A DECOMPOSITION INTO ATOMS OF TENT SPACES ASSOCIATED WITH GENERAL APPROACH REGIONS

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Abstract. We first introduce a space of homogeneous type $X$, and develop the theory of the tent spaces on the generalized upper half-space $X \times (0, \infty)$. The goal of this paper is to study that every element of the tent spaces $T^p_\Omega(X \times (0, \infty), 0 < p \leq 1)$ can be decomposed into elementary particles which are called "atoms."

1. Introduction

The theory of the tent spaces on the upper half-space $\mathbb{R}^{n+1}_+$ was introduced from the work of R. R. Coifman, Y. Meyer and E. M. Stein [1]. In this paper we study the theory of the tent spaces on the generalized upper half-space $X \times (0, \infty)$, where $X$ is a space of homogeneous type.

We begin by introducing the notion of a space of homogeneous type [2]: Let $X$ be a topological space endowed with Borel measure $\mu$. Assume that $d$ is a pseudo-metric on $X$, that is, a nonnegative function defined on $X \times X$ satisfying

1. $d(x, x) = 0; d(x, y) > 0$ if $x \neq y$,
2. $d(x, y) = d(y, x)$, and
3. $d(x, z) \leq K(d(x, y) + d(y, z))$, where $K$ is some fixed constant.

Assume further that

(a) the balls $B(x, \rho) = \{y \in X : d(x, y) < \rho\}, \rho > 0$, form a basis of open neighborhoods at $x \in X$,
and that $\mu$ satisfies the doubling property:

(b) $0 < \mu(B(x, 2\rho)) \leq A\mu(B(x, \rho)) < \infty$, where $A$ is some fixed constant.

Then we call $X$ a space of homogeneous type.

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Property (iii) will be referred to as the “triangle inequality.” Note that property (b) implies that for every $C > 0$ there is a constant $A_C < \infty$ such that 
\[ \mu(B(x, C\rho)) \leq A_C \mu(B(x, \rho)) \]
for all $x \in X$ and $\rho > 0$.

Note that the volume of balls will be proportional to a fixed power of the radius. Thus assume there are a $\alpha \in \mathbb{R}$ and constants $C_1$ and $C_2$ such that 
\[ C_1 \rho^\alpha \leq \mu(B(x, \rho)) \leq C_2 \rho^\alpha. \]
We will denote $\mu(B(x, \rho)) \approx \rho^\alpha$ for the simplicity of the notation.

Now consider the space $X \times (0, \infty)$, which is a kind of generalized upper half-space over $X$. Suppose that there is a given set $\Omega_x \subset X \times (0, \infty)$ for each $x \in X$. Let $\Omega$ denote the family $\{\Omega_x\}_{x \in X}$. Thus at each $x \in X$, $\Omega$ determines a collection of balls, namely, $\{B(y, t) : (y, t) \in \Omega_x\}$.

For a measurable function $f$ defined on $X \times (0, \infty)$, and real number $\alpha$, we define an area function $S_{\Omega, \alpha}(f)$ of $f$, with respect to $\Omega$, as
\[
S_{\Omega, \alpha}(f)(x) = \left( \int_{\Omega_x} |f(y, t)|^2 \frac{d\mu(y)dt}{t^{\alpha+1}} \right)^{1/2}
\]
for $x \in X$. Throughout this paper we will always assume that $\Omega$ is chosen so that $S_{\Omega, \alpha}(f)$ is a measurable function on $X$, and that $\Omega = \{\Omega_x\}_{x \in X}$ is a symmetric family, that is, if $x \in \Omega_y(t)$, then $y \in \Omega_x(t)$, where $\Omega_x(t) = \{y \in X : (y, t) \in \Omega_x\}$.

For any set $E \subset X$, the tent over $E$, with respect to $\Omega$, is the set
\[
\hat{E}_\Omega = (X \times (0, \infty)) \setminus \bigcup_{x \not\in E} \Omega_x.
\]

The tent space $T^p_{\Omega}$ is defined as the space of functions $f$ on $X \times (0, \infty)$, so that $S_{\Omega, \alpha}(f) \in L^p(d\mu)$, $0 < p < \infty$, and set
\[
||f||_{T^p_{\Omega}} = ||S_{\Omega, \alpha}(f)||_{L^p(d\mu)}.
\]

For $0 < p \leq 1$, a function $a$, supported in $B_\Omega$ for some ball $B$ in $X$, is said to be an $(\Omega, p)$-atom if
\[
\int_{B_\Omega} |a(x, t)|^2 \frac{d\mu(x)dt}{t} \leq [\mu(B)]^{1-2/p}.
\]

We need the notion of points of density: Let $F$ be a closed subset of $X$ whose complement has finite measure. Let $\gamma$ be a fixed parameter,
0 < \gamma < 1. Then we say that a point \( x \in X \) has \textit{global} \( \gamma \)-density with respect to \( F \) if
\[
\frac{\mu(F \cap B(x, \rho))}{\mu(B(x, \rho))} \geq \gamma
\]
for all balls \( B(x, \rho) \) in \( X \). Observe that if \( F^* \) is the set of points of global \( \gamma \)-density with respect to \( F \); then \( F^* \) is closed, \( F^* \subset F \), and
\[
\{ x \in X : M(\chi_{c^*F})(x) > 1 - \gamma \},
\]
where \( \chi_{c^*F} \) is the characteristic function of the open set \( c^*F \), and \( M \) is the Hardy-Littlewood maximal operator on \( X \).

**2. Main result**

**Lemma 2.1.** The Hardy-Littlewood maximal operator \( M \) is of weak type \((1,1)\). More precisely, if \( f \in L^1_{\text{loc}}(d\mu) \), then there is a constant \( C \) so that
\[
\mu(\{ x \in X : M(f)(x) > \lambda \}) \leq C ||f||_1 / \lambda
\]
for all \( \lambda > 0 \).

**Lemma 2.2.** Assume \( F \) is a closed subset of \( X \). Then there is a constant \( C \) such that
\[
\mu(\c^*F) \leq C \mu(c^*F),
\]
where \( F^\ast \) is the set of points of global \( \gamma \)-density with respect to \( F \).

**Proof.** Since the Hardy-Littlewood maximal operator \( M \) is of weak type \((1,1)\) by Lemma 1, there is a constant \( C_\gamma \) so that
\[
\mu(\{ x \in X : M(\chi_{c^*F})(x) > 1 - \gamma \}) \leq C_\gamma ||\chi_{c^*F}||_1 / 1 - \gamma.
\]
But the left side of (3) is equal to \( \mu(\c^*F) \) by (2) and so the proof is complete.

**Lemma 2.3.** There are constants \( C_\gamma \) and \( \gamma \), \( 0 < \gamma < 1 \), sufficiently close to 1, so that whenever \( F \) is a closed subset of \( X \) whose complement has finite measure and \( \Phi \) is a nonnegative measurable function defined on \( X \times (0, \infty) \), then
\[
\int_{\cup x \in F^* \Omega_x} \Phi(y, t)^\alpha d\mu(y) dt \leq C_\gamma \int_F \left( \int_{\Omega_x} \Phi(y, t) d\mu(y) dt \right) d\mu(x),
\]
where \( \alpha \) is given as in (1), and \( F^* \) is the set of points of global \( \gamma \)-density with respect to \( F \).
Proof. Observe that Fubini’s theorem gives
\[
\int_F \left( \int_{\Omega_x} \Phi(y, t) d\mu(y) dt \right) d\mu(x) \\
= \int_X \Phi(y, t) \left( \int_F \chi_{B(y,t)}(x) d\mu(x) \right) d\mu(y) dt,
\]
where \( \chi_{B(y,t)} \) is the characteristic function of the ball \( B(y, t) \). Thus it will suffice to show that if
\[
(y, t) \in \bigcup_{x \in F^*} \Omega_x,
\]
then there is a constant \( C_\gamma \) so that
\[
(4) \quad \int_F \chi_{B(y,t)}(x) d\mu(x) \geq C_\gamma t^\alpha.
\]
Let
\[
(y, t) \in \bigcup_{x \in F^*} \Omega_x.
\]
Then there is a point \( x \in F^* \) so that \( d(x, y) < t \). Now it is obvious by geometric observation that
\[
(5) \quad \mu(B(x, t) \cap cB(y, t)) \leq C \mu(B(x, t)),
\]
where \( C < 1 \). However, it is true that
\[
(6) \quad \mu(F \cap B(y, t)) + \mu(B(x, t) \cap cB(y, t)) \\
\geq \mu(F \cap B(x, t) \cap B(y, t)) + \mu(F \cap B(x, t) \cap cB(y, t)) \\
= \mu(F \cap B(x, t)).
\]
By the global \( \gamma \)-density property, we have
\[
(7) \quad \mu(F \cap B(x, t)) \geq \gamma \mu(B(x, t)).
\]
Thus (5), (6) and (7) imply that
\[
\mu(F \cap B(y, t)) \\
\geq \mu(F \cap B(x, t)) - \mu(B(x, t) \cap cB(y, t)) \\
\geq (\gamma - C) \mu(B(x, t)) \\
= C_\gamma \mu(B(x, t)),
\]
and so, if \( \gamma \) is chosen sufficiently close to 1, then we have
\[
\int_F \chi_{B(y,t)}(x) d\mu(x) \geq C_\gamma t^\alpha,
\]
since $\mu(B(x, t)) \approx t^\alpha$. Thus we get (4). The proof is therefore complete.

The next lemma is of the type due to Whitney.

**Lemma 2.4.** Let $O$ be an open subset of $X$. Then there are positive constants $A, h_1 > 1, h_2 > 1$ and $h_3 < 1$ which depend only on the space $X$, and a sequence $\{B(x_i, \rho_i)\}$ of balls such that

(i) $\bigcup_i B(x_i, \rho_i) = O$,
(ii) $B(x_i, h_2 \rho_i) \subset O$ and $B(x_i, h_1 \rho_i) \cap (X \setminus O) \neq \emptyset$,
(iii) the balls $B(x_i, h_3 \rho_i)$ are pairwise disjoint, and
(iv) no point in $O$ lies in more than $A$ of the balls $B(x_i, h_2 \rho_i)$.

As the main result of this paper, the following theorem means that every element of the tent spaces $T_{\Omega}^p$, $0 < p \leq 1$, can be decomposed into elementary particles which are called “atoms.”

**Theorem 2.5.** Let a function $f$ belong to the tent spaces $T_{\Omega}^p$, $0 < p \leq 1$. Then

$$|f(x, t)| \leq \sum_{j=0}^{\infty} \lambda_j a_j(x, t),$$

where the $a_j$’s are $(\Omega, p)$-atoms, and the $\lambda_j$’s are positive numbers. Moreover,

$$\sum_{j=0}^{\infty} \lambda_j^p \leq C ||S_{\Omega, \alpha}(f)||_{L^p(d\mu)}^{p}$$

for some constant $C$.

**Proof.** For each integer $k$, let $O_k$ be the open set

$$O_k = ^c F_k = \{x \in X : S_{\Omega, \alpha}(f)(x) > 2^k\}.$$  

Let $O^*_k = ^c F^*_k$. Then it follows from the notion of global $\gamma$-density (with $\gamma$ sufficiently close to 1) that

$$O^*_k = \{x \in X : M(\chi_{O_k})(x) > 1 - \gamma\}.$$ 

Observe that for each integer $k$,

$$O_k \supset O_{k+1},$$

$$O^*_k \supset O_k,$$

and

$$\hat{O}_k \supset \hat{O}_k.$$ 

Moreover, $\bigcup_{k=-\infty}^{\infty} O^*_k$ contains the support of $f$ in $X \times (0, \infty)$. We distinguish two cases:
Case 1. For every integer $k$, $O_k^* \neq X$. Let

$$O_k^* = \bigcup_{j=0}^{\infty} B_{k,j}$$

be a Whitney decomposition of the open set $O_k^*$, where

$$B_{k,j} = B(x_{k,j}, \rho_{k,j}).$$

Let

$$\tilde{B}_{k,j} = B(x_{k,j}, Ch_1 \rho_{k,j}),$$

where $h_1$ is given in (ii) of Lemma 4, and $C$ will be chosen sufficiently large in a moment. If $(x, t) \in \hat{O}_k^*$, then $B(x, t) \subset O_k^*$, and $x \in B_{k,j}$ for some $j$. Let

$$y \in B(x_{k,j}, h_1 \rho_{k,j}) \cap (X \setminus O_k^*).$$

Then we have

$$t \leq d(x, y) \leq K(d(x, x_{k,j}) + d(x_{k,j}, y)) \leq K(1 + h_1) \rho_{k,j},$$

where $K$ is the constant in the triangle inequality. Hence if $z \in B(x, t)$, then it follows from (8) that

$$d(x_{k,j}, z) \leq K(d(x_{k,j}, x) + d(x, z)) \leq K(\rho_{k,j} + t) \leq K(\rho_{k,j} + K(1 + h_1) \rho_{k,j}) = K(1 + K(1 + h_1)) \rho_{k,j}.$$ 

Thus if we choose $C$ so that

$$K(1 + K(1 + h_1)) < Ch_1,$$

then it follows that

$$B(x, t) \subset B(x_{k,j}, Ch_1 \rho_{k,j}),$$

and hence

$$(x, t) \in \overline{\chi_{B_{k,j}}}.$$ 

Thus we have

$$\hat{O}_k^* \setminus O_{k+1}^* = \bigcup_{j} \Delta_{k,j},$$

where

$$\Delta_{k,j} = \overline{\chi_{B_{k,j}}} \cap (\hat{O}_k^* \setminus O_{k+1}^*).$$
If we let $\chi_{k,j}$ be the characteristic function of the set $\Delta_{k,j}$, then
\[
|f(y,t)| \leq \sum_{k,j} |f(y,t)|\chi_{k,j}(y,t)
\]
\[
= \sum_{k,j} \lambda_{k,j} a_{k,j}(y,t),
\]
where
\[
a_{k,j}(y,t) = \mu(\tilde{B}_{k,j})^{1/2-1/p}|f(y,t)|\chi_{k,j}(y,t) \left( \int_{\Delta_{k,j}} |f(y,t)|^2 \frac{d\mu(y)dt}{t} \right)^{-1/2},
\]
and
\[
\lambda_{k,j} = \mu(\tilde{B}_{k,j})^{-1/2+1/p} \left( \int_{\Delta_{k,j}} |f(y,t)|^2 \frac{d\mu(y)dt}{t} \right)^{1/2}.
\]
Now $a_{k,j}$ is an $(\Omega, p)$-atom associated to the ball $\tilde{B}_{k,j}$ since $|f(y,t)| \leq 2^{k+1}$ in $\Omega \times (0, \infty) \setminus \hat{O}_{k+1}$. Also, put
\[
F = \mathring{c}O_{k+1},
\]
\[
\bigcup_{x \in F^*} \Omega_x = O_{k+1}^*,
\]
\[
F^* = \mathring{c}O_{k+1}^*,
\]
and
\[
\Phi(y,t) = |f(y,t)|^2 \frac{1}{t^{\alpha+1}} \chi_{\tilde{B}_{k,j}}(y,t),
\]
and apply Lemma 3 to get that
\[
\int_{\Delta_{k,j}} |f(y,t)|^2 \frac{d\mu(y)dt}{t} \leq \int_{\tilde{B}_{k,j} \setminus O_{k+1}^*} |f(y,t)|^2 \frac{d\mu(y)dt}{t} \leq \int_{O_{k+1}^*} \chi_{\tilde{B}_{k,j}}(y,t) |f(y,t)|^2 \frac{d\mu(y)dt}{t} \leq C \gamma \int_{O_{k+1}^*} \int_{\Omega_x} |f(y,t)|^2 \frac{d\mu(y)dt}{t^{\sigma+1}} d\mu(x)
\[ \leq C_\gamma \int_{\tilde{O}_{k+1} \cap \tilde{B}_{k,j}} (S_{\Omega,a}(f)(x))^2 d\mu(x) \]

\[ \leq C_\gamma (2^{k+1})^2 \mu(\tilde{B}_{k,j}). \]

Thus we have

\[ \sum_{k,j} \lambda_{k,j}^p = \sum_{k,j} \mu(\tilde{B}_{k,j})^{1-p/2} \left( \int_{\Delta_{k,j}} |f(y,t)|^2 \frac{d\mu(y)dt}{t} \right)^{p/2} \]

\[ \leq C \sum_{k,j} 2^{pk} \mu(\tilde{B}_{k,j})^{1-p/2} \mu(\tilde{B}_{k,j})^{p/2} \]

\[ \leq C \sum_{k,j} 2^{pk} \mu(B_{k,j}) \] (by the doubling property)

\[ \leq C \sum_k 2^{pk} \mu(O_k) \] (by Lemma 4)

\[ \leq C \sum_k 2^{pk} \mu(O_k) \] (by Lemma 2)

\[ \leq C \|S_{\Omega,a}(f)\|_{L^p(d\mu)}^p. \]

**Case 2.** \( O_k^* = X \) for some integer \( k \). Since \( \|S_{\Omega,a}(f)\|_{L^p(d\mu)} < \infty \), there is an integer \( n \) so that \( O_k^* = X \) for \( k \leq n \), and \( O_k^* \neq X \) for \( k > n \).

For \( k = n \), let

\[ \Delta_n = (X \times (0, \infty)) \setminus O_{n+1}^* \]

\[ \lambda_n = \mu(X)^{-1/2+1/p} \left( \int_{\Delta_n} |f(y,t)|^2 \frac{d\mu(y)dt}{t} \right)^{1/2} \]

and

\[ a_n(y,t) = \mu(X)^{-1/p+1/2} |f(y,t)| \chi_{\Delta_n}(y,t) \left( \int_{\Delta_n} |f(y,t)|^2 \frac{d\mu(y)dt}{t} \right)^{-1/2}. \]

Then \( a_n \) is an \( (\Omega, p) \)-atom since \( |f(y,t)| \leq 2^{n+1} \) in \( (X \times (0, \infty)) \setminus O_{n+1}^* \).

For \( k > n \), define \( \chi_{k,j}, \lambda_{k,j}, \) and \( a_{k,j} \) as before. Then we have

\[ |f(y,t)| \leq |f(y,t)| \chi_{\Delta_n}(y,t) + \sum_{k>n,j} |f(y,t)| \chi_{k,j}(y,t) \]

\[ = \lambda_n a_n(y,t) + \sum_{k>n,j} \lambda_{k,j} a_{k,j}(y,t). \]
Finally we have
\[ \lambda_n^p = \mu(X)^{-p/2+1} \left( \int_{\Delta_n} |f(y,t)|^2 \frac{d\mu(y)dt}{t^{p+1}} \right)^{p/2} \]
\[ \leq C \mu(X)^{-p/2+1} \left( \int_{O_{n+1}} \int_{\Omega_x} |f(y,t)|^2 \frac{d\mu(y)dt}{t^{p+1}} d\mu(x) \right)^{p/2} \]
\[ \leq C \mu(X)^{-p/2+1} \left( \int_{cO_{n+1}} (S_{\Omega,\alpha}(f)(x))^2 d\mu(x) \right)^{p/2} \]
\[ \leq C \mu(\Omega) \quad \text{(by Lemma 2)} \]
\[ \leq C \|S_{\Omega,\alpha}(f)\|_{L^p(d\mu)}^p \quad \text{(by the Chebycheff’s inequality).} \]

Thus, for \( k > n \), we have as before
\[ \sum_{k,j} \lambda_{k,j}^p \leq C \|S_{\Omega,\alpha}(f)\|_{L^p(d\mu)}^p, \]
and the proof is complete. \( \square \)

References


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