FUZZY PAIRWISE STRONG PRE-IRRESOLUTE CONTINUOUS MAPPINGS

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Abstract. We define and characterize a fuzzy pairwise strong pre-irresolute continuous mapping and a fuzzy pairwise strong pre-irresolute open mapping on a fuzzy bitopological space.

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1. Introduction

Singal and Prakash [9] introduced a fuzzy preopen set and studied characteristic properties of a fuzzy precontinuous mapping on a fuzzy topological space. Later, Sampath Kumar [7] defined a \((\tau_i, \tau_j)\)-fuzzy preopen set and characterized a fuzzy pairwise precontinuous mapping on a fuzzy bitopological space as a natural generalization of a fuzzy topological space. Also, Im [2] characterized a fuzzy pairwise pre-irresolute mapping on a fuzzy bitopological space.

Krsteska [3, 4] also defined a fuzzy strongly preopen set and studied a fuzzy strongly precontinuous mapping (a fuzzy strong preopen mapping) on a fuzzy topological space. In particular, he defined and characterized a fuzzy strongly pre-irresolute mapping and a fuzzy strong pre-irresolute open mapping on a fuzzy topological space.

Recently, Park, Lee and Im [8] characterized a fuzzy pairwise strong precontinuous mapping and a fuzzy pairwise strong preopen(preclosed) mapping on a fuzzy bitopological space. One purpose of this paper is to find more stronger mapping than we studied in [8].
In this paper, we define a fuzzy pairwise strong pre-irresolute continuous mapping and a fuzzy pairwise pre-irresolute open mapping (fuzzy pairwise pre-irresolute closed mapping) on a fuzzy bitopological space and study their properties. We also give an example is a fuzzy pairwise strong precontinuous mapping but not a fuzzy pairwise strong pre-irresolute continuous mapping.

2. Preliminaries

Let $X$ be a set and let $\tau_1$ and $\tau_2$ be fuzzy topologies on $X$. Then we call $\langle X, \tau_1, \tau_2 \rangle$ a fuzzy bitopological space $[\text{fbts}]$.

A mapping $f : \langle X, \tau_1, \tau_2 \rangle \rightarrow \langle Y, \tau^*_1, \tau^*_2 \rangle$ is fuzzy pairwise continuous $[\text{fpc}]$ if the induced mapping $f : \langle X, \tau_k \rangle \rightarrow \langle Y, \tau^*_k \rangle$ is fuzzy continuous for $k = 1, 2$.

A mapping $f : \langle X, \tau_1, \tau_2 \rangle \rightarrow \langle Y, \tau^*_1, \tau^*_2 \rangle$ is fuzzy pairwise open $[\text{fp open}]$ (fuzzy pairwise closed $[\text{fp closed}]$) if the induced mapping $f : \langle X, \tau_k \rangle \rightarrow \langle Y, \tau^*_k \rangle$ is fuzzy open (fuzzy closed) for $k = 1, 2$.

Notations. (1) Throughout this paper, we take an ordered pair $\langle \tau_i, \tau_j \rangle$ with $i, j \in \{1, 2\}$ and $i \neq j$.

(2) For simplicity, we abbreviate a $\tau_i$-fuzzy open set $\mu$ and a $\tau_j$-fuzzy closed set $\mu$ with a $\tau_i$-fo set $\mu$ and a $\tau_j$-fc set $\mu$ respectively. Also, we denote the interior and the closure of $\mu$ for a fuzzy topology $\tau_i$ with $\tau_i - \text{Int} \mu$ and $\tau_i - \text{Cl} \mu$ respectively.

Definition 2.1. [7] Let $\mu$ be a fuzzy set on a fbts $X$. Then we call $\mu$;
(1) a $\langle \tau_i, \tau_j \rangle$-fuzzy preopen $[\text{(\tau_i, \tau_j)-fpo}]$ set on $X$ if
$$\mu \leq \tau_i - \text{Int}(\tau_j - \text{Cl} \mu)$$
and
(2) a $\langle \tau_i, \tau_j \rangle$-fuzzy preclosed $[\text{(\tau_i, \tau_j)-fpc}]$ set on $X$ if
$$\tau_i - \text{Cl}(\tau_j - \text{Int} \mu) \leq \mu.$$

Definition 2.2. [7] Let $\mu$ be a fuzzy set on a fbts $X$.
(1) The $\langle \tau_i, \tau_j \rangle$-preinterior of $\mu$, $[\langle \tau_i, \tau_j \rangle - \text{pInt} \mu]$ is
$$\bigvee \{\nu \mid \nu \leq \mu, \ \nu \text{ is a } \langle \tau_i, \tau_j \rangle - \text{fpo set}\}.$$
(2) The $\langle \tau_i, \tau_j \rangle$-preclosure of $\mu$, $[\langle \tau_i, \tau_j \rangle - \text{pCl} \mu]$ is
$$\bigwedge \{\nu \mid \nu \geq \mu, \ \nu \text{ is a } \langle \tau_i, \tau_j \rangle - \text{fpc set}\}.$$
Definition 2.3. [8] Let $\mu$ be a fuzzy set on a $fbts$ $X$. Then we call $\mu$;
(1) a $(\tau_i, \tau_j)$-fuzzy strongly preopen $[(\tau_i, \tau_j) - fspo]$ set on $X$ if
\[ \mu \leq \tau_i - \text{Int}((\tau_j, \tau_i) - \text{pCl} \mu) \] and
(2) a $(\tau_i, \tau_j)$-fuzzy strongly preclosed $[(\tau_i, \tau_j) - fspc]$ set on $X$ if
\[ \tau_i - \text{Cl}((\tau_j, \tau_i) - \text{pInt} \mu) \leq \mu. \]

It is clear that a $\tau_i - fo$ set is a $(\tau_i, \tau_j) - fspo$ set and a $(\tau_i, \tau_j) - fspo$ set is a $(\tau_i, \tau_j) - fpo$ set on a $fbts$ $X$. But the converses are not true in general [8].

Proposition 2.4. [8] (1) A union of $(\tau_i, \tau_j) - fspo$ sets is a $(\tau_i, \tau_j) - fspo$ set.
(2) An intersection of $(\tau_i, \tau_j) - fspc$ sets is a $(\tau_i, \tau_j) - fspc$ set.

We remark an intersection of two $(\tau_i, \tau_j) - fspo$ sets need not be a $(\tau_i, \tau_j) - fspo$ set and a union of two $(\tau_i, \tau_j) - fspc$ sets need not be a $(\tau_i, \tau_j) - fspo$ set [8].

Definition 2.5. [8] Let $\mu$ be a fuzzy set on a $fbts$ $X$.
(1) The $(\tau_i, \tau_j)$-strongly preinterior of $\mu$, $[(\tau_i, \tau_j) - spInt \mu]$ is
\[ \bigwedge \{\nu | \nu \leq \mu, \nu \text{ is a (} \tau_i, \tau_j \text{) - } fspo \text{ set}\}. \]
(2) The $(\tau_i, \tau_j)$-strongly preclosure of $\mu$, $[(\tau_i, \tau_j) - spCl \mu]$ is
\[ \bigwedge \{\nu | \nu \geq \mu, \nu \text{ is a (} \tau_i, \tau_j \text{) - } fspc \text{ set}\}. \]

Obviously, $(\tau_i, \tau_j) - spCl \mu$ is the smallest $(\tau_i, \tau_j) - fspc$ set which contains $\mu$, and $(\tau_i, \tau_j) - spInt \mu$ is the largest $(\tau_i, \tau_j) - fspo$ set which is contained in $\mu$. Therefore, $(\tau_i, \tau_j) - spCl \mu = \mu$ for every $(\tau_i, \tau_j) - fspc$ set $\mu$ and $(\tau_i, \tau_j) - spInt \mu = \mu$ for every $(\tau_i, \tau_j) - fspo$ set $\mu$.
Moreover, we have
\[ \tau_i - \text{Int} \mu \leq (\tau_i, \tau_j) - spInt \mu \leq (\tau_i, \tau_j) - \text{pInt} \mu \leq \mu, \]
\[ \mu \leq (\tau_i, \tau_j) - \text{pCl} \mu \leq (\tau_i, \tau_j) - spCl \mu \leq (\tau_i, \tau_j) - \text{Cl} \mu. \]

We state the following lemma from the above definition, which will be used later.

Lemma 2.6. [8] Let $\mu$ be a fuzzy set on a $fbts$ $X$. Then
\[ (\tau_i, \tau_j) - spInt(\mu^c) = ((\tau_i, \tau_j) - spCl \mu)^c \]
and
\[ (\tau_i, \tau_j) - spCl(\mu^c) = ((\tau_i, \tau_j) - spInt \mu)^c. \]

Definition 2.7. [8] Let $f : (X, \tau_1, \tau_2) \to (Y, \tau_1^*, \tau_2^*)$ be a mapping. Then $f$ is called a fuzzy pairwise strong precontinuous $[fpspc]$ mapping if $f^{-1}(\nu)$ is a $(\tau_i, \tau_j) - fspo$ set on $X$ for each $\tau_i^* - fo$ set $\nu$ on $Y$. 
It is clear that every \( fpc \) mapping is a \( fpspc \) mapping and every \( fpspc \) mapping is a \( fpcs \) mapping on \( fbtts \). But the converses are not true in general [8].

**Definition 2.8.** [8] Let \( f: (X, \tau_1, \tau_2) \rightarrow (Y, \tau^*_1, \tau^*_2) \) be a mapping. Then \( f \) is called:

1. a fuzzy pairwise strong preopen \( [fpsp \text{ open}] \) mapping if \( f(\mu) \) is a \( (\tau^*_i, \tau^*_j) - fspo \) set on \( Y \) for each \( \tau_i - fo \) set \( \mu \) on \( X \) and
2. a fuzzy pairwise strong preclosed \( [fpsp \text{ closed}] \) mapping if \( f(\mu) \) is a \( (\tau^*_i, \tau^*_j) - fspc \) set on \( Y \) for each \( \tau_i - fc \) set \( \mu \) on \( X \).

It is clear that every \( fp \text{ open}(fp \text{ closed}) \) mapping is a \( fpsp \text{ open}(fpsp \text{ closed}) \) mapping and every \( fpsp \text{ open}(fpsp \text{ closed}) \) mapping is a \( fpp \text{ open}(fpp \text{ closed}) \) mapping on \( fbtts \). But the converses are not true in general [8].

3. Fuzzy pairwise strong pre-irresolute continuous mappings

In this section, we introduce a fuzzy pairwise strong pre-irresolute continuous mapping and a fuzzy pairwise strong pre-irresolute open mapping which are stronger than a fuzzy pairwise strong precontinuous mapping and a fuzzy pairwise strong preopen mapping respectively. And we characterize a fuzzy pairwise strong pre-irresolute continuous mapping and a fuzzy pairwise pre-irresolute open mapping.

**Definition 3.1.** Let \( f: (X, \tau_1, \tau_2) \rightarrow (Y, \tau^*_1, \tau^*_2) \) be a mapping. Then \( f \) is called a fuzzy pairwise strong pre-irresolute continuous \( [fpsp \text{-irresolute continuous}] \) mapping if \( f^{-1}(\nu) \) is a \( (\tau_i, \tau_j) - fspo \) set on \( X \) for each \( (\tau^*_i, \tau^*_j) - fspo \) set \( \nu \) on \( Y \).

It is clear that every \( fp \text{ open}(fp \text{ closed}) \) mapping is a \( fpsp \text{ open}(fpsp \text{ closed}) \) mapping and every \( fpsp \text{ open}(fpsp \text{ closed}) \) mapping is a \( fpp \text{ open}(fpp \text{ closed}) \) mapping on \( fbtts \). But the converses are not true in general [8].

**Example 3.2.** Let \( \mu_1, \mu_2, \mu_3, \mu_4, \mu_5 \) and \( \mu_6 \) be fuzzy sets on \( X = \{a, b, c\} \) with

- \( \mu_1(a) = 0.9, \mu_1(b) = 0.5, \mu_1(c) = 0.9, \)
- \( \mu_2(a) = 0.5, \mu_2(b) = 0.7, \mu_2(c) = 0.5, \)
- \( \mu_3(a) = 0.8, \mu_3(b) = 0.5, \mu_3(c) = 0.8, \)
- \( \mu_4(a) = 0.8, \mu_4(b) = 0.5, \mu_4(c) = 0.7, \)
- \( \mu_5(a) = 0.5, \mu_5(b) = 0.5, \mu_5(c) = 0.5 \) and
- \( \mu_6(a) = 0.3, \mu_6(b) = 0.4, \mu_6(c) = 0.3. \)

Let

\[
\tau_1 = \{0_X, \mu_4, \mu_6, 1_X\}, \tau_2 = \{0_X, \mu_3, \mu_6, 1_X\} \quad \text{and} \quad \tau^*_1 = \{0_X, \mu_1, 1_X\}, \tau^*_2 = \{0_X, \mu_2, 1_X\}.
\]

Let
be fuzzy topologies on \( X \).

Then we can show that the identity mapping \( i_X : (X, \tau_1, \tau_2) \rightarrow (X, \tau_1^*, \tau_2^*) \) is \( fp\sp\sp c \) but not \( f\sp p\sp c \)-irresolute continuous and \( \mu_5 \) is a \((\tau_1^*, \tau_2^*) - f\sp{spo} \) set but not a \((\tau_i, \tau_j) - f\sp{spo} \) set. □

**Theorem 3.3.** Let \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \tau_1^*, \tau_2^*) \) be a mapping. Then the following statements are equivalent:

1. \( f \) is \( fp\sp\sp c \)-irresolute continuous.
2. The inverse image of each \((\tau_i^*, \tau_j^*) - f\sp{sp\sp c} \) set on \( Y \) is a \((\tau_i, \tau_j) - f\sp{sp\sp c} \) set on \( X \).
3. \( f((\tau_i, \tau_j)) - spCl(\mu) \leq (\tau_i^*, \tau_j^*) - spCl(f(\mu)) \) for each fuzzy set \( \mu \) on \( X \).
4. \( (\tau_i, \tau_j) - spCl(f^{-1}(\nu)) \leq f^{-1}((\tau_i^*, \tau_j^*) - spCl(\nu)) \) for each fuzzy set \( \nu \) on \( Y \).
5. \( f^{-1}((\tau_i^*, \tau_j^*) - spInt(\nu)) \leq (\tau_i, \tau_j) - spInt(f^{-1}(\nu)) \) for each fuzzy set \( \nu \) on \( Y \).

**Proof.**

(1) implies (2): Let \( \nu \) be a \((\tau_i^*, \tau_j^*) - f\sp{sp\sp c} \) set on \( Y \). Then \( \nu^c \) is a \((\tau_i^*, \tau_j^*) - f\sp{sp\sp c} \) set on \( Y \). Since \( f \) is \( fp\sp\sp c \)-irresolute continuous, \( f^{-1}(\nu^c) = (f^{-1}(\nu))^c \) is a \((\tau_i, \tau_j) - f\sp{sp\sp c} \) set on \( X \). Hence \( f^{-1}(\nu) \) is a \((\tau_i, \tau_j) - f\sp{sp\sp c} \) set on \( X \).

(2) implies (3): Let \( \mu \) be a fuzzy set on \( X \). Then \( f^{-1}((\tau_i^*, \tau_j^*) - spCl(f(\mu))) \) is a \((\tau_i, \tau_j) - f\sp{sp\sp c} \) set on \( X \). Thus

\[
(\tau_i, \tau_j) - spCl(\mu) \leq (\tau_i, \tau_j) - spCl(f^{-1}(f(\mu)))
\]

\[
\leq (\tau_i, \tau_j) - spCl(f^{-1}((\tau_i^*, \tau_j^*) - spCl(f(\mu))))
\]

\[
= f^{-1}((\tau_i^*, \tau_j^*) - spCl(f(\mu))).
\]

Hence

\[
f((\tau_i, \tau_j) - spCl(\mu)) \leq f(f^{-1}((\tau_i^*, \tau_j^*) - spCl(f(\mu))))
\]

\[
\leq (\tau_i^*, \tau_j^*) - spCl(f(\mu)).
\]

(3) implies (4): Let \( \nu \) be a fuzzy set on \( Y \). Then

\[
f((\tau_i, \tau_j) - spCl(f^{-1}(\nu))) \leq (\tau_i^*, \tau_j^*) - spCl(f(f^{-1}(\nu))) \leq (\tau_i^*, \tau_j^*) - spCl(\nu).
\]

Hence

\[
(\tau_i, \tau_j) - spCl(f^{-1}(\nu)) \leq f^{-1}(f((\tau_i, \tau_j) - spCl(f^{-1}(\nu))))
\]

\[
\leq f^{-1}((\tau_i^*, \tau_j^*) - spCl(\nu)).
\]

(4) implies (5): Let \( \nu \) be a fuzzy set on \( Y \). Then

\[
(\tau_i, \tau_j) - spCl(f^{-1}(\nu^c)) \leq f^{-1}((\tau_i^*, \tau_j^*) - spCl(\nu^c)).
\]
Hence, by Lemma 2.6,
\[
f^{-1}((\tau_i^*, \tau_j^*) - \text{spInt}\ \nu) = f^{-1}(((\tau_i^* \tau_j^*) - \text{spCl}(\nu^c))\cap)
\leq ((\tau_i, \tau_j) - \text{spCl}(f^{-1}(\nu^c)))^c
= (\tau_i, \tau_j) - \text{spInt}(f^{-1}(\nu)).
\]

(5) implies (1): Let \( \nu \) be a \((\tau_i^*, \tau_j^*) - fspo\) set on \( Y \). Then
\[
f^{-1}(\nu) = f^{-1}((\tau_i^* \tau_j^*) - \text{spInt}\ \nu) \leq (\tau_i, \tau_j) - \text{spInt}(f^{-1}(\nu)).
\]
Hence \( f^{-1}(\nu) \) is a \((\tau_i, \tau_j) - fspo\) set on \( X \) and therefore, \( f \) is \( fpsp\)-irresolute continuous.
\( \Box \)

**Theorem 3.4.** Let \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \tau_1^*, \tau_2^*) \) be a bijection. \( f \) is \( fpsp\)-irresolute continuous if and only if for each fuzzy set \( \mu \) on \( X \),
\[
(\tau_i^*, \tau_j^*) - \text{spInt}(f(\mu)) \leq f(\tau_i, \tau_j) - \text{spInt}(\mu).
\]

**Proof.** Let \( \mu \) be a fuzzy set on \( X \). Then, by Theorem 3.3,
\[
f^{-1}((\tau_i^* \tau_j^*) - \text{spInt}(f(\mu))) \leq (\tau_i, \tau_j) - \text{spInt}(f^{-1}(f(\mu))).
\]
Since \( f \) is a bijection,
\[
(\tau_i^*, \tau_j^*) - \text{spInt}(f(\mu)) = f(f^{-1}((\tau_i^* \tau_j^*) - \text{spInt}(f(\mu)))) \leq f((\tau_i, \tau_j) - \text{spInt}(\mu)).
\]
Conversely, let \( \nu \) be a fuzzy set on \( Y \). Then
\[
(\tau_i^*, \tau_j^*) - \text{spInt}(f(f^{-1}(\nu))) \leq f((\tau_i, \tau_j) - \text{spInt}(f^{-1}(\nu))).
\]
Recall that \( f \) is a bijection. Hence
\[
(\tau_i^*, \tau_j^*) - \text{spInt}\ \nu = (\tau_i^*, \tau_j^*) - \text{spInt}(f(f^{-1}(\nu))) \leq f((\tau_i, \tau_j) - \text{spInt}(f^{-1}(\nu))).
\]
and
\[
f^{-1}((\tau_i^*, \tau_j^*) - \text{spInt}\ \nu) \leq f^{-1}(f((\tau_i, \tau_j) - \text{spInt}(f^{-1}(\nu))))
= (\tau_i, \tau_j) - \text{spInt}(f^{-1}(\nu)).
\]
Therefore, by Theorem 3.3, \( f \) is \( fpsp\)-irresolute continuous.
\( \Box \)

**Definition 3.5.** Let \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \tau_1^*, \tau_2^*) \) be a mapping. Then \( f \) is called
1. a fuzzy pairwise strong pre-irresolute open \([fpsp\)-irresolute open\] mapping if \( f(\mu) \) is a \((\tau_i^*, \tau_j^*) - fspo\) set on \( Y \) for each \((\tau_i, \tau_j) - fspo\) set \( \mu \) on \( X \) and
2. a fuzzy pairwise strong pre-irresolute closed \([fpsp\)-irresolute closed\] mapping if \( f(\mu) \) is a \((\tau_i^*, \tau_j^*) - fspc\) set on \( Y \) for each \((\tau_i, \tau_j) - fspc\) set \( \mu \) on \( X \).
It is clear that every \textit{fpsp-irresolute open} mapping and every \textit{fpsp-irresolute closed} mapping are \textit{fpsp open} and \textit{fpsp closed} respectively. But the converses are not true in general.

In fact, in Example 3.2, the identity mapping \(i_X : (X, \tau^*_1, \tau^*_2) \to (X, \tau_1, \tau_2)\) is \textit{fpsp open}(fpsp \textit{closed}) but not \textit{fpsp-irresolute open}(fpsp-irresolute closed).

**Theorem 3.6.** Let \( f : (X, \tau_1, \tau_2) \to (Y, \tau^*_1, \tau^*_2) \) be a mapping. Then the following statements are equivalent:

1. \( f \) is \textit{fpsp-irresolute open},
2. \( f((\tau_1, \tau_2)) - \text{spInt} \leq (\tau^*_1, \tau^*_2) - \text{spInt}(f(\mu)) \) for each fuzzy set \( \mu \) on \( X \).
3. \( (\tau_1, \tau_2) - \text{spInt}(f^{-1}(\nu)) \leq f^{-1}((\tau^*_1, \tau^*_2) - \text{spInt} \nu) \) for each fuzzy set \( \nu \) on \( Y \).

\[
(1) \implies (2): \text{Let } \mu \text{ be a fuzzy set on } X. \text{ Then } f((\tau_1, \tau_2)) - \text{spInt} \leq (\tau^*_1, \tau^*_2) - \text{spInt}(f(\mu)) \text{ for each fuzzy set } \mu \text{ on } X.
\]

\[
(2) \implies (3): \text{Let } \nu \text{ be a fuzzy set on } Y. \text{ Then } f((\tau_1, \tau_2)) - \text{spInt}(f^{-1}(\nu)) \leq (\tau^*_1, \tau^*_2) - \text{spInt}(f(\mu)).
\]

\[
(3) \implies (1): \text{Let } \mu \text{ be a fuzzy set on } X. \text{ Then } f((\tau_1, \tau_2)) - \text{spInt}(f^{-1}(\nu)) \leq (\tau^*_1, \tau^*_2) - \text{spInt} \nu. \]

\[
(\tau_1, \tau_2) - \text{spInt}(f^{-1}(\nu)) \leq f^{-1}((\tau^*_1, \tau^*_2) - \text{spInt} \nu).
\]

\[
\frac{1}{2}
\]

**Theorem 3.7.** A mapping \( f : (X, \tau_1, \tau_2) \to (Y, \tau^*_1, \tau^*_2) \) is \textit{fpsp-irresolute closed} if and only if \( (\tau^*_1, \tau^*_2) - \text{spCl}(f(\mu)) \leq f((\tau_1, \tau_2) - \text{spCl}(\mu)) \) for each fuzzy set \( \mu \) on \( X \).
Proof. Let $\mu$ be a fuzzy set on $X$. Then $f((\tau_i, \tau_j) - \text{spCl} \mu)$ is a $(\tau^*_i, \tau^*_j) - \text{fspCl}$ set on $Y$ and $f(\mu) \leq f((\tau_i, \tau_j) - \text{spCl} \mu)$. Hence

$$(\tau^*_i, \tau^*_j) - \text{spCl}(f(\mu)) \leq (\tau^*_i, \tau^*_j) - \text{spCl}(f((\tau_i, \tau_j) - \text{spCl} \mu)) = f((\tau_i, \tau_j) - \text{spCl} \mu).$$

Conversely, let $\mu$ be a $(\tau_i, \tau_j) - \text{fspCl}$ set on $X$. Then

$$(\tau^*_i, \tau^*_j) - \text{spCl}(f(\mu)) \leq f((\tau_i, \tau_j) - \text{spCl} \mu) = f(\mu).$$

Consequently, $f(\mu)$ is a $(\tau^*_i, \tau^*_j) - \text{fspCl}$ set on $Y$ and therefore $f$ is a $\text{fspCl}$-irresolute closed mapping.

\hfill $\Box$

**Theorem 3.8.** Let $f : (X, \tau_1, \tau_2) \to (Y, \tau^*_1, \tau^*_2)$ be a bijection. Then the following statements are equivalent:

1. $f$ is $\text{fspCl}$-irresolute closed.
2. $f^{-1}((\tau^*_i, \tau^*_j) - \text{spCl} \nu) \leq (\tau_i, \tau_j) - \text{spCl}(f^{-1}(\nu))$ for each fuzzy set $\nu$ on $Y$.
3. $f$ is $\text{fspCl}$-irresolute open.
4. $f^{-1}$ is $\text{fspCl}$-irresolute continuous.

**Proof.** (1) implies (2): Let $\nu$ be a fuzzy set on $Y$. Then, by Theorem 3.7,

$$(\tau^*_i, \tau^*_j) - \text{spCl}(f(f^{-1}(\nu))) \leq f((\tau_i, \tau_j) - \text{spCl}(f^{-1}(\nu))).$$

Hence

$$f^{-1}((\tau^*_i, \tau^*_j) - \text{spCl}(f(f^{-1}(\nu)))) \leq f^{-1}(f((\tau_i, \tau_j) - \text{spCl}(f^{-1}(\nu)))).$$

Since $f$ is a bijection,

$$f^{-1}((\tau^*_i, \tau^*_j) - \text{spCl} \nu) \leq (\tau_i, \tau_j) - \text{spCl}(f^{-1}(\nu)).$$

(2) implies (1): Let $\mu$ be a fuzzy set on $X$. Then

$$f^{-1}((\tau^*_i, \tau^*_j) - \text{spCl}(f(\mu))) \leq (\tau_i, \tau_j) - \text{spCl}(f^{-1}(f(\mu))).$$

Hence

$$f(f^{-1}((\tau^*_i, \tau^*_j) - \text{spCl}(f(\mu)))) \leq f((\tau_i, \tau_j) - \text{spCl}(f^{-1}(f(\mu)))).$$

Since $f$ is a bijection,

$$(\tau^*_i, \tau^*_j) - \text{spCl}(f(\mu)) \leq f((\tau_i, \tau_j) - \text{spCl} \mu).$$

Therefore, by Theorem 3.7, $f$ is $\text{fspCl}$-irresolute closed.

(2) implies (3): Let $\nu$ be a fuzzy set on $Y$. Then

$$f^{-1}((\tau^*_i, \tau^*_j) - \text{spCl}(\nu)) \leq (\tau_i, \tau_j) - \text{spCl}(f^{-1}(\nu)).$$
By Lemma 2.6,
\[
(\tau_i, \tau_j) - \text{spInt}(f^{-1}(\nu)) = ((\tau_i, \tau_j) - \text{spCl}(f^{-1}(\nu^c)))^c \\
\leq f^{-1}(((\tau_i^*, \tau_j^*) - \text{spCl}(\nu^c))^c) \\
= f^{-1}((\tau_i^*, \tau_j^*) - \text{spInt} \nu).
\]
Hence \( f \) is \( \text{fpsp} \)-irresolute open from Theorem 3.6.

(3) implies (4): Let \( \nu \) be a fuzzy set on \( Y \). Then
\[
(\tau_i, \tau_j) - \text{spInt}(f^{-1}(\nu)) \leq f^{-1}((\tau_i^*, \tau_j^*) - \text{spInt} \nu).
\]
Since \( f \) is a bijection, by Theorem 3.4, \( f^{-1} \) is \( \text{fpsp} \)-irresolute continuous.

(4) implies (2): It is clear from Theorem 3.3.

We have the following corollaries from Theorem 3.3, Theorem 3.7 and Theorem 3.6.

**Corollary 3.9.** Let \( f : (X, \tau_1, \tau_2) \to (Y, \tau_1^*, \tau_2^*) \) be a mapping. Then, \( f \) is a \( \text{fpsp} \)-irresolute closed and \( \text{fpsp} \)-irresolute continuous if and only if \( f((\tau_i, \tau_j) - \text{spCl} \mu) = (\tau_i^*, \tau_j^*) - \text{spCl}(f(\mu)) \) for each fuzzy set \( \mu \) on \( X \).

**Corollary 3.10.** Let \( f : (X, \tau_1, \tau_2) \to (Y, \tau_1^*, \tau_2^*) \) be a mapping. Then, \( f \) is \( \text{fpsp} \)-irresolute open and \( \text{fpsp} \)-irresolute continuous if and only if \( f^{-1}((\tau_i^*, \tau_j^*) - \text{spCl} \nu) = (\tau_i, \tau_j) - \text{spCl}(f^{-1}(\nu)) \) for each fuzzy set \( \nu \) on \( Y \).

A bijection \( f : (X, \tau_1, \tau_2) \to (Y, \tau_1^*, \tau_2^*) \) is called a fuzzy pairwise strong pre-irresolute homeomorphism if \( f \) and \( f^{-1} \) are \( \text{fpsp} \)-irresolute continuous mappings.

**Theorem 3.11.** Let \( f : (X, \tau_1, \tau_2) \to (Y, \tau_1^*, \tau_2^*) \) be a bijection. Then the following statements are equivalent:

1. \( f \) is a fuzzy pairwise strong pre-irresolute homeomorphism.
2. \( f^{-1} \) is a fuzzy pairwise strong pre-irresolute homeomorphism.
3. \( f \) and \( f^{-1} \) are \( \text{fpsp} \)-irresolute open (\( \text{fpsp} \)-irresolute closed).
4. \( f \) is \( \text{fpsp} \)-irresolute continuous and \( \text{fpsp} \)-irresolute open (\( \text{fpsp} \)-irresolute closed).
5. \( f((\tau_i, \tau_j) - \text{spCl} \mu) = (\tau_i^*, \tau_j^*) - \text{spCl}(f(\mu)) \) for each fuzzy set \( \mu \) on \( X \).
6. \( f((\tau_i, \tau_j) - \text{spInt} \mu) = (\tau_i^*, \tau_j^*) - \text{spInt}(f(\mu)) \) for each fuzzy set \( \mu \) on \( X \).
7. \( f^{-1}((\tau_i^*, \tau_j^*) - \text{spInt} \nu) = (\tau_i, \tau_j) - \text{spInt}(f^{-1}(\nu)) \) for each fuzzy set \( \nu \) on \( Y \).
8. \( (\tau_i, \tau_j) - \text{spCl}(f^{-1}(\nu)) = f^{-1}((\tau_i^*, \tau_j^*) - \text{spCl} \nu) \) for each fuzzy set \( \nu \) on \( Y \).
Proof. (1) implies (2): It follows immediately from the definition of a fuzzy pairwise strong pre-irresolute homeomorphism.

(2) implies (3) and (3) implies (4): It follows from Theorem 3.8.

(4) implies (5): It follows from Theorem 3.8 and Corollary 3.9.

(5) implies (6): Let \( \mu \) be a fuzzy set on \( X \). Then, by Lemma 2.6,
\[
\begin{align*}
f((\tau_i, \tau_j) - \text{spInt} \mu) &= (f((\tau_i, \tau_j) - \text{spCl}(\mu^c))^c \\
&= ((\tau^*_i, \tau^*_j) - \text{spCl}(f(\mu^c)))^c \\
&= (\tau^*_i, \tau^*_j) - \text{spInt} f(\mu).
\end{align*}
\]

(6) implies (7): Let \( \nu \) be a fuzzy set on \( Y \). Then
\[
\begin{align*}
f((\tau_i, \tau_j) - \text{spInt}(f^{-1}(\nu))) &= (\tau^*_i, \tau^*_j) - \text{spInt}(f(f^{-1}(\nu))) \\
&= (\tau^*_i, \tau^*_j) - \text{spInt} \nu.
\end{align*}
\]

Hence
\[
f^{-1}(f((\tau_i, \tau_j) - \text{spInt}(f^{-1}(\nu)))) = f^{-1}((\tau^*_i, \tau^*_j) - \text{spInt} \nu).
\]
Therefore,
\[
(\tau_i, \tau_j) - \text{spInt}(f^{-1}(\nu)) = f^{-1}((\tau^*_i, \tau^*_j) - \text{spInt} \nu).
\]

(7) implies (8): Let \( \nu \) be a fuzzy set on \( Y \). Then, by Lemma 2.6,
\[
\begin{align*}
(\tau_i, \tau_j) - \text{spCl}(f^{-1}(\nu)) &= (f^{-1}(((\tau^*_i, \tau^*_j) - \text{spInt}(\nu^c)))^c \\
&= (((\tau_i, \tau_j) - \text{spInt}(f^{-1}(\nu^c)))^c \\
&= f^{-1}(((\tau^*_i, \tau^*_j) - \text{spCl} \nu).
\end{align*}
\]

(8) implies (1): It follows from Theorem 3.8 and Corollary 3.10.

\( \square \)

References
1. Y. B. Im, Fuzzy pairwise \( \gamma \)-irresoluteness, International J. fuzzy and Intelligent Systems, 7 (2007), 188-192.
6. Y. B. Im, E. P. Lee and S. W. Park, Fuzzy pairwise \( \gamma \)-continuous mappings J. Fuzzy Math. 10 (2002), 695-709.

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