are similar to that of Theorem 3.2. □

4. *IFμ*-separation axioms

Definition 4.1. Let (X,τ,τ^*) be an IFTS, $\mu\in I^X$ and $r\in I_0$, $s\in I_1$ with $r+s\leq 1$. μ is said to be (r,s)-$IF\mu T_2$-space if for each $x_\alpha, y_\beta(x\neq y)\in\mu$, there are $\nu_1,\nu_2\in\mathcal{A}_\mu$ with $\tau_\mu(\nu_1)\geq r$, $\tau_\mu(\nu_2)\geq r$, $\tau^*_\mu(\nu_1)\leq s$ and $\tau^*_\mu(\nu_2)\leq s$ such that $x_\alpha\in\nu_1$, $y_\beta\in\nu_2$ and $\nu_1\ \not{q}\nu_2[\mu]$.

Theorem 4.1. *Let (X,τ,τ^*) be an IFTS, $\mu\in I^X$ and $r\in I_0$, $s\in I_1$ with $r+s\leq 1$. μ is (r,s)-$IF\mu T_2$-space if and only if for each $x_\alpha, y_\beta(x\neq y)\in\mu$, we have*

$$y_\beta\notin\{C_{\tau_\mu,\tau^*_\mu}(\nu,r,s) : \tau_\mu(\nu)\geq r, \tau^*_\mu(\nu)\leq s, x_\alpha\in\nu\}.$$

Proof. Let $x_\alpha, y_\beta(x\neq y)\in\mu$ and $m=\mu(y)-\beta$. Then $x_\alpha, y_m(x\neq y)\in\mu$. Since μ is (r,s)-$IF\mu T_2$-space, there are $\nu_1,\nu_2\in\mathcal{A}_\mu$ with $\tau_\mu(\nu_1)\geq r$, $\tau_\mu(\nu_2)\geq r$, $\tau^*_\mu(\nu_1)\leq s$ and $\tau^*_\mu(\nu_2)\leq s$ such that $x_\alpha\in\nu_1$, $y_m\in\nu_2$ and $\nu_1\ \not{q}\nu_2[\mu]$. Thus $\nu_1\leq\mu-\nu_2$, $\tau(\mu-(\mu-\nu_2))\geq r$ and $\tau^*(\mu-(\mu-\nu_2))\leq s$ which implies, $C_{\tau_\mu,\tau^*_\mu}(\nu_1,r,s)\leq\mu-\nu_2$. Since $y_m\in\nu_2$,

$$\beta=\mu(y)-m>\mu(y)-\nu_2(y)\geq(C_{\tau_\mu,\tau^*_\mu}(\nu_1,r,s))(y)$$

and hence $y_\beta\notin C_{\tau_\mu,\tau^*_\mu}(\nu_1,r,s)$. Thus $y_\beta\notin\{C_{\tau_\mu,\tau^*_\mu}(\nu,r,s):\tau_\mu(\nu)\geq r,\tau^*_\mu(\nu)\leq s, x_\alpha\in\nu\}$.

Conversely, let $x_\alpha, y_\beta(x\neq y)\in\mu$. Then, $x_\alpha, y_{\mu(y)-\beta}(x\neq y)\in\mu$. By hypothesis, there is $\nu_1\in\mathcal{A}_\mu$ with $\tau_\mu(\nu_1)\geq r$, $\tau^*_\mu(\nu_1)\leq s$ such that $x_\alpha\in\nu_1$ and $y_{\mu(y)-\beta}\notin C_{\tau_\mu,\tau^*_\mu}(\nu_1,r,s)$ and hence, $\mu(y)-\beta>(C_{\tau_\mu,\tau^*_\mu}(\nu_1,r,s))(y)$ which implies $y_\beta\in\mu-C_{\tau_\mu,\tau^*_\mu}(\nu_1,r,s)=\nu_2$ and $\tau_\mu(\nu_2)\geq r$, $\tau^*_\mu(\nu_2)\leq s$. Since $\nu_2=\mu-C_{\tau_\mu,\tau^*_\mu}(\nu_1,r,s)\leq\mu-\nu_1$, $\nu_1\ \not{q}\nu_2[\mu]$. Hence μ is (r,s)-$IF\mu T_2$-space. □

Definition 4.2. Let (X,τ,τ^*) be an IFTS, $\mu\in I^X$ and $r\in I_0$, $s\in I_1$ with $r+s\leq 1$. μ is said to be (r,s)-$IF\mu$-regular space if for each $\xi\in\mathcal{A}_\mu$ with $\tau_\mu(\mu-\xi)\geq r$, $\tau^*_\mu(\mu-\xi)\leq s$ and for each fuzzy point $x_t\in\mu$ with $x_t\ \not{q}\xi[\mu]$, there are $\nu,\eta\in\mathcal{A}_\mu$ with $\tau_\mu(\nu)\geq r$, $\tau_\mu(\eta)\geq r$, $\tau^*_\mu(\nu)\leq s$ and $\tau^*_\mu(\eta)\leq s$ such that $x_t\in\nu$, $\xi\leq\eta$ and $\nu\ \not{q}\eta[\mu]$.

Theorem 4.2. *Let (X,τ,τ^*) be an IFTS, $\mu\in I^X$ and $r\in I_0$, $s\in I_1$ with $r+s\leq 1$. Then the following are equivalent*:

(i) *μ is (r,s)-$IF\mu$-regular.*

(ii) *For each fuzzy point* $x_t \in \mu$ *and each* $\nu \in \mathcal{A}_\mu$ *with* $x_t \in \nu$, $\tau_\mu(\nu) \geq r$, $\tau^*_\mu(\nu) \leq s$ *there is* $\eta \in \mathcal{A}_\mu$ *with* $\tau_\mu(\eta) \geq r$, $\tau^*_\mu(\eta) \leq s$ *such that* $x_t \in \eta \leq C_{\tau_\mu,\tau^*_\mu}(\eta, r, s) \leq \nu$.

(iii) *For each fuzzy point* $x_t \in \mu$ *and each* $\xi \in \mathcal{A}_\mu$ *with* $\tau_\mu(\mu - \xi) \geq r$ *and* $\tau^*_\mu(\mu - \xi) \leq s$ *such that* $x_t \not q \xi[\mu]$, *there are* $\nu, \eta \in \mathcal{A}_\mu$ *with* $\tau_\mu(\nu) \geq r$, $\tau_\mu(\eta) \geq r$, $\tau^*_\mu(\nu) \leq s$ *and* $\tau^*_\mu(\eta) \leq s$ *such that* $x_t \in \nu$, $\xi \leq \eta$ *and* $C_{\tau_\mu,\tau^*_\mu}(\nu, r, s) \not q \eta[\mu]$.

Proof. (i)⇒ (ii) Let $x_t \in \mu$ be a fuzzy point, $r \in I_0$, $s \in I_1$ with $r + s \leq 1$ and $\nu \in \mathcal{A}_\mu$ with $\tau_\mu(\nu) \geq r$, $\tau^*_\mu(\nu) \leq s$ and $x_t \in \nu$. Then $\tau_\mu(\mu - (\mu - \nu)) \geq r$, $\tau^*_\mu(\mu - (\mu - \nu)) \leq s$ and $x_t \not q(\mu - \nu)[\mu]$. By (i), there are $\eta, \nu_1 \in \mathcal{A}_\mu$ with $\tau_\mu(\eta) \geq r$, $\tau_\mu(\nu_1) \geq r$, $\tau^*_\mu(\eta) \leq s$ and $\tau^*_\mu(\nu_1) \leq s$ such that $x_t \in \eta$, $\mu - \nu \leq \nu_1$ and $\eta \not q \nu_1[\mu]$. Since $\eta \not q \nu_1[\mu]$, $\eta \leq \mu - \nu_1 \leq \nu$ and hence $C_{\tau_\mu,\tau^*_\mu}(\eta, r, s) \leq \mu - \nu_1 \leq \nu$. Thus $x_t \in \eta \leq C_{\tau_\mu,\tau^*_\mu}(\eta, r, s) \leq \nu$.

(ii)⇒ (iii) Let $x_t \in \mu$ be a fuzzy point, $r \in I_0$, $s \in I_1$ with $r + s \leq 1$ and $\xi \in \mathcal{A}_\mu$ with $\tau_\mu(\mu - \xi) \geq r$, $\tau^*_\mu(\mu - \xi) \leq s$ and $x_t \not q \xi[\mu]$. Then $x_t \in \mu - \xi$. By (ii), there is $\eta_1 \in \mathcal{A}_\mu$ with $\tau_\mu(\eta_1) \geq r$ and $\tau^*_\mu(\eta_1) \leq s$ such that $x_t \in \eta_1 \leq C_{\tau_\mu,\tau^*_\mu}(\eta_1, r, s) \leq \mu - \xi$. By (ii) again, there is $\nu \in \mathcal{A}_\mu$ with $\tau_\mu(\nu) \geq r$ and $\tau^*_\mu(\nu) \leq s$ such that

$$x_t \in \nu \leq C_{\tau_\mu,\tau^*_\mu}(\nu, r, s) \leq \eta_1 \leq C_{\tau_\mu,\tau^*_\mu}(\eta_1, r, s) \leq \mu - \xi.$$

Put $\eta = \mu - C_{\tau_\mu,\tau^*_\mu}(\eta_1, r, s)$. Hence there are $\nu, \eta \in \mathcal{A}_\mu$ with $\tau_\mu(\nu) \geq r$, $\tau_\mu(\eta) \geq r$, $\tau^*_\mu(\nu) \leq s$ and $\tau^*_\mu(\eta) \leq s$ such that $x_t \in \nu$, $\xi \leq \eta$ and $C_{\tau_\mu,\tau^*_\mu}(\nu, r, s) \not q \eta[\mu]$.

(iii)⇒ (i) It is clear. □

5. $IF\mu$-compactness

Definition 5.1. Let (X, τ, τ^*) be an IFTS, $\mu \in I^X$ and $r \in I_0$, $s \in I_1$ with $r + s \leq 1$. μ is said to be (r, s)-$IF\mu$-compact if for every family $\{\nu_i : i \in J\}$ in $\{\nu : \nu \in \mathcal{A}_\mu, \tau_\mu(\nu) \geq r, \tau^*_\mu(\nu) \leq s\}$ such that $(\bigvee_{i\in J} \nu_i)(x) = \mu(x)$ for each $x \in X$, there exists a finite subset J_0 of J such that $(\bigvee_{i\in J_0} \nu_i)(x) = \mu(x)$.

Definition 5.2. Let X be a non-empty set and $\mu \in I^X$. A collection $\beta \subseteq \mathcal{A}_\mu$ is said to form a fuzzy μ-filterbasis if for each finite subcollection $\{\nu_1, \nu_2, \ldots, \nu_n\}$ of β, $(\bigwedge_{i=1}^n \nu_i)(x) > 0$ for some $x \in X$.

Theorem 5.1. *Let* (X, τ, τ^*) *be an IFTS,* $\mu \in I^X$ *and* $r \in I_0$, $s \in I_1$ *with* $r + s \leq 1$. μ *is* (r, s)-$IF\mu$*-compact if and only if for each fuzzy* μ*-filterbasis* $\beta \subseteq \mathcal{A}_\mu$, $(\bigwedge_{\nu\in\beta} C_{\tau_\mu,\tau^*_\mu}(\nu, r, s))(x) > 0$ *for some* $x \in X$.

Proof. Let $\{\nu_i : i \in J\}$ be a family in $\Lambda = \{\nu : \nu \in \mathcal{A}_\mu, \tau_\mu(\nu) \geq r, \tau^*_\mu(\nu) \leq s\}$ such that $(\bigvee_{i\in J} \nu_i)(x) = \mu(x)$. Suppose that there is no finite subset J_0 of J such that $(\bigvee_{i\in J_0} \nu_i)(x) = \mu(x)$. Then for each finite subcollection $\{\nu_1, \nu_2, \ldots, \nu_n\}$ of Λ, there exists $x \in X$ such that $\nu_i(x) < \mu(x)$ for each $i = 1, 2, \ldots, n$. Then $\mu(x) - \nu_i(x) > 0$ for each $i = 1, 2, \ldots, n$. So, $\bigwedge_{i=1}^n (\mu - \nu_i)(x) > 0$ and hence $\beta = \{\mu - \nu_i : \nu_i \in \Lambda, i \in J\}$ forms a fuzzy μ-filterbasis.

Since $(\bigvee_{i\in J}\nu_i)(x)=\mu(x)$ for each $x\in X$ and $\tau_\mu(\nu_i)\geq r$, $\tau^*_\mu(\nu_i)\leq s$ for each $i\in J$ we have

$$(\bigwedge_{i\in J} C_{\tau_\mu,\tau^*_\mu}(\mu-\nu_i,r,s))(x)=(\bigwedge_{i\in J}(\mu-\nu_i))(x)=(\mu-\bigvee_{i\in J}\nu_i)(x)=0$$

for each $x\in X$. It is a contradiction. Thus there exists a finite subset J_0 of J such that $(\bigvee_{i\in J_0}\nu_i)(x)=\mu(x)$ for all $x\in X$. Thus μ is (r,s)-$IF\mu$-compact.

Conversely, Suppose that there exists fuzzy μ-filterbasis β such that

$$(\bigwedge_{\nu\in\beta} C_{\tau_\mu,\tau^*_\mu}(\nu,r,s))(x)=0$$

for each $x\in X$. Then

$$(\bigvee_{\nu\in\beta}(\mu-C_{\tau_\mu,\tau^*_\mu}(\nu,r,s))(x)=\mu(x)\quad\text{for each}\quad x\in X.$$

Since $\tau_\mu(\mu-C_{\tau_\mu,\tau^*_\mu}(\nu,r,s))\geq r$ and $\tau^*_\mu(\mu-C_{\tau_\mu,\tau^*_\mu}(\nu,r,s))\leq s$ and μ is (r,s)-$IF\mu$-compact, there exists a finite subcollection $\{\mu-C_{\tau_\mu,\tau^*_\mu}(\nu_i,r,s)) : i=1,2,\ldots,n\}$ of $\{\mu-C_{\tau_\mu,\tau^*_\mu}(\nu,r,s)) : \nu\in\beta\}$ such that

$$(\bigvee_{i=1}^{n}(\mu-C_{\tau_\mu,\tau^*_\mu}(\nu_i,r,s)))(x)=\mu(x)\quad\text{for each}\quad x\in X.$$

Since $\nu_i(x)\leq C_{\tau_\mu,\tau^*_\mu}(\nu_i,r,s)(x)$ for each $x\in X$ we have

$$(\bigvee_{i=1}^{n}(\mu-\nu_i))(x)\geq(\bigvee_{i=1}^{n}(\mu-C_{\tau_\mu,\tau^*_\mu}(\nu_i,r,s)))(x)=\mu(x)\quad\text{for each}\quad x\in X.$$

Then $(\bigvee_{i=1}^{n}(\mu-\nu_i))(x)=\mu(x)$ for each $x\in X$. Thus $(\bigwedge_{i=1}^{n}\nu_i)(x)=0$ for each $x\in X$. It is a contradiction. Hence $(\bigwedge_{\nu\in\beta} C_{\tau_\mu,\tau^*_\mu}(\nu,r,s))(x)>0$ for some $x\in X$. □

Theorem 5.2. *Let (X,τ,τ^*), (Y,σ,σ^*) be two IFTSs, $f : X\to Y$ be an $IF\mu$-continuous bijective mapping. For $r\in I_0$, $s\in I_1$ with $r+s\leq 1$, if μ is (r,s)-$IF\mu$-compact, then $f(\mu)$ is (r,s)-$IFf(\mu)$-compact.*

Proof. Let $\{\lambda_i : i\in J\}$ be a family in $\{\lambda : \lambda\in\mathcal{A}_{f(\mu)},\sigma_{f(\mu)}(\lambda)\geq r,\sigma^*_{f(\mu)}(\lambda)\leq s\}$ such that $(\bigvee_{i\in J}\lambda_i)(y)=(f(\mu))(y)$ for all $y\in Y$. Since f is $IF\mu$-continuous for each $i\in J$ we have,

$$\tau_\mu(f^{-1}(\lambda_i)\wedge\mu)\geq\sigma_{f(\mu)}(\lambda_i)\geq r,$$
$$\tau^*_\mu(f^{-1}(\lambda_i)\wedge\mu)\leq\sigma^*_{f(\mu)}(\lambda_i)\leq s.$$

Since f is injective,

$$\bigvee_{i\in J}(f^{-1}(\lambda_i)\wedge\mu)=f^{-1}(\bigvee_{i\in J}\lambda_i)\wedge\mu=f^{-1}(f(\mu))\wedge\mu=\mu.$$

Since μ is (r,s)-$IF\mu$-compact, there exists a finite subset J_0 of J such that $(\bigvee_{i\in J_0}(f^{-1}(\lambda_i)\wedge\mu))(x)=\mu(x)$ for all $x\in X$. This implies that

$f(\bigvee_{i\in J_0}(f^{-1}(\lambda_i)\wedge\mu)) = f(\mu)$ and hence $\bigvee_{i\in J_0}(f(f^{-1}(\lambda_i)))\wedge f(\mu) = f(\mu)$. Since f is surjective, $\bigvee_{i\in J_0}\lambda_i\wedge f(\mu) = f(\mu)$. Thus $(\bigvee_{i\in J_0}\lambda_i)(y) = f(\mu)(y)$ for all $y\in Y$. Hence $f(\mu)$ is (r,s)-$IFf(\mu)$-compact. □

6. $IF\mu$-connected sets

Definition 6.1. Let (X,τ,τ^*) be an IFTS, $\mu\in I^X$ and $r\in I_0$, $s\in I_1$ with $r+s\le 1$. Then $\nu,\eta\in\mathcal{A}_\mu$ are said to be (r,s)-$IF\mu$-separated if $\nu \not{q} C_{\tau_\mu,\tau^*_\mu}(\eta,r,s)[\mu]$ and $\eta \not{q} C_{\tau_\mu,\tau^*_\mu}(\nu,r,s)[\mu]$.

Theorem 6.1. *Let (X,τ,τ^*) be an IFTS, $\mu\in I^X$ and $r\in I_0$, $s\in I_1$ with $r+s\le 1$. Then for $\nu,\eta\in\mathcal{A}_\mu$,*

(i) *If ν and η are (r,s)-$IF\mu$-separated and $\nu_1,\eta_1\in\mathcal{A}_\mu$ such that $\underline{0}\ne\nu_1\le\nu$, $\underline{0}\ne\eta_1\le\eta$, then ν_1 and η_1 are (r,s)-$IF\mu$-separated.*

(ii) *If $\nu \not{q}\eta[\mu]$ and either $\tau_\mu(\nu)\ge r$, $\tau_\mu(\eta)\ge r$, $\tau^*_\mu(\nu)\le s$ and $\tau^*_\mu(\eta)\le s$ or $\tau_\mu(\mu-\nu)\ge r$, $\tau_\mu(\mu-\eta)\ge r$, $\tau^*_\mu(\mu-\nu)\le s$ and $\tau^*_\mu(\mu-\eta)\le s$, then ν and η are (r,s)-$IF\mu$-separated.*

(iii) *If $\tau_\mu(\nu)\ge r$, $\tau_\mu(\eta)\ge r$, $\tau^*_\mu(\nu)\le s$ and $\tau^*_\mu(\eta)\le s$ or $\tau_\mu(\mu-\nu)\ge r$, $\tau_\mu(\mu-\eta)\ge r$, $\tau^*_\mu(\mu-\nu)\le s$ and $\tau^*_\mu(\mu-\eta)\le s$, then $\nu\wedge(\mu-\eta)$ and $\eta\wedge(\mu-\nu)$ are (r,s)-$IF\mu$-separated.*

Proof. (i) Since $\nu_1\le\nu$, $C_{\tau_\mu,\tau^*_\mu}(\nu_1,r,s)\le C_{\tau_\mu,\tau^*_\mu}(\nu,r,s)$. Since ν, η are (r,s)-$IF\mu$-separated, $\eta \not{q} C_{\tau_\mu,\tau^*_\mu}(\nu,r,s)[\mu]$. Thus

$$\eta_1\le\eta\le\mu-C_{\tau_\mu,\tau^*_\mu}(\nu,r,s)\le\mu-C_{\tau_\mu,\tau^*_\mu}(\nu_1,r,s).$$

Then $\eta_1 \not{q} C_{\tau_\mu,\tau^*_\mu}(\nu_1,r,s)[\mu]$. Similarly, $\nu_1 \not{q} C_{\tau_\mu,\tau^*_\mu}(\eta_1,r,s)[\mu]$. Hence ν_1 and η_1 are (r,s)-$IF\mu$-separated.

(ii) Let $\nu \not{q}\eta[\mu]$, $\tau_\mu(\nu)\ge r$, $\tau_\mu(\eta)\ge r$, $\tau^*_\mu(\nu)\le s$ and $\tau^*_\mu(\eta)\le s$. Since $\nu \not{q}\eta[\mu]$, $\nu\le\mu-\eta$. Thus

$$C_{\tau_\mu,\tau^*_\mu}(\nu,r,s)\le C_{\tau_\mu,\tau^*_\mu}(\mu-\eta,r,s)=\mu-\eta.$$

Then $C_{\tau_\mu,\tau^*_\mu}(\nu,r,s) \not{q}\eta[\mu]$. Similarly, $C_{\tau_\mu,\tau^*_\mu}(\eta,r,s) \not{q}\nu[\mu]$. Then ν and η are (r,s)-$IF\mu$-separated. Let $\tau_\mu(\mu-\nu)\ge r$, $\tau_\mu(\mu-\eta)\ge r$, $\tau^*_\mu(\mu-\nu)\le s$ and $\tau^*_\mu(\mu-\eta)\le s$. By Theorem 2.5, $\nu=C_{\tau_\mu,\tau^*_\mu}(\nu,r,s)$ and $\eta=C_{\tau_\mu,\tau^*_\mu}(\eta,r,s)$. Since $\nu \not{q}\eta[\mu]$ we have, $C_{\tau_\mu,\tau^*_\mu}(\nu,r,s) \not{q}\eta[\mu]$ and $C_{\tau_\mu,\tau^*_\mu}(\eta,r,s) \not{q}\nu[\mu]$. Then ν and η are (r,s)-$IF\mu$-separated.

(iii) Let $\tau_\mu(\nu)\ge r$, $\tau_\mu(\eta)\ge r$, $\tau^*_\mu(\nu)\le s$ and $\tau^*_\mu(\eta)\le s$. Since $\nu\wedge(\mu-\eta)\le\mu-\eta$,

$$C_{\tau_\mu,\tau^*_\mu}(\nu\wedge(\mu-\eta),r,s)\le C_{\tau_\mu,\tau^*_\mu}((\mu-\eta),r,s)=\mu-\eta.$$

Then

$$\eta\le\mu-C_{\tau_\mu,\tau^*_\mu}(\nu\wedge(\mu-\eta),r,s).$$

Since $\eta\wedge(\mu-\nu)\le\eta$,

$$\eta\wedge(\mu-\nu)\le\mu-C_{\tau_\mu,\tau^*_\mu}(\nu\wedge(\mu-\eta),r,s).$$

Thus $C_{\tau_\mu,\tau^*_\mu}(\nu\wedge(\mu-\eta),r,s)\ \not q(\eta\wedge(\mu-\nu))[\mu]$. Similarly, we can show $C_{\tau_\mu,\tau^*_\mu}(\eta\wedge(\mu-\nu),r,s)\ \not q(\nu\wedge(\mu-\eta))[\mu]$. Hence, $\nu\wedge(\mu-\eta)$ and $\eta\wedge(\mu-\nu)$ are (r,s)-$IF\mu$-separated. Let $\tau_\mu(\mu-\nu)\geq r$, $\tau_\mu(\mu-\eta)\geq r$, $\tau^*_\mu(\mu-\nu)\leq s$ and $\tau^*_\mu(\mu-\eta)\leq s$. By Theorem 2.5, we have $\nu=C_{\tau_\mu,\tau^*_\mu}(\nu,r,s)$ and $\eta=C_{\tau_\mu,\tau^*_\mu}(\eta,r,s)$. Then

$$\nu\wedge(\mu-\eta)\leq\mu-\eta=\mu-C_{\tau_\mu,\tau^*_\mu}(\eta,r,s).$$

Thus $C_{\tau_\mu,\tau^*_\mu}(\eta,r,s)\leq\mu-(\nu\wedge(\mu-\eta))$. Since $\eta\wedge(\mu-\nu)\leq\eta$,

$$C_{\tau_\mu,\tau^*_\mu}((\eta\wedge(\mu-\nu)),r,s)\leq C_{\tau_\mu,\tau^*_\mu}(\eta,r,s)\leq\mu-(\nu\wedge(\mu-\eta)).$$

Thus $C_{\tau_\mu,\tau^*_\mu}(\eta\wedge(\mu-\nu),r,s)\ \not q(\nu\wedge(\mu-\eta))[\mu]$. Similarly,

$$C_{\tau_\mu,\tau^*_\mu}(\nu\wedge(\mu-\eta),r,s)\ \not q(\eta\wedge(\mu-\nu))[\mu].$$

Hence, $\nu\wedge(\mu-\eta)$ and $\eta\wedge(\mu-\nu)$ are (r,s)-$IF\mu$-separated. □

Theorem 6.2. *Let (X,τ,τ^*) be an IFTS, $\mu\in I^X$ and $r\in I_0$, $s\in I_1$ with $r+s\leq 1$. Then $\nu,\eta\in\mathcal{A}_\mu$ are (r,s)-$IF\mu$-separated if and only if there exist $\nu_1,\eta_1\in\mathcal{A}_\mu$ with $\tau_\mu(\nu_1)\geq r$, $\tau_\mu(\eta_1)\geq r$, $\tau^*_\mu(\nu_1)\leq s$ and $\tau^*_\mu(\eta_1)\leq s$ such that $\nu\leq\nu_1$, $\eta\leq\eta_1$, $\nu\ \not q\eta_1[\mu]$ and $\eta\ \not q\nu_1[\mu]$.*

Proof. Let $\nu,\eta\in\mathcal{A}_\mu$ are (r,s)-$IF\mu$-separated. Then

$$\nu\leq\mu-C_{\tau_\mu,\tau^*_\mu}(\eta,r,s)=\nu_1(\text{say})\ \text{and}\quad \eta\leq\mu-C_{\tau_\mu,\tau^*_\mu}(\nu,r,s)=\eta_1(\text{say}).$$

Then $\tau_\mu(\nu_1)\geq r$, $\tau_\mu(\eta_1)\geq r$, $\tau^*_\mu(\nu_1)\leq s$ and $\tau^*_\mu(\eta_1)\leq s$. Since $\nu_1=\mu-C_{\tau_\mu,\tau^*_\mu}(\eta,r,s)\leq\mu-\eta$ and $\eta_1=\mu-C_{\tau_\mu,\tau^*_\mu}(\nu,r,s)\leq\mu-\nu$, we have $\nu_1\ \not q\eta[\mu]$ and $\eta_1\ \not q\nu[\mu]$.

Conversely, let $\nu_1,\eta_1\in\mathcal{A}_\mu$ with $\tau_\mu(\nu_1)\geq r$, $\tau_\mu(\eta_1)\geq r$, $\tau^*_\mu(\nu_1)\leq s$ and $\tau^*_\mu(\eta_1)\leq s$ such that $\nu\leq\nu_1$, $\eta\leq\eta_1$, $\nu\ \not q\eta_1[\mu]$ and $\eta\ \not q\nu_1[\mu]$. Then

$$C_{\tau_\mu,\tau^*_\mu}(\nu,r,s)\leq C_{\tau_\mu,\tau^*_\mu}(\mu-\eta_1,r,s)=\mu-\eta_1\leq\mu-\eta$$

and

$$C_{\tau_\mu,\tau^*_\mu}(\eta,r,s)\leq C_{\tau_\mu,\tau^*_\mu}(\mu-\nu_1,r,s)=\mu-\nu_1\leq\mu-\nu.$$

Thus $C_{\tau_\mu,\tau^*_\mu}(\nu,r,s)\ \not q\eta[\mu]$ and $C_{\tau_\mu,\tau^*_\mu}(\eta,r,s)\ \not q\nu[\mu]$. Hence ν,η are (r,s)-$IF\mu$-separated. □

Definition 6.2. Let (X,τ,τ^*) be an IFTS, $\mu\in I^X$ and $r\in I_0$, $s\in I_1$ with $r+s\leq 1$. Then $\nu\in\mathcal{A}_\mu$ is said to be (r,s)-$IF\mu$-connected if it can't be expressed as the union of two (r,s)-$IF\mu$-separated sets.

Theorem 6.3. *Let (X,τ,τ^*) and (Y,σ,σ^*) be IFTS, $\mu\in I^X$ and $f:X\to Y$ be $IF\mu$-continuous injective mapping. For $r\in I_0$, $s\in I_1$ with $r+s\leq 1$, if λ is (r,s)-$IF\mu$-connected, then $f(\lambda)$ is (r,s)-$IFf(\mu)$-connected.*

Proof. Suppose that $f(\lambda)$ is not (r,s)-$IFf(\mu)$-connected. Then there exist two (r,s)-$IFf(\mu)$-separated sets $\nu,\eta\in\mathcal{A}_{f(\mu)}$ such that $f(\lambda)=\nu\vee\eta$. By Theorem 6.2, there exist $\nu_1,\eta_1\in\mathcal{A}_{f(\mu)}$ with $\sigma_{f(\mu)}(\nu_1)\geq r$, $\sigma_{f(\mu)}(\eta_1)\geq r$, $\sigma^*_{f(\mu)}(\nu_1)\leq s$ and $\sigma^*_{f(\mu)}(\eta_1)\leq s$ such that $\nu\leq\nu_1$, $\eta\leq\eta_1$, $\nu\ \not q\eta_1[f(\mu)]$ and

$\eta \not q \nu_1[f(\mu)]$. Since f is injective and $\nu_1 \leq f(\mu)$, $f^{-1}(\nu_1) \leq f^{-1}(f(\mu)) = \mu$ and hence $f^{-1}(\nu) \leq f^{-1}(\nu_1) \wedge \mu$. Similarly, $f^{-1}(\eta) \leq f^{-1}(\eta_1) \wedge \mu$. Since f is $IF\mu$-continuous, we have

$$\tau_\mu(f^{-1}(\nu_1) \wedge \mu) \geq \sigma_{f(\mu)}(\nu_1) \geq r \quad \text{and} \quad \tau^*_\mu(f^{-1}(\nu_1) \wedge \mu) \leq \sigma^*_{f(\mu)}(\nu_1) \leq s.$$

Similarly, $\tau_\mu(f^{-1}(\eta_1) \wedge \mu) \geq r$, $\tau^*_\mu(f^{-1}(\eta_1) \wedge \mu) \leq s$. Since f is injective,

$$\begin{aligned} f^{-1}(\nu)(x) + (f^{-1}(\eta_1) \wedge \mu)(x) &= f^{-1}(\nu)(x) + f^{-1}(\eta_1)(x) \\ &= \nu(f(x)) + \eta_1(f(x)) \\ &= \nu(y) + \eta_1(y) \leq f(\mu)(y) = \mu(x) \end{aligned}$$

and hence $f^{-1}(\nu) \not q (f^{-1}(\eta_1) \wedge \mu)[\mu]$. Similarly, $f^{-1}(\eta) \not q (f^{-1}(\nu_1) \wedge \mu)[\mu]$. Then by Theorem 6.2, $f^{-1}(\nu)$ and $f^{-1}(\eta)$ are (r, s)-$IF\mu$-separated. Since f is injective,

$$\lambda = f^{-1}(f(\lambda)) = f^{-1}(\nu \vee \eta) = f^{-1}(\nu) \vee f^{-1}(\eta).$$

It is a contradiction with λ is (r, s)-$IF\mu$-connected. Hence $f(\lambda)$ is (r, s)-$IFf(\mu)$-connected. □

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