PROPERTY $(D_k)$ IN BANACH SPACES

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Abstract. In this paper, we define property $(D_k)$ and get the following strict implications.

$$(\text{UC}) \Rightarrow (D_2) \Rightarrow (D_3) \Rightarrow \cdots \Rightarrow (D_{\infty}) \Rightarrow (\text{BS}).$$

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1. Introduction

Let $(X, \| \cdot \|)$ be a real Banach space and $X^*$ the dual space of $X$. By $B_X$, we denote the closed unit ball of $X$. For a Banach space $X$ with a usual unit basis $(e_n)$, if $x = \sum_{n=1}^{\infty} a_n e_n$, we define the support of $x$, $\text{supp}(x) = \{ n : a_n \neq 0 \}$. For $x, y \in X$, we write $x < y$ for $\max \text{supp}(x) < \min \text{supp}(y)$.

$(X, \| \cdot \|)$ is said to be uniformly convex (UC) if for all $\epsilon > 0$, there exists a $\delta > 0$ such that for $x, y \in B_X$ with $\| x - y \| \geq \epsilon$,

$$\frac{1}{2} \| x + y \| \leq 1 - \delta.$$

A Banach space is said to have Banach-Saks property (BS) if any bounded sequence in the space admits a subsequence whose arithmetic means converges in norm. S. Kakutani [4] showed that Uniform convexity implies Banach-Saks property. And T. Nishiura and D. Waterman [5] proved that Banach-Saks property implies reflexivity in Banach spaces. A Banach space $X$ is said to have weak Banach-Saks property if every weakly null sequence $(x_n)$ in $X$ admits a subsequence whose arithmetic means converges in norm. It is easy to see that Banach-Saks property implies weak Banach-Saks property. Since every bounded sequence in reflexive Banach spaces has weakly convergent subsequence, weak
Banach-Saks property is equivalent to Banach-Saks property in reflexive Banach spaces.

2. Main Results

We start with the following definition.

**Definition 1.** A Banach space $X$ is said to have property $(D_k)$, where $k \geq 2$ if it is reflexive and there exists a number $\alpha$, $0 < \alpha < 1$, such that for a weakly null sequence $(x_n)$ in $B_X$, there exist $n_1 < n_2 < \cdots < n_k$ with

$\left\| \frac{1}{k} \sum_{i=1}^{k} (-1)^{i+1} x_{n_i} \right\| < \alpha.$

It is easy to see that property $(D_k)$ implies property $(D_{k+1})$.

**Proposition 2.** If $X$ has property $(D_k)$, then it has property $(D_{k+1})$.

*Proof.* Suppose that $X$ has property $(D_k)$. Then $X$ is reflexive and there exists a number $\alpha$, $0 < \alpha < 1$, such that for a weakly null sequence $(x_n)$ in $B_X$, there exist $n_1 < n_2 < \cdots < n_k$ with

$\left\| \frac{1}{k} \sum_{i=1}^{k} (-1)^{i+1} x_{n_i} \right\| < \alpha.$

Let $n_{k+1} = n_k + 1$. Then

$\left\| \frac{1}{k+1} \sum_{i=1}^{k+1} (-1)^{i+1} x_{n_i} \right\| \leq \frac{k}{k+1} \left\| \frac{1}{k} \sum_{i=1}^{k} (-1)^{i+1} x_{n_i} \right\| + \frac{1}{k+1} \left\| x_{n_{k+1}} \right\|$

$\leq \frac{k}{k+1} \alpha + \frac{1}{k+1} = \frac{1}{k+1} (k\alpha + 1) < 1.$

Letting $\beta = \frac{1}{k+1} (k\alpha + 1)$, we get the result. \(\square\)

The following Proposition 3 can be found in [2].

**Proposition 3.** If $X$ is uniformly convex, then it has property $(D_2)$. The converse does not hold.

The following Definition 4 and Lemma 5 can be found in [1].
Definition 4. A Banach space $X$ is said to have alternate signs weak Banach-Saks property if every weakly null sequence $(x_n)$ in $X$ there exists a subsequence $(x'_{n})$ of $(x_n)$ and a sequence $(\epsilon_n)$ of $\{\pm 1\}$ such that $(1/n) \sum_{i=1}^{n} \epsilon_i x'_i$ converges in norm.

Lemma 5. A Banach space has weak Banach-Saks property if and only if it has alternate signs weak Banach-Saks property.

Banach spaces with property $(D_k)$ have alternate Banach-Saks property.

Proposition 6. If $X$ has property $(D_k)$, it has alternate signs weak Banach-Saks property.

Proof. Suppose that $X$ has property $(D_k)$. Then there exists $0 < \alpha < 1$ such that for all weakly null sequence $(x_n)$ in $B_X$, there exist $n_1 < n_2 < \cdots < n_k$ with

$$\left\| \frac{1}{k} \sum_{i=1}^{k} (-1)^{i+1} x_{n_i} \right\| < \alpha.$$ 

Suppose $(x_n)$ is a weakly null sequence in $X$. Without loss of generality, we may assume that $\|x_n\| \leq 1$. Then there exist $n_1 < n_2 < \cdots < n_k$ such that

$$\left\| \frac{1}{k} \sum_{i=1}^{k} (-1)^{i+1} x_{n_i} \right\| < \alpha.$$ 

Since $(x_n)_{n>n_k}$ is weakly null and $\|x_n\| \leq 1$ for $n > n_k$, there exist $(n_k <) n_{k+1} < n_{k+2} < \cdots < n_{2k}$ such that

$$\left\| \frac{1}{k} \sum_{i=k+1}^{2k} (-1)^{i+1} x_{n_i} \right\| < \alpha.$$ 

Continue this process, we obtain a subsequence $(x'_{n_m})$ for which given any $k \in \mathbb{N}$

$$\left\| \frac{1}{k} \sum_{i=jk+1}^{(j+1)k} (-1)^{i+1} x_{n_i} \right\| < \alpha,$$ 

for all $j \in \mathbb{N} \cup \{\emptyset\}$. Now, using Kakutani’s result [4], we conclude that there exists a subsequence $(x'_{n})$ of $(x_n)$ such that

$$\left\| \frac{1}{n} \sum_{i=1}^{n} (-1)^{i+1} x'_i \right\| \to 0 \quad \text{as} \quad n \to \infty.$$ 

This means that $X$ has alternate weak Banach-Saks property. \qed
By Lemma 5 and Proposition 6, if $X$ has property $(D_k)$ then $X$ has weak Banach-Saks property. Since weak Banach-Saks property is equivalent to Banach-Saks property in reflexive Banach spaces, we get the following.

**Corollary 7.** If $X$ has property $(D_k)$, then it has Banach-Saks property.

By Proposition 2, Proposition 3 and Corollary 7, we get the following implications.

$$(UC) \Rightarrow (D_2) \Rightarrow (D_3) \Rightarrow \cdots \Rightarrow (D_\infty) \Rightarrow (BS).$$

We will now show that the implications are not reversible. The following Example 8 can be found in [6].

**Example 8.** For $x = (a_n) \in l_2$, we define a norm $\|x\|_{(s)}$ by

$$\|x\|_{(s)} = \left( \left\| \left( \sum_{i=1}^{s} |a_{n_i}| \right)^2 + \sum_{n\neq n_1, n_2, \ldots, n_s} |a_n|^2 \right\| \right)^{\frac{1}{2}}.$$  

Then $\|x\|_2 \leq \|x\|_{(s)} \leq \sqrt{s} \|x\|_2$. Let $X_s = (l_2, \| \cdot \|_{(s)})$.

The following Lemma 9 can be found in [3].

**Lemma 9.** If $X$ is a Banach space with basis $(e_n)$ and $(x_n)$ is a weakly null sequence in $X$, then for all $\epsilon > 0$ there exists a subsequence $(x_{n_i})$ of $(x_n)$ and block sequence $(u_i)$ of $(e_n)$ such that $\|x_{n_i} - u_i\| < \frac{\epsilon}{\sqrt{s}}$.

We need the following lemma.

**Lemma 10.** If $x_1, x_2, \ldots, x_k, x_{k+1} \in B_{X_k}$ and $x_1 < x_2 < \cdots < x_k < x_{k+1}$ then

$$\left\| \sum_{i=1}^{k+1} (-1)^{i+1} x_i \right\|_{(k)} \leq \sqrt{k^2 + 1}.$$
Proof. This is proved by straightforward computation using the following inequality
\[(n - 1) \sum_{i=1}^{n} a_i^2 \geq 2 \sum_{1 \leq i < j \leq n} a_i a_j,\]
where \((a_i)\) is a real sequence. For simplicity, we give the proof in case \(k = 2\).
Suppose that \(x = (a_n), y = (b_n), z = (c_n) \in B_{X_2}\) and \(x < y < z\). Without loss of generality, it suffices to consider the following two cases.

Case 1: \(\|x - y + z\|_{(2)} = \sup_{n_1, n_2} (|a_{n_1}| + |a_{n_2}|)^2 + \sum_{n \neq n_1, n_2} |a_n|^2 + \sum_n |b_n|^2 + \sum_n |c_n|^2\).

\[
\|x - y + z\|_{(2)} = \sup_{n_1, n_2} (|a_{n_1}| + |a_{n_2}|)^2 + \sum_{n \neq n_1, n_2} |a_n|^2 + \sum_n |b_n|^2 + \sum_n |c_n|^2 \\
\leq \|x\|_{(2)}^2 + \|y\|_{(2)}^2 + \|z\|_{(2)}^2 \leq \|x\|_{(2)}^2 + \|y\|_{(2)}^2 + \|z\|_{(2)}^2 = 3.
\]

Case 2: \(\|x - y + z\|_{(2)} = \sup_{n_1, n_2} (|a_{n_1}| + |b_{n_2}|)^2 + \sum_{n \neq n_1, n_2} |a_n|^2 + \sum_{n \neq n_2} |b_n|^2 + \sum_n |c_n|^2\).

\[
\|x - y + z\|_{(2)} = \sup_{n_1, n_2} (|a_{n_1}| + |b_{n_2}|)^2 + \sum_{n \neq n_1, n_2} |a_n|^2 + \sum_n |b_n|^2 + \sum_n |c_n|^2 \\
\leq 2 \sup_{n_1, n_2} (|a_{n_1}|^2 + |b_{n_2}|^2) + \sum_{n \neq n_1} |a_n|^2 + \sum_{n \neq n_2} |b_n|^2 + \sum_n |c_n|^2 \leq \sum_{n_1, n_2} (|a_{n_1}|^2 + |b_{n_2}|^2) + \sum_n |a_n|^2 + \sum_n |b_n|^2 + \sum_n |c_n|^2 \\
\leq \|x\|_{2}^2 + \|y\|_{2}^2 + \|z\|_{2}^2 = 5.
\]

This implies that \(\|x - y + z\|_{(2)} \leq \sqrt{5}\). \(\square\)

By the above lemmas, we get the following.

**Proposition 11.** Property \((D_{k+1})\) does not imply Property \((D_k)\)

*Proof.* Since the space \(X_k\) is isomorphic to \(l_2\), unit vector basis \((e_n)\) is weakly null in \(X_k\). But

\[
\left\| \sum_{i=1}^{k} (-1)^{i+1} e_{n_i} \right\|_{(k)} = k
\]

for all choice of \(n_i\). This means that \(X_k\) does not have property \((D_k)\).
Let \( (x_n) \) be a weak null sequence in \( B_{X_k} \). By Lemma 9, for all \( \epsilon > 0 \) there exists a subsequence \( (x_{n_i}) \) of \( (x_n) \) and block sequence \( (u_i) \) of \( (e_n) \) such that \( \|x_{n_i} - u_i\| < \frac{\epsilon}{2^{i+1}} \). We note that
\[
\|\sum_{i=1}^{k+1} (-1)^{i+1} u_i\|_k \leq \sqrt{k^2 + 1},
\]
by Lemma 10. For some large \( i_1 < i_2 < \cdots < i_k < i_{k+1} \),
\[
\|x_{n_{i_j}} - u_{i_j}\| < \frac{1}{k+1} \left( \sqrt{k^2 + 2} - \sqrt{k^2 + 1} \right),
\]
where \( j = 1, 2, \ldots, k + 1 \). Then we have
\[
\|\sum_{j=1}^{k+1} (-1)^{i+1} x_{n_{i_j}}\| \leq \|\sum_{j=1}^{k+1} x_{n_{i_j}} - u_{i_j}\| + \|\sum_{j=1}^{k+1} (-1)^{i+1} u_{n_{i_j}}\|
\leq \sqrt{k^2 + 2}.
\]
Let \( \alpha = \frac{\sqrt{k^2 + 2}}{k+1} \). Then \( \alpha < 1 \) and this leads that the space \( X_k \) has property \( (D_{k+1}) \).

To get that the following implications hold and strict, the remaining proof is that Banach-Saks property does not imply Property \( (D_\infty) \).

\[
(UC) \Rightarrow (D_2) \Rightarrow (D_3) \Rightarrow \cdots \Rightarrow (D_\infty) \Rightarrow (BS).
\]

**Proposition 12.** Banach-Saks property does not imply Property \( (D_\infty) \).

**Proof.** Consider \( \left( \prod_{s \geq 2} C_s \right)_{l_2} \). Then \( \left( \prod_{s \geq 2} C_s \right)_{l_2} \) has Banach-Saks property [6].

Let \( k \in \mathbb{N} \). If \( x^{(n)} = (0, 0, \ldots, 0, e_n, 0, \ldots) \) where usual unit vector \( e_n \) in \( k \)-th coordinate is only nonzero element of \( x^{(n)} \), then \( x^{(n)} \in \left( \prod_{s \geq 2} C_s \right)_{l_2} \) and
\[
\|x^{(n)}\|_{\left( \prod_{s \geq 2} C_s \right)_{l_2}} = 1.
\]
We note that \( x^{(n)} \) is weakly null in \( \left( \prod_{s \geq 2} C_s \right)_{l_2} \). But
\[
\left\| \sum_{j=1}^{k} (-1)^{i+1} x^{(n)}_{i} \right\|_{\left( \prod_{s \geq 2} C_s \right)_{l_2}} = \left\| \sum_{j=1}^{k} (-1)^{i+1} e_{n_{i_{j}}} \right\|_{(k)} = k.
\]
This means that \( \left( \prod_{s \geq 2} C_s \right)_{l_2} \) has no property \( (D_\infty) \). \( \square \)
References


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