STRONGLY C-CONVEX FUNCTIONS AND ALMOST C-CONVEX FUNCTIONS ON PRECONVEXITY SPACES

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Abstract. In this paper, we introduce the concepts of strongly c-convex (strongly c-concave) function, almost c-convex function on preconvex spaces. We investigate the relationships between such concepts and several types of preconvex sets.

1. Introduction

In [1], Guay introduced the concept of preconvexity spaces defined by a binary relation on the power set $P(X)$ of a set $X$ and investigated some properties. He showed that a preconvexity on a set yields a convexity space in the same manner as a proximity [3] yields a topological space.

The purpose of this paper is to introduce the concepts of strongly c-convex function and almost c-convex function on preconvex spaces. We show that every strongly c-convex function is c-convex and every c-convex function is almost c-convex. In particular, we investigate the relationships between such concepts and several types of preconvex sets.

Definition 1.1 ([1]). Let $X$ be a nonempty set. A binary relation $\sigma$ on $P(X)$ is called a preconvexity on $X$ if the relation satisfies the following properties; we write $x\sigma A$ for $\{x\}\sigma A$:

1. If $A \subseteq B$, then $A\sigma B$.
2. If $A\sigma B$ and $B = \emptyset$, then $A = \emptyset$.
3. If $A\sigma B$ and $b\sigma C$ for all $b \in B$, then $A\sigma C$.
4. If $A\sigma B$ and $x \in A$, then $x\sigma B$.

The pair $(X, \sigma)$ is called a preconvexity space.

In a preconvexity space $(X, \sigma)$, $G(A) = \{x \in X : x\sigma A\}$ is called the convexity hull of a subset $A$. $A$ is called convex [1] if $G(A) = A$. 

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Let \((A, \sigma)\) be a preconvexity space and \(A \subseteq X\). Then \(A\) is a co-convex set iff \(A^c\) is a convex set.

**Theorem 1.3 ([2]).** Let \((X, \sigma)\) be a preconvexity space and \(A \subseteq X\). Thus from Theorem 1.3(1), we have

**Theorem 2.2** Let \((X, \sigma)\) and \((Y, \mu)\) be two preconvexity spaces. A function \(f : (X, \sigma) \rightarrow (Y, \mu)\) is said to be strongly \(c\)-convex if \(A \sigma B\) implies \(f(A) \mu f(B)\).

**Remark 2.2.** Let \((X, \sigma)\) and \((Y, \mu)\) be two preconvexity spaces. A function \(f : X \rightarrow Y\) is said to be \(c\)-convex [1] if \(A \sigma B\) implies \(f(A) \mu f(B)\).

**Lemma 2.3 ([3]).** Let \((X, \sigma)\) be a preconvexity space. For all \(A \subseteq X\), \(G(A) \sigma A\).

**Theorem 2.4.** Let \(f : X \rightarrow Y\) be a function on two preconvexity spaces \((X, \sigma)\) and \((Y, \mu)\). Then the following things are equivalent:

1. \(f\) is strongly \(c\)-convex.
2. \(f(G(A)) \subset G_\mu(I_\mu(f(A)))\) for all \(A \subseteq X\).
3. \(G_\sigma(f^{-1}(B)) \subset f^{-1}(G_\mu(I_\mu(B)))\) for all \(B \subseteq Y\).
4. \(f^{-1}(G_\mu(I_\mu(B))) \subset G_\sigma(f^{-1}(B))\) for all \(B \subseteq Y\).

**Proof.** (1) \(\Rightarrow\) (2) For each \(A \subset X\), since \(G(A) \sigma A\) and \(f\) is strongly \(c\)-convex, we have that \(f(G(A)) \mu I_\mu(f(A))\). Thus from Theorem 1.3(1), it follows \(f(G(A)) \subset G_\mu(I_\mu(f(A)))\).

(2) \(\Rightarrow\) (1) Let \(A \sigma B\) for \(A, B \subset X\). Then \(A \subset G_\sigma(A) \subset G_\sigma(B)\). From (2), it follows \(f(A) \subset f(G_\sigma(A)) \subset f(G_\sigma(B)) \subset G_\mu(I_\mu(f(B)))\). So \(f(A) \subset G_\mu(I_\mu(f(B)))\). By Theorem 1.3(1), we have \(f(A) \mu I_\mu(f(B))\).

(2) \(\Rightarrow\) (3) For \(B \subset Y\), by (2), \(f(G_\sigma(f^{-1}(B))) \subset G_\mu(I_\mu(f(f^{-1}(B)))) \subset G_\mu(I_\mu(B))\). Thus \(G_\sigma(f^{-1}(B)) \subset f^{-1}(G_\mu(I_\mu(B)))\).
(3) ⇒ (2) For \( A \subset X \), \( G_\sigma(A) \subset G_\sigma(f^{-1}(f(A))) \subset f^{-1}(G_\mu(I_\mu(f(A)))) \).
This implies \( f(A) \subset G_\mu(I_\mu(f(B))) \).

(3) ⇒ (4) For \( B \subset Y \), by (3), \( G_\sigma(f^{-1}(Y-B)) \subset f^{-1}(G_\mu(I_\mu(Y-B))) \). From Theorem 2.2, it follows \( G_\sigma(X-f^{-1}(B)) \subset f^{-1}(Y-I_\mu(G_\mu(B))) = X - f^{-1}(I_\mu(G_\mu(B))) \).
Thus \( f^{-1}(I_\mu(G_\mu(B))) \subset X - G_\sigma(X - f^{-1}(B)) = I_\sigma(f^{-1}(B)) \).
(4) ⇒ (3) It is similar to the proof of (3) ⇒ (4) \( \square \)

Let \( (X, \sigma) \) be a preconvexity space and \( A \subset X \). \( A \) is said to be \( p \)-preconvex [4] (resp., \( \alpha \)-preconvex) [6] if \( A \subset I_\sigma(G_\sigma(A)) \) (resp., \( A \subset I_\sigma(G_\sigma(I_\sigma(A))) \)). And \( A \) is said to be \( \text{cop-preconvex} \) (resp., \( \text{coa-preconvex} \) if the complement of \( A \) is a \( p \)-preconvex (resp., \( \alpha \)-preconvex) set.

**Lemma 2.5.** Let \( (X, \sigma) \) be a preconvexity space and \( A \subset X \).
(1) \( A \) is \( \text{cop-preconvex} \) iff \( G_\sigma(I_\sigma(A)) \subset A \) [4].
(2) \( A \) is \( \text{coa-preconvex} \) iff \( G_\sigma(I_\sigma(G_\sigma(A))) \subset A \) [6].

**Theorem 2.6.** Let \( f : X \to Y \) be a function on two preconvexity spaces \( (X, \sigma) \) and \( (Y, \mu) \). If \( f \) is strongly c-convex, then \( f^{-1}(B) \) is convex for every \( \text{cop-preconvex} \) set \( B \) in \( Y \).

**Proof.** Let \( B \) be a \( \text{cop-preconvex} \) set in \( Y \), since \( f \) is strongly c-convex, by Lemma 2.5, \( G_\sigma(f^{-1}(B)) \subset f^{-1}(G_\mu(I_\mu(B))) \subset f^{-1}(B) \). It implies \( f^{-1}(B) \) is convex. \( \square \)

**Theorem 2.7.** Let \( f : X \to Y \) be a function on two preconvexity spaces \( (X, \sigma) \) and \( (Y, \mu) \). If \( f \) is strongly c-convex, then \( f^{-1}(B) \) is convex for every \( \text{coa-preconvex} \) set \( B \) in \( Y \).

**Proof.** Let \( B \) be a \( \text{coa-preconvex} \) set in \( Y \). Then by Lemma 2.5, \( G_\sigma(f^{-1}(B)) \subset f^{-1}(G_\mu(I_\mu(B))) \subset f^{-1}(G_\mu(I_\mu(G_\mu(B)))) \subset f^{-1}(B) \). Hence \( f^{-1}(B) \) is convex. \( \square \)

**Definition 2.8.** Let \( (X, \sigma) \) and \( (Y, \mu) \) be two preconvexity spaces. A function \( f : X \to Y \) is said to be \( \text{strongly c-concave} \) if for \( C, D \subset Y \) whenever \( C \mu D, f^{-1}(C) \sigma I_\sigma(f^{-1}(D)) \).

**Remark 2.9.** Let \( (X, \sigma) \) and \( (Y, \mu) \) be two preconvexity spaces. A function \( f : X \to Y \) is said to be \( \text{c-concave} \) [3] if for \( C, D \subset Y \) whenever \( C \mu D, f^{-1}(C) \sigma f^{-1}(D) \). Obviously it is that every \( \text{strongly c-concave} \) function is \( \text{c-concave} \).

**Theorem 2.10.** Let \( f : X \to Y \) be a function on two preconvexities \( (X, \sigma) \) and \( (Y, \mu) \). Then the following things are equivalent:
(1) \( f \) is \( \text{strongly c-concave} \).
(2) \( f^{-1}(G_{\mu}(A)) \subset G_{\sigma}(I_{\sigma}(f^{-1}(A))) \) for all \( A \subset Y \).

(3) \( I_{\sigma}(G_{\sigma}(f^{-1}(A))) \subset f^{-1}(I_{\mu}(A)) \) for all \( A \subset Y \).

**Proof.** (1) \( \Rightarrow \) (2) Let \( f \) be strongly \( c \)-concave and \( A \subset Y \). Since \( G_{\mu}(A) \mu A \) and \( f \) is strongly \( c \)-concave, \( f^{-1}(G_{\mu}(A)) \sigma I_{\sigma}(f^{-1}(A)) \). Thus by Theorem 1.3, \( f^{-1}(G_{\mu}(A)) \subset G_{\sigma}(I_{\sigma}(f^{-1}(A))) \).

(2) \( \Rightarrow \) (1) If \( C_{\mu}D \) for \( C, D \subset Y \), then \( C \subset G_{\mu}(D) \). By hypothesis, \( f^{-1}(C) \subset f^{-1}(G_{\mu}(D)) \subset G_{\sigma}I_{\sigma}(f^{-1}(D)) \). So \( f^{-1}(C) \sigma I_{\sigma}(f^{-1}(D)) \).

(2) \( \Rightarrow \) (3) For \( A \subset Y \), from Theorem 1.3 and the condition (2), it follows \( X - f^{-1}(I_{\mu}(A)) \subset f^{-1}(G_{\mu}(Y - A)) \subset G_{\sigma}(I_{\sigma}(f^{-1}(Y - A))) = X - I_{\sigma}(G_{\sigma}(f^{-1}(A))) \). So \( I_{\sigma}(G_{\sigma}(f^{-1}(A))) \subset f^{-1}(I_{\mu}(A)) \).

Similarly, it is obtained that (3) \( \Rightarrow \) (2). \( \square \)

**Definition 2.11.** Let \((X, \sigma)\) be a preconvexity space and \( A \subset X \). \( A \) is called a regular-preconvex set (briefly, \( r \)-preconvex set) if \( A = I_{\sigma}(G_{\sigma}(A)) \). And \( A \) is called a coregular-preconvex set (briefly, \( \sigma \)-preconvex set) if the complement of \( A \) is a \( r \)-preconvex set.

**Definition 2.12.** Let \((X, \sigma)\) and \((Y, \mu)\) be two preconvexity spaces. A function \( f : X \to Y \) is said to be almost \( c \)-convex if \( f^{-1}(A) \) is convex for every cor-preconvex set \( A \) of \( X \).

**Theorem 2.13.** Let \( f : X \to Y \) be a function on two preconvexity spaces \((X, \sigma)\) and \((Y, \mu)\). Then the following things are equivalent:

(1) \( f \) is almost \( c \)-convex.

(2) \( G_{\sigma}(f^{-1}(G_{\mu}(I_{\mu}(F)))) \subset f^{-1}(F) \) for every convex set \( F \subset Y \).

(3) \( G_{\sigma}(f^{-1}(G_{\mu}(I_{\mu}(B)))) \subset f^{-1}(G_{\mu}(B)) \) for every \( B \subset Y \).

**Proof.** (1) \( \Rightarrow \) (2) For a convex subset \( F \subset Y \), then \( G_{\mu}(I_{\mu}(G_{\mu}(I_{\mu}(F)))) = G_{\mu}(I_{\mu}(F)) \), that is, \( G_{\mu}(I_{\mu}(F)) \) is cor-preconvex. By (1) and \( G_{\mu}(I_{\mu}(F)) \subset F \), \( G_{\sigma}(f^{-1}(G_{\mu}(I_{\mu}(F)))) = f^{-1}(G_{\mu}(I_{\mu}(F))) \subset f^{-1}(F) \).

(2) \( \Rightarrow \) (3) Obvious.

(3) \( \Rightarrow \) (1) Let \( A \) be cor-preconvex in \( Y \). Since \( G_{\mu}(I_{\mu}(G_{\mu}(A))) = A \) and \( G_{\mu}(A) = A \), by (3), we have \( G_{\sigma}(f^{-1}(B)) \subset f^{-1}(B) \), and \( f^{-1}(B) \) is convex. Hence \( f \) is almost \( c \)-convex. \( \square \)

Easily we have the following:

**Theorem 2.14.** Let \( f : X \to Y \) be a function on two preconvexity spaces \((X, \sigma)\) and \((Y, \mu)\). Then the following things are equivalent:

(1) \( f \) is almost \( c \)-convex.

(2) \( f^{-1}(A) \) is co-convex for every \( r \)-preconvex set \( A \) of \( X \).

(3) \( f^{-1}(U) \subset I_{\mu}(f^{-1}(I_{\mu}(G_{\mu}(U)))) \) for every co-convex set \( U \subset Y \).
Theorem 2.15. Let $f : X \to Y$ be a function on two preconvexity spaces $(X, \sigma)$ and $(Y, \mu)$. Then if $f$ is strongly c-convex, then it is almost c-convex.

Proof. Let $A$ be co-$\sigma$-preconvex in $Y$. Then $G_\mu(I_\mu(A)) = A$. From Theorem 2.4 (3), it follows $G_\sigma(f^{-1}(A)) \subset f^{-1}(G_\mu(I_\mu(A))) = f^{-1}(A)$. It implies $f^{-1}(A)$ is convex, and hence $f$ is almost c-convex. \hfill $\square$

Let $(X, \sigma)$ be a preconvexity space and $A \subset X$. $A$ is said to be semi-preconvex [5] (resp., $\beta$-preconvex) [7] if $A \sigma I_\sigma(A)$ (resp., $A \sigma I_\sigma(G_\sigma(A))$).

Remark 2.16. For a function $f : X \to Y$ on two preconvexity spaces $(X, \sigma)$ and $(Y, \mu)$, from Theorem 2.4 and Theorem 2.10, the following things are obtained:

1. If $f$ is strongly c-convex, then for every convex set $A$ of $X$, $f(A)$ is semi-preconvex ($\beta$-preconvex) set $B$ in $Y$.
2. If $f$ is strongly c-concave, then for every convex set $B$ of $Y$, $f^{-1}(B)$ is semi-preconvex ($\beta$-preconvex) set $B$ in $Y$.
3. If $f$ is strongly c-convex, then for every co-$\sigma$-preconvex set $B$ of $Y$, $f^{-1}(B)$ is convex.

References

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