ON HILBERT 2-CLASS FIELD TOWERS
OF REAL QUADRATIC FUNCTION FIELDS

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Abstract. In this paper we prove that real quadratic function field $F$ over $\mathbb{F}_q(T)$ has infinite 2-class field tower if the 4-rank of narrow ideal class group of $F$ is equal to or greater than 4 when $q \equiv 3$ mod 4.

1. Introduction and statement of main result

Let $k = \mathbb{F}_q(T)$ be the rational function field over the finite field $\mathbb{F}_q$ of $q$ elements and $\mathcal{A} = \mathbb{F}_q[T]$. Let $\infty$ be the prime of $k$ associated to $(1/T)$. For a finite separable extension $F$ of $k$, let $\mathcal{O}_F$ be the integral closure of $\mathcal{A}$ in $F$ and $H_F$ be the Hilbert class field of $F$ with respect to $\mathcal{O}_F$ ([5]). Let $\ell$ be a prime number. Let $F_1^{(\ell)}$ be the Hilbert $\ell$-class field of $F_0 = F$ (i.e., $F_1^{(\ell)}$ is the maximal $\ell$-extension of $F$ inside $H_F$) and inductively, $F_{n+1}^{(\ell)}$ be the Hilbert $\ell$-class field of $F_n^{(\ell)}$ for $n \geq 1$. Then we obtain a sequence of fields $F_0^{(\ell)} = F \subset F_1^{(\ell)} \subset \cdots \subset F_n^{(\ell)} \subset \cdots$, which is called the Hilbert $\ell$-class field tower of $F$. We say that the Hilbert $\ell$-class field tower of $F$ is infinite if $F_n^{(\ell)} \neq F_{n+1}^{(\ell)}$ for each $n \geq 0$. For any multiplicative abelian group $A$, write $r_\ell(A) = \dim_{\mathbb{F}_\ell}(A/A^\ell)$, which is called the the $\ell$-rank of $A$. In [6], Schoof has proved that the Hilbert $\ell$-class field tower of $F$ is infinite if

$$r_\ell(\mathcal{C}(F)) \geq 2 + 2\sqrt{r_\ell(\mathcal{O}_F^*) + 1},$$

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where \( \mathcal{C}(F) \) and \( \mathcal{O}_F^* \) are the ideal class group and the group of units of \( \mathcal{O}_F \), respectively. This is a function field analog of the theorem of Golod-Shafarevich.

Assume that \( q \) is odd. Throughout the paper, by a \textit{real quadratic function field}, we always mean a quadratic extension \( F \) of \( k \) in which \( \infty \) splits. Any real quadratic function field \( F \) can be written uniquely as \( F = k(\sqrt{D}) \), where \( D \in \mathbb{A} \) is a nonconstant square-free monic polynomial of even degree. In this paper, we study the infiniteness of Hilbert 2-class field tower of such a real quadratic function field \( F \). Since \( \mathcal{O}_F^* \cong \mathbb{F}_q^* \times \mathbb{Z} \), \( r_2(\mathcal{O}_F^*) = 2 \), so the Hilbert 2-class field tower of \( F \) is infinite if \( r_2(\mathcal{C}(F)) \geq 6 \) by Schoof’s theorem. Let \( \mathcal{C}^+(F) \) be the narrow ideal class group of \( \mathcal{O}_F \) (cf. §2.1). Write \( r_4(\mathcal{C}^+(F)) = r_2(\mathcal{C}^+(F)^2) \), which is called the 4-rank of \( \mathcal{C}^+(F) \). The main result of this paper is the following theorem.

\textbf{Theorem 1.1.} Assume that \( q \equiv 3 \mod 4 \). Let \( F \) be a real quadratic function field over \( k \). If \( r_4(\mathcal{C}^+(F)) \geq 4 \), then \( F \) has infinite Hilbert 2-class field tower.

In classical case, Lemmermeyer [3] has proved a similar result for the real quadratic number field \( F \). Our method is elementary since we only use the Rédei matrix \( M_F^+ \) associated to \( F \) and this method also works for real quadratic number field case.

\section{Preliminaries}

\subsection{Narrow ideal class group \( \mathcal{C}^+(F) \)}

Let \( k_\infty = \mathbb{F}_q((1/T)) \) be the completion of \( k \) at \( \infty \). Let \( \text{sgn} : k_\infty^* \to \mathbb{F}_q^* \) be the sign function satisfying \( \text{sgn}(1/T) = 1 \) and define \( s(x) = \text{sgn}(x)^{\frac{x}{2+\infty}} \) for any \( x \in k_\infty^* \). Let \( F \) be a real quadratic function field over \( k \). Let \( \infty_1 \) and \( \infty_2 \) be primes of \( F \) lying above \( \infty \). Define a homomorphism

\[ s : F^* \to \{ \pm 1 \} \times \{ \pm 1 \}, \quad x \mapsto (s_1(x), s_2(x)), \]

where \( s_i(x) = s(\eta_i(x)) \) and \( \eta_i \) is the embedding of \( F \) into \( k_\infty \) associated to \( \infty_i \) for \( i = 1, 2 \). An element \( x \in F^* \) is said to be \textit{positive} if \( s(x) = (1, 1) \). Put \( F^+ = \text{Ker}(s) \), which is the subgroup of \( F^* \) consisting of all positive elements of \( F^* \). Let \( I(F) \) be the group of fractional ideals of \( \mathcal{O}_F \) and \( P^+(F) \) be the subgroup of \( I(F) \) consisting of principal ideals generated by an element of \( F^+ \). The narrow ideal class group \( \mathcal{C}^+(F) \) of \( \mathcal{O}_F \) is defined as \( \mathcal{C}^+(F) = I(F)/P^+(F) \).
2.2. 4-rank of $\mathcal{C}^+(F)$ and Rédei matrix $M_F^+$

Consider a real quadratic function field $F = k(\sqrt{D})$ with $D = P_1 \cdots P_t$, where $P_i$ is a monic irreducible polynomial in $\mathbb{A}$ for $1 \leq i \leq t$. By genus theory, $r_2(\mathcal{C}^+(F)) = t - 1$. Let $s = s(D)$ denote the number of the $P_i$ with odd degree. Since $\deg(D)$ is even, $s$ is even. From now on we always assume that $2 \nmid \deg(P_i)$ for $1 \leq i \leq s$ and $2|\deg(P_i)$ for $s+1 \leq i \leq t$. For $1 \leq i \neq j \leq t$, let $e_{ij} \in \mathbb{F}_2$ be defined by $(-1)^{ij} = (\ov{P_i} \ov{P_j})$, where $\ov{P_i} = (-1)^{\deg(P_i)}P_i$ and $e_{ii}$ is defined to satisfy $\sum_{i=1}^t e_{ij} = 0$. Let $M_F' = (e_{ij})_{1 \leq i,j \leq t}$. We associate a matrix $M_F^*$ to $F$ defined as follows: If there is an ideal $\mathfrak{a}$ of $F$ such that $a^{1-s} = \alpha \mathcal{O}_F$ with $N_{F/k}(\alpha) = (A)$ and $e_{iA} \in \mathbb{F}_2$ is defined to satisfy $(-1)^{e_{iA}} = (\ov{P_i}^2)$, and $M_F^* = M_F'$ otherwise. We remark that if $q \equiv 3 \mod 4$, we always have $M_F^{+} = M_F'$. Then $r_4(\mathcal{C}^+(F))$ satisfies the following equality ($[1]$)

$$r_4(\mathcal{C}^+(F)) = t - 1 - \text{rank}(M_F^+)$$

2.3. Martinet’s inequality

Let $E$ and $K$ be finite (geometric) separable extensions of $k$ such that $E/K$ is a cyclic extension of degree $\ell$ with $\Delta = \text{Gal}(E/K)$, where $\ell$ is a prime number not dividing $q$. Then $H^0(\Delta, \mathcal{O}_E^*)$ and $H^1(\Delta, \mathcal{O}_E^*)$ are elementary abelian $\ell$-groups with

$$\frac{|H^0(\Delta, \mathcal{O}_E^*)|}{|H^1(\Delta, \mathcal{O}_E^*)|} = \ell^{-1} \prod_{p_{\infty} \in S_{\infty}(K)} |\Delta_{p_{\infty}}|,$$

where $S_{\infty}(K)$ is the set of primes of $K$ lying above $\infty$ and $\Delta_{p_{\infty}}$ denotes the decomposition group of $p_{\infty}$ in $\Delta$. Note that $\Delta_{p_{\infty}} = \Delta$ if $p_{\infty}$ ramifies or inert in $E$ and $\Delta_{p_{\infty}} = \{1\}$ otherwise. Following the arguments in [4, §2], we get the following.

**Proposition 2.1.** Let $E$ and $K$ be finite (geometric) separable extensions of $k$ such that $E/K$ is a cyclic extension of degree $\ell$, where $\ell$ is a prime number not dividing $q$. Let $\gamma_{E/K}$ be the number of prime ideals of $\mathcal{O}_K$ that ramify in $E$ and $\rho_{E/K}$ be the number of primes $p_{\infty}$ in $S_{\infty}(K)$ that ramify or inert in $E$. If $\gamma_{E/K}$ satisfies the inequality

$$\gamma_{E/K} \geq |S_\infty(K)| - \rho_{E/K} + 3 + 2\sqrt{\ell |S_\infty(K)|} + (1 - \ell)\rho_{E/K} + 1,$$

then

$$\gamma_{E/K} \geq |S_\infty(K)| - \rho_{E/K} + 3 + 2\sqrt{\ell |S_\infty(K)|} + (1 - \ell)\rho_{E/K} + 1,$$
then the Hilbert ℓ-class field tower of \( E \) is infinite.

The inequality (2.2) is called the Martinet’s inequality for \( E/K \). Let \( F \) be a real quadratic function field over \( k \). We remark that if there exists an extension \( E \) of \( F \) which has infinite Hilbert 2-class field tower and \( F \subset E \subset F^{(2)}_1 \), then \( F \) also has infinite Hilbert 2-class field tower.

**Corollary 2.2.** Let \( F = k(\sqrt{D}) \) be a real quadratic function field over \( k \). If \( D \) has a nonconstant monic divisor \( D_1 \) of even degree satisfying \((\frac{D_1}{Q_j}) = 1\) for monic irreducible divisors \( Q_j \) \((1 \leq j \leq 5)\) of \( D \), then \( F \) has infinite Hilbert 2-class field tower.

**Proof.** Put \( K = k(\sqrt{D_1}) \), which is a real quadratic extension of \( k \) in which \( Q_1, Q_2, Q_3, Q_4 \) and \( Q_5 \) split. Let \( E = KF \). Applying Proposition 2.1 on \( E/K \) with \( \gamma_{E/K} = 10 \) and \((|S_{\infty}(K)|, \rho_{E/K}) = (2, 0)\), we see that \( E \) has infinite Hilbert 2-class field tower, so \( F \) also has infinite Hilbert 2-class field tower.

**Corollary 2.3.** Let \( F = k(\sqrt{D}) \) be a real quadratic function field over \( k \). If \( D \) has a two distinct nonconstant monic divisors \( D_1 \) and \( D_2 \) of even degrees satisfying \((\frac{D_1}{Q_j}) = (\frac{D_2}{Q_j}) = 1\) for monic irreducible divisors \( Q_j \) \((1 \leq j \leq 4)\) of \( D \), then \( F \) has infinite Hilbert 2-class field tower.

**Proof.** Put \( K = k(\sqrt{D_1}, \sqrt{D_2}) \), which is a real biquadratic extension of \( k \) in which \( Q_1, Q_2, Q_3 \) and \( Q_4 \) split completely. Let \( E = KF \). Applying Proposition 2.1 on \( E/K \) with \( \gamma_{E/K} \geq 16 \) and \((|S_{\infty}(K)|, \rho_{E/K}) = (4, 0)\), we see that \( E \) has infinite Hilbert 2-class field tower, so \( F \) also has infinite Hilbert 2-class field tower.

**Corollary 2.4.** Let \( F = k(\sqrt{D}) \) be a real quadratic function field over \( k \). If \( D \) has a two distinct nonconstant monic divisors \( D_1 \) and \( D_2 \) of even degrees satisfying \((\frac{D_1}{Q_j}) = (\frac{D_2}{Q_j}) = 1\) for monic irreducible divisors \( Q_j \) \((1 \leq j \leq 3)\) of \( D \) and there is a monic irreducible divisor \( Q \) of \( D \) which is different from \( Q_1, Q_2, Q_3 \) and \( Q \nmid D_1D_2 \), then \( F \) has infinite Hilbert 2-class field tower.

**Proof.** Put \( K = k(\sqrt{D_1}, \sqrt{D_2}) \), which is a real biquadratic extension of \( k \) in which \( Q_1, Q_2 \) and \( Q_3 \) split completely. Let \( E = KF \). Since \( Q \) splits in at least one quadratic subfield of \( K \), we have \( \gamma_{E/K} \geq 14 \). Applying Proposition 2.1 on \( E/K \) with \( \gamma_{E/K} \geq 14 \) and \((|S_{\infty}(K)|, \rho_{E/K}) = (4, 0)\), we see that \( E \) has infinite Hilbert 2-class field tower, so \( F \) also has infinite Hilbert 2-class field tower.
3. Proof of Theorem 1.1

Consider a real quadratic function field \( F = k(\sqrt{D}) \) with \( D = P_1 \cdots P_t \), where \( P_i \) is a monic irreducible polynomial in \( \mathbb{A} \) for \( 1 \leq i \leq t \). Recall that \( s = s(D) \) is the number of the \( P_i \) with odd degree and we assume that \( 2 \nmid \deg(P_i) \) for \( 1 \leq i \leq s \) and \( 2 | \deg(P_i) \) for \( s + 1 \leq i \leq t \). Assume that \( q \equiv 3 \mod 4 \). In this section, we are going to prove the infiniteness of Hilbert 2-class field tower of \( F \) under the condition \( r_2(C(F)) \geq 4 \). Note that \( r_2(C(F)) = t - 1 \) if \( s = 0 \) and \( t - 2 \) if \( s \geq 2 \). Hence, if \( r_2(C(F)) \geq 6 \) when \( s = 0 \) or \( r_2(C(F)) \geq 7 \) when \( s \geq 2 \), then \( r_2(C(F)) \geq 6 \), so \( F \) has infinite Hilbert 2-class field tower. If \( r_2(C(F)) = r_4(C(F)) \), then rank \( M^+_F = 0 \), i.e., \( M^+_F = 0 \), so \( e_{12} = e_{21} \). Thus the case \( r_2(C(F)) = r_4(C(F)) \) with \( s \geq 2 \) can’t occur by the quadratic reciprocity law. Thus we only need to consider the cases

\[
(r_2(C(F)), r_4(C(F))) = \begin{cases} 
(4, 4), (5, 4), (5, 5) & \text{if } s = 0, \\
(5, 4), (6, 4), (6, 5) & \text{if } s \geq 2.
\end{cases}
\]

Let \( r_i(F) \) denote the \( i \)-th row of \( M^+_F \) and \( 0 \) denote the zero one in \( \mathbb{F}_2^t \).

- **Case** \((r_2(C(F)), r_4(C(F))) = (4, 4)\) with \( D = P_1P_2P_3P_4P_5 \) and \( s = 0 \). Since \( M^+_F = O \), \((\frac{P_i}{F}) = (\frac{P_6}{F}) = 1\) for \( 3 \leq i \leq 5 \), so \( P_3, P_4 \) and \( P_5 \) split completely in \( K = k(\sqrt{P_1}, \sqrt{P_2}) \). Let \( E = KF \). Since \( F^*_q = F^*_q \cap N_{E/K}(F^*) \), \( F^*_q \) is contained in \( O^*_K \cap N_{E/K}(E^*) \) and so \((O^*_K : O^*_K \cap N_{E/K}(E^*)) \leq 2^3 \). Since \((\frac{P_3}{F}) = 1\), the ideal class number \( h(O_K) \) of \( O_K \) is even. Since \( \gamma_{E/K} = 12 \), by the ambiguous class number formula ([2, Lemma 2.2]), \( r_2(C(E)) \geq 9 \). Since \( r_2(O^*_E) = 8 \), by Schoof’s theorem, \( E \) has infinite Hilbert 2-class field tower. Since \( F \subset E \subset F^{(2)}_1 \), \( F \) also has infinite Hilbert 2-class field tower.

- **Case** \((r_2(C(F)), r_4(C(F))) = (5, 5)\) with \( D = P_1P_2P_3P_4P_5P_6 \) and \( s = 0 \). Since \( M^+_F = 0 \), \((\frac{P_i}{F}) = 1\) for \( 1 \leq i \leq 5 \), so \( F \) has infinite Hilbert 2-class field tower by Corollary 2.2.

- **Case** \((r_2(C(F)), r_4(C(F))) = (5, 4)\) with \( D = P_1P_2P_3P_4P_5P_6 \) and \( s \in \{0, 2, 4, 6\} \). In this case rank \( M^+_F = 1 \), so at least one row of \( M^+_F \) is nonzero and the other ones are multiple of this row. Assume first \( s = 0 \). Since at least two rows of \( M^+_F \) are equal, we may assume \( r_5(F) = r_6(F) \). Then \( e_{5j} = e_{6j} \) for \( 1 \leq j \leq 4 \), so \( P_1, P_2, P_3 \) and \( P_4 \) split in \( K = k(\sqrt{P_5}, \sqrt{P_6}) \). Let \( E = KF \). Since \( F^*_q = F^*_q \cap N_{E/K}(F^*) \), \( F^*_q \) is contained in \( O^*_K \cap N_{E/K}(E^*) \) and so \((O^*_K : O^*_K \cap N_{E/K}(E^*)) \leq 2 \). Since \( \gamma_{E/K} = 8 \) and \( r_2(C(K)) = 1 \), by the ambiguous class number formula, \( r_2(C(E)) \geq 7 \). Since \( r_2(O^*_E) = 4 \), by (1.1), \( E \) has infinite Hilbert 2-class
field tower. Since $F \subset E \subset F_1^{(2)}$, $F$ also has infinite Hilbert 2-class field tower.

Assume that $s = 2$ or 4. If $r_i(F) = 0$ for some $s + 1 \leq i \leq 6$, say $r_6(F) = 0$, then $\left( \frac{P_{ij}}{P_{ij}} \right) = 1$ for $1 \leq j \leq 5$, so $F$ has infinite Hilbert 2-class field tower by Corollary 2.2. Now we may assume that $r_i(F) \neq 0$ for $s + 1 \leq i \leq 6$, so they are all equal. If $r_i(F) = r_j(F) = 0$ for some $1 \leq i \neq j \leq s$, say $r_1(F) = r_2(F) = 0$, then $e_{12} = e_{21} = 0$ which is a contradiction. Thus at most one row of $M_F^+$ is zero. Then all rows of $M_F^+$ are nonzero and they are all equal. If $e_{12} = 1$, then all rows of $M_F^+$ are $(1 \ 0 \ 1 \ 1 \ 1)$, but then $e_{26} = 1 \neq 0 = e_{62}$ which is a contradiction. If $e_{12} = 0$, then all rows of $M_F^+$ are $(0 \ 1 \ 0 \ 0 \ 0)$, but then $e_{26} = 0 \neq 1 = e_{62}$ which is a contradiction.

Consider the case $s = 6$. If $r_i(F) = r_j(F) = 0$ for some $1 \leq i \neq j \leq 6$, say $r_1(F) = r_2(F) = 0$, then $e_{12} = e_{21} = 0$ which is a contradiction. Thus at most one row of $M_F^+$ is zero. Then all rows of $M_F^+$ are nonzero and they are all equal, so we can get a contradiction as above. Thus this case can’t occur.

- Case $(r_2(C^+(F)), r_4(C^+(F))) = (6, 4)$ with $D = P_1 \cdots P_7$ and $s \in \{2, 4, 6\}$. In this case, rank $M_F^+ = 2$, so two rows of $M_F^+$ are independent over $\mathbb{F}_2$ and the others are $\mathbb{F}_2$-linear combinations of these two rows. If $r_7(F) = 0$, then $\left( \frac{P_{ij}}{P_{ij}} \right) = 1$ for $1 \leq j \leq 6$, so $F$ has infinite Hilbert 2-class field tower by Corollary 2.2. Thus we may assume $r_7(F) \neq 0$. Consider first the case $s = 2$. At least two of $r_3(F), r_4(F), r_5(F), r_6(F)$ and $r_7(F)$ are equal, say $r_6(F) = r_7(F)$, then $\left( \frac{P_{ij}}{P_{ij}} \right) = 1$ for $1 \leq j \leq 5$, so $F$ has infinite Hilbert 2-class field tower by Corollary 2.2.

Assume $s = 4$. If two of $r_1(F), r_2(F), r_3(F)$ and $r_4(F)$ are equal, say $r_1(F) = r_2(F)$, then $\left( \frac{P_{ij}}{P_{ij}} \right) = 1$ for $3 \leq j \leq 7$, so $F$ has infinite Hilbert 2-class field tower by Corollary 2.2. Hence, we may assume that $r_1(F), r_2(F), r_3(F)$ and $r_4(F)$ are all distinct (†). If $r_i(F) = 0$ for $i = 5$ or 6, say $r_6(F) = 0$, then $\left( \frac{P_{ij}}{P_{ij}} \right) = 1$ for $1 \leq j \leq 5$, so $F$ has infinite Hilbert 2-class field tower by Corollary 2.2. If two of $r_5(F), r_6(F)$ and $r_7(F)$ are equal, say $r_6(F) = r_7(F)$, then $\left( \frac{P_{ij}}{P_{ij}} \right) = 1$ for $1 \leq j \leq 5$, so $F$ has infinite Hilbert 2-class field tower by Corollary 2.2. Thus, we may assume that $r_5(F), r_6(F)$ and $r_7(F)$ are all distinct and nonzero (‡). From (†) and (‡), without loss of generality, we may assume that $r_5(F), r_6(F)$ are linearly independent over $\mathbb{F}_2$ and $r_1(F) = 0$, $r_2(F) = r_5(F)$, $r_3(F) = r_6(F)$, $r_4(F) = r_7(F) = r_5(F) + r_6(F)$. But, since $r_1(F) = 0$, we have $e_{21} = e_{31} = e_{41} = 1$ and $e_{51} = e_{61} = e_{71} = 0$, which is a contradiction.
Assume $s = 6$. Since $r_7(F) \neq 0$, at least two rows of $M_F^+$ except the 7-th row are equal, say $r_1(F) = r_2(F)$. Then $(\frac{P_iP_j}{P_7}) = 1$ for $3 \leq j \leq 7$, so $F$ has infinite Hilbert 2-class field tower by Corollary 2.2.

• Case $(r_2(C^+(F)), r_4(C^+(F))) = (6, 5)$ with $D = P_1 \cdots P_7$ and $s \in \{2, 4, 6\}$. In this case, rank $M_F^+ = 1$, so at least one row of $M_F^+$ is nonzero and the other ones are multiple of this row. If $r_7(F) = 0$, then $(\frac{P_7}{P_j}) = 1$ for $1 \leq j \leq 6$, so $F$ has infinite Hilbert 2-class field tower by Corollary 2.2. Thus we may assume $r_7(F) \neq 0$ and the other rows of $M_F^+$ are multiple of $r_7(F)$.

Assume that $s = 2$ or 4. If $r_i(F) = 0$ for $i = 5$ or 6, say $r_6(F) = 0$, then $(\frac{P_i}{P_7}) = 1$ for $1 \leq j \leq 5$, so $F$ has infinite Hilbert 2-class field tower by Corollary 2.2. We may assume that $r_i(F) \neq 0$ for $i = 5, 6$, so $r_5(F) = r_6(F) = r_7(F)$. Then $(\frac{P_iP_k}{P_j^7}) = (\frac{P_iP_k}{P_j^5}) = 1$ for $1 \leq j \leq 4$, so $F$ has infinite Hilbert 2-class field tower by Corollary 2.4.

Consider the case $s = 6$. At least three rows of $M_F^+$ except the 7-th row are equal, say $r_4(F) = r_5(F) = r_6(F)$. Then $(\frac{P_iP_k}{P_j^5}) = (\frac{P_iP_k}{P_j^3}) = 1$ for $1 \leq j \leq 3$, so $F$ has infinite Hilbert 2-class field tower by Corollary 2.4. We complete the proof of Theorem 1.1.

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