COMMUTATIVE MONOID OF THE SET OF
k-ISOMORPHISM CLASSES OF SIMPLE CLOSED
k-SURFACES IN $\mathbb{Z}^3$

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Abstract. In this paper we prove that with some hypothesis the set of $k$-isomorphism classes of simple closed $k$-surfaces in $\mathbb{Z}^3$ forms a commutative monoid with an operation derived from a digital connected sum, $k \in \{18, 26\}$. Besides, with some hypothesis the set of $k$-homotopy equivalence classes of closed $k$-surfaces in $\mathbb{Z}^3$ is also proved to be a commutative monoid with the above operation, $k \in \{18, 26\}$.

1. Introduction

In order to study some properties of the set of $k$-isomorphism classes of simple closed $k$-surfaces in $\mathbb{Z}^3$, we need to recall some notions, as follows. In algebra, a monoid is defined to be a set $X$ with a binary operation $*: X \times X \rightarrow X$, obeying the following axioms:

- $(X, *)$ has the associative law,
- there is an element $e \in X$ such that for any element $x \in X$, $x * e = e * x = x$ and further,
- if $x * y = y * x$ for any elements $x, y \in X$, then we say that $(X, *)$ is a commutative monoid.

Let $\mathbb{N}$ and $\mathbb{Z}$ be the sets of natural numbers and integers, respectively. Let $\mathcal{L}$ be the set of lattice points in Euclidean $n$-dimensional space, $n \in \mathbb{N}$. In [27] a closed $k$-surface was studied in $\mathbb{Z}^3$, $k \in \{6, 26\}$ and in [1] a closed 18-surface was introduced in $\mathbb{Z}^3$. Besides, the study of
various properties of a closed $k$-surface in $\mathbb{Z}^3$ and digital space includes the papers [1, 9, 10, 14, 25].

The connected sum in geometric topology cannot be available in discrete (or digital) geometry. Thus we need its digital version to study a digital $k$-surface. Motivated by the notion of connected sum in geometric topology, its digital version was established in [9] (see also [7, 14]). Thus, the notion of digital connected sum of two simple closed $k$-surfaces was introduced and further, its digital topological properties were partially studied [9, 15]. In [5] a geometric realization of a digital space $X \subset \mathbb{Z}^3$ has been introduced. Moreover, in [15] the Euler characteristic of a digital space was studied in relation with a digital connected sum in [9] (see also [15]). In [14] two types of simple, closed 18-surfaces in $\mathbb{Z}^3$ were introduced. One is 18-contractible, denoted by $MSS_{18}'$ (see 3.1) and the other is not 18-contractible, denoted by $MSS_{18}$ (see 3.1). Especially, $MSS_{18}'$ plays an important role in establishing the monoid structure of the set of $k$-isomorphism classes of simple closed $k$-surfaces in $\mathbb{Z}^3$.

In this paper we prove that with some hypothesis the set of $k$-isomorphism classes of simple closed $k$-surfaces in $\mathbb{Z}^3$ forms a commutative monoid with an operation derived from a digital connected sum in [9], $k \in \{18, 26\}$. Besides, we prove that both $MSS_{18}'$ and $MSS_{18}$ are 26-surfaces and further, $MSS_{18}'$ is proved 26-contractible. Moreover, $k$-contractibility of $MSS_{k}'$ allows us to establish a commutative monoid of the set of $k$-isomorphism classes of simple closed $k$-surfaces with an operation derived from a digital connected sum, $k \in \{18, 26\}$. In other words, the $k$-isomorphism class of $MSS_{k}'$, denoted by $[MSS_{k}']$, is proved to be the identity element for the above-mentioned monoid, $k \in \{18, 26\}$. Similarly, with some hypothesis we also form another commutative monoid of the set of $k$-homotopy classes of closed $k$-surfaces in $\mathbb{Z}^3$, $k \in \{18, 26\}$. This kinds of two monoids of the sets of $k$-isomorphism classes of simple closed $k$-surfaces in $\mathbb{Z}^3$ and $k$-homotopy equivalence classes of closed $k$-surfaces can be used in classifying simple closed $k$-surfaces in $\mathbb{Z}^3$.

This paper is organized as follows. Section 2 provides basic notions. Section 3 investigates some properties of a closed $k$-surface and a relative $k$-homotopy, $k \in \{18, 26\}$. Section 4 establishes a commutative monoid of the set of $k$-isomorphism classes of closed $k$-surfaces with an operation derived from a digital connected sum, $k \in \{18, 26\}$. Section 5 shows that with some hypothesis the set of $k$-homotopy equivalence classes of closed $k$-surfaces with an operation forms a commutative monoid, $k \in \{18, 26\}$. 
2. Preliminaries

In order to make this paper self-contained, we recall some necessary terminology from earlier literature in [1, 3, 25, 28]. Since a closed $k$-surface in $\mathbb{Z}^3$ can be studied with a digital $k$-graph structure in $\mathbb{Z}^3$, we now use the $k(m,n)$ (or $k_m$)-adjacency relations of $\mathbb{Z}^n$, $n \in \mathbb{N}$ [8] (see also [12]):

Let $m$ be a positive integer with $1 \leq m \leq n$. Then we say that two distinct points $p = (p_1, p_2, \ldots, p_n)$ and $q = (q_1, q_2, \ldots, q_n) \in \mathbb{Z}^n$ are $k(m,n)$-adjacent according to $m$ if

1. there are at most $m$ distinct indices $i$ such that $|p_i - q_i| = 1$; and
2. for all indices $i$ such that $|p_i - q_i| \neq 1, p_i = q_i$.

In terms of this operator the number $m$ determines one of the $k(m,n)$-adjacency relations of $\mathbb{Z}^n$, we may use $k := k(m,n)$. Precisely, by $N_k(p)$ we denote the set of the points $q \in \mathbb{Z}^n$ which are $k_m$-adjacent to a given point $p$ and the number $k := k(m,n)$ is the cardinal number of $N_k(p)$. Consequently, we obtain the following $k$-adjacency relations of $\mathbb{Z}^n$ [8] (see also [9, 15]).

**Proposition 2.1.** [19] $k := k(m,n) = \sum_{i=n-m}^{n-1} \binom{n}{i} 2^{n-i} C_i^n$, where $C_i^n = \frac{n!}{(n-i)! i!}$.

In general, for a subset $X \subset \mathbb{Z}^n$ with $k$-adjacency, $n \in \mathbb{N}$, we call it a *digital space with $k$-adjacency*, denoted by $(X,k)$, and further, $(X,k)$ is usually considered in a digital picture $\langle \mathbb{Z}^n, k, \bar{k}, X \rangle$ [27, 28], $k$ and $\bar{k}$ are related to the adjacencies of $X$ and $\mathbb{Z}^n - X$, respectively. In this paper, we assume $(k, \bar{k}) \in \{(k, 2n), (2n, 3^n-1)\}$. Hereafter, we call briefly $(X,k)$ a *space* if not confused. Owing to the *digital $k$-connectivity paradox* in [26], we commonly assume that $k \neq \bar{k}$ except for the case $(\mathbb{Z}, 2, 2, X)$. For $a, b \in \mathbb{Z}$ with $a \leq b$, the set $[a, b]_\mathbb{Z} = \{n \in \mathbb{Z} | a \leq n \leq b\}$ is called a *digital interval* [3].

A digital space $(X,k)$ is a digital graph $G_k$ [13] (see also [15, 16, 18]). To be specific, the vertex set of $G_k$ can be considered as the set of points of $X$. Besides, two points $x_1, x_2 \in X$ determine a $k$-edge of $G_k$ if and only if $x_1$ and $x_2$ are $k$-adjacent in $X$.

A $k$-path from $x$ to $y$ in $X$ is a sequence $(x = x_0, x_1, x_2, \ldots, x_m-1, x_m = y)$ in $X$ such that each point $x_i$ is $k$-adjacent to $x_{i+1}$ for $m \geq 1$ and $i \in [0, m-1]_\mathbb{Z}$. Then, the number $m$ is called the *length* of this path [26]. If $x_0 = x_m$, then the $k$-path is said to be *closed* [26]. A set of lattice points is *$k$-connected* if it is not a union of two disjoint non-empty sets.
that are not $k$-adjacent to each other [25]. Thus a singleton set with $k$-adjacency is $k$-connected. For a digital space $(X, k)$, two distinct points $x, y \in X$ are $k$-connected [22] if there is a $k$-path from $x$ to $y$ in $X$. For an adjacency relation $k$ of $\mathbb{Z}^n$, a simple $k$-path with $m$ elements in $\mathbb{Z}^n$ is assumed to be a sequence $(x_i)_{i \in [0, m-1]} \subset \mathbb{Z}^n$ such that $x_i$ and $x_j$ are $k$-adjacent if and only if either $j = i + 1$ or $i = j + 1$ [25]. Furthermore, a simple closed $k$-curve with $l$ elements in $\mathbb{Z}^n$ is a sequence $(x_i)_{i \in [0, l-1]} \subset \mathbb{Z}^n$ derived from a simple $k$-curve $(x_i)_{i \in [0, l]} \subset \mathbb{Z}^n$ with $x_0 = x_l$, where $x_i$ and $x_j$ are $k$-adjacent if and only if $j = i + 1 (mod l)$ or $i = j + 1 (mod l)$ [25]. By $\text{SC}_{k}^{n,l}$ we denote a simple closed $k$-curve with $l$ elements in $\mathbb{Z}^n$, $n \in \mathbb{N} - \{1\}$ [12].

Motivated by both the digital continuity of [28] and the $(k_0, k_1)$-continuity of [2], we say that a function $f : X \to Y$ is $(k_0, k_1)$-continuous at a point $x_0 \in X$. Let $(X, k_0)$ and $(Y, k_1)$ be spaces in $\mathbb{Z}^{n_0}$ and $\mathbb{Z}^{n_1}$, respectively. A function $f : X \to Y$ is $(k_0, k_1)$-continuous at a point $x_0 \in X$ if and only if $f(N_{k_0}(x_0, 1)) \subset N_{k_1}(f(x_0), 1)$, where $N_{k_0}(x_0, 1) \subset X$ and $N_{k_1}(f(x_0), 1) \subset Y$.

Unlike the pasting property of classical continuity in topology, the $(k_0, k_1)$-continuity has some intrinsic features [24]: $(k_0, k_1)$-continuity has the almost pasting property instead of the pasting property of classical topology.

For a $k$-adjacency relation of $\mathbb{Z}^n$, we recall that a simple closed $k$-curve with $l$ elements in $X \subset \mathbb{Z}^n$ is the image of a $(2, k)$-continuous function $f : [0, l-1]_Z \to X$ such that $f(i)$ and $f(j)$ are $k$-adjacent if and only if either $j = i + 1 (mod l)$ or $i = j + 1 (mod l)$ [26]. Thus, we may use the notation $\text{SC}_{k}^{n,l} := (c_i)_{i \in [0, l-1]} \subset \mathbb{Z}^n$ with $f(i) = c_i$ [12].

Recently, digital graph versions of $(k_0, k_1)$-continuity, $(k_0, k_1)$-homeomorphism, $(k_0, k_1)$-covering, and $(k_0, k_1)$-homotopy in digital topology were established in [13]. Consequently, we may use the term a $(k_0, k_1)$-isomorphism as in [4, 13] rather than a $(k_0, k_1)$-homeomorphism as in [3]:

**Definition 1.** [13] (see also [4]) For two spaces $(X, k_0)$ in $\mathbb{Z}^{n_0}$ and $(Y, k_1)$ in $\mathbb{Z}^{n_1}$, a map $h : X \to Y$ is called a $(k_0, k_1)$-isomorphism if $h$ is a $(k_0, k_1)$-continuous bijection and further, $h^{-1} : Y \to X$ is $(k_1, k_0)$-continuous. Then, we use the notation $X \approx_{(k_0, k_1)} Y$. If $n_0 = n_1$ and $k_0 = k_1$, then we call it a $k_0$-isomorphism and use the notation $X \approx_{k_0} Y$ or $X \approx Y$ if not confused.
3. Some properties of a simple closed \( k \)-surface in \( \mathbb{Z}^3 \), \( k \in \{18, 26\} \)

For a space \( (X, k) \) and its subset \( A \), we call \( ((X, A), k) \) a \textit{digital space pair} with \( k \)-adjacency. Furthermore, if \( A \) is a singleton set \( \{x_0\} \), then \( (X, x_0) \) is called a \textit{pointed space} \[3\]. Motivated by the \( k \)-homotopy of \[3\], the \textit{homotopy relative to a subset} \( A \subset X \) was established in \[9\] and has been used in studying digital spaces in relation with a strong \( k \)-deformation retract, a \( k \)-homotopic thinning \[10\] (see also \[16\]), and a \( k \)-contractibility \[17\]. As special case of the \((k_0, k_1)\)-homotopy in \[3\], we use the following \( k \)-homotopy in this paper.

**Definition 2.** \[9\] (see also \[16\]) Let \( (X, k) \) and \( (Y, k) \) be spaces in \( \mathbb{Z}^n \), and \( A \subset X \). Let \( f, g : X \to Y \) be \((k, k)\)-continuous functions. Suppose the existence of both \( m \in \mathbb{N} \) and a function \( F : X \times [0, m] \mathbb{Z} \to Y \) such that

- for all \( x \in X \), \( F(x, 0) = f(x) \) and \( F(x, m) = g(x) \);
- for all \( x \in X \), the induced function \( F_x : [0, m] \mathbb{Z} \to Y \) defined by \( F_x(t) = F(x, t) \) is \((2, k)\)-continuous for all \( t \in [0, m] \mathbb{Z} \);
- for all \( t \in [0, m] \mathbb{Z} \), the induced function \( F_t : X \to Y \) defined by \( F_t(x) = F(x, t) \) is \( k \)-continuous for all \( x \in X \).

Then, \( F \) is called a \( k \)-homotopy between \( f \) and \( g \), and \( f \) and \( g \) are \( k \)-homotopic in \( Y \).

- Furthermore, for all \( t \in [0, m] \mathbb{Z} \), then suppose the induced map \( F_t \) on \( A \) is a constant which is the prescribed function from \( A \) to \( Y \). In other words, \( F_t(x) = f(x) = g(x) \) for all \( x \in A \) and for all \( t \in [0, m] \mathbb{Z} \).

Then, we call \( F \) a \( k \)-homotopy relative to \( A \) between \( f \) and \( g \), and we say that \( f \) and \( g \) are \( k \)-homotopic relative to \( A \) in \( Y \) denoted by \( f \simeq_{k \text{-rel.} A} g \).

In Definition 2, if \( A = \{x_0\} \subset X \), then we say that \( F \) is a pointed \( k \)-homotopy at \( \{x_0\} \) in \[3\].

**Definition 3.** \[3\] If, for some \( x_0 \in X \), \( 1_X \) is \( k \)-homotopic to the constant map with space \( x_0 \) relative to \( \{x_0\} \), then we say that \( (X, x_0) \) is \textit{pointed} \( k \)-contractible.

Indeed, the notion of \( k \)-contractibility is slightly different from both the contractibility in Euclidean topology \[3, 12\] and the contractibility of \[3\].

In classical topology, the notions of \textit{interior} and \textit{exterior} have been essentially used in studying a topological space. By analogy, we obtain the following from the view point of digital topology.
Definition 4. [9] Let $c^* = (x_0, x_1, \ldots, x_n)$ be a closed $k$-curve in $\mathbb{Z}^2$. Let $c^*$ be the complement of $c^*$ in $\mathbb{Z}^2$. A point $x$ of $c^*$ is said to be interior to $c^*$ if it belongs to the bounded $\bar{k}$-connected component of $c^*$. Otherwise, it is called exterior to $c^*$. The set of all interior (respectively exterior) points to $c^*$ is denoted by $\text{Int}(c^*)$ (respectively $\text{Ext}(c^*)$).

We now recall the terminology for the study of a digital $k$-surface in $\mathbb{Z}^3$. A point $x \in X \subset \mathbb{Z}^3$ is called a $k$-corner if $x$ is $k$-adjacent to two and only two points $y, z \in X$ such that $y$ and $z$ are $k$-adjacent to each other [1]. The $k$-corner $x$ is called simple if $y$ and $z$ are not $k$-corners and if $x$ is the only point $k$-adjacent to both $y, z$. $X$ is called a generalized simple closed $k$-curve if what is obtained by removing all simple $k$-corners of $X$ is a simple closed $k$-curve [1]. For a $k$-connected space $(X, k)$ in $\mathbb{Z}^3$, we recall $|X|^* = N_{26}(x) \cap X, N_{26}^*(x) = \{x' | x$ and $x'$ are $26$-adjacent$. In other words, $|X|^* = N_{26}(x, 1) - \{x\}$ [9, 10, 14].

By using the above terminology, the notion of closed $k$-surface was introduced:

Definition 5. [1] Let $(X, k)$ be a space in $\mathbb{Z}^3$, and $\bar{X} = \mathbb{Z}^3 - X$. Then, $X$ is called a closed $k$-surface if it satisfies the following:

1. In case $(k, \bar{k}) \in \{(26, 6), (6, 26)\}$, then
   (a) for each point $x \in X$, $|X|^*$ has exactly one $k$-component $k$-adjacent to $x$;
   (b) $|X|^*$ has exactly two $\bar{k}$-components which are $\bar{k}$-adjacent to $x$; we denote by $C^{x \bar{x}}$ and $D^{x \bar{x}}$ these two components; and
   (c) for any point $y \in N_k(x) \cap X, N_k^*(y) \cap C^{x \bar{x}} \neq \emptyset$ and $N_k(y) \cap D^{x \bar{x}} \neq \emptyset$, where $N_k(x) = N_k^*(x) \cup \{x\}$ and $N_k^*(x) = \{x' | x$ and $x'$ are $k$-adjacent$\}$.
2. In case $(k, \bar{k}) = (18, 6)$, then
   (a) $X$ is $k$-connected,
   (b) for each point $x \in X$, $|X|^*$ is a generalized simple closed $k$-curve.

In (1) and (2), for $k \in \{18, 26\}$ if the image $|X|^*$ is a simple closed $k$-curve, then $X$ is called simple.

Obviously, we observe that each closed 6-surface is simple (see $MSS_6$ in Figure 1). Furthermore, in this paper we will not consider the orientability of a closed $k$-surface in [27].

The paper [14] establishes the following:

$$
\begin{cases}
MSS_{18} \approx_{18} (MSC_8 \times \{1\}) \cup (\text{Int}(MSC_8) \times \{0, 2\}); \\
MSS'_{18} \approx_{18} (MSC_8' \times \{1\}) \cup (\text{Int}(MSC_8') \times \{0, 2\}),
\end{cases}
$$

(3.1)
where ‘×’ means the Cartesian product (or digital product) and
\[ MSC_8 := ((0, 0), (1, -1), (2, -1), (3, 0), (2, 1), (1, 1)) \] and,
\[ MSC'_8 := ((0, 0), (1, 1), (0, 2), (-1, 1)). \]

Remark 3.1. The space MSS'_{18} in (3.1) can be represented as the set
\[ \{(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)\} \] in \( \mathbb{Z}^3, 18, 6, MSS'_{18} \).

In [14], it turns out that MSS_{18} is a simple closed 18-surface not 18-contractible (see Figure 1) and further, MSS'_{18} is a simple closed 18-surface which is 18-contractible. In this paper each of MSS_{18} and MSC'_{18} is considered with an (18, 6) or a (26, 6)-structure instead of the others in [6].

Both 18-surfaces MSS_{18} and MSS'_{18} have some useful properties, as follows.

Lemma 3.2. (1) MSS_{k} is unique up to k-isomorphism, \( k \in \{18, 26\} \).
(2) MSS_{18} is also a simple closed 26-surface not 26-contractible.
(3) MSS'_{18} is also a simple closed 26-surface which is 26-contractible.
Proof: (1) Trivial.

(2) Since \( MSS_{18} \) is obviously a simple closed 26-surface with a (26, 6)-structure and further, there is no 26-homotopy on \( MSS_{18} \) making \( \text{const}_{p_i} \) \( 26 \)-homotopic to a constant map \( c \), where \( p_i \) is an arbitrary point in \( MSS_{18} \). Thus, \( MSS_{18} \) cannot be 26-homotopy equivalent to a singleton in \( MSS_{18} \), the proof is completed.

(3) \( MSS'_{18} \) is obviously a simple closed 26-surface with a (26, 6)-structure. Furthermore, \( MSS'_{18} \) is 26-contractible due to the 18-contractibility of \( MSS'_{18} \) in [14]. Hereafter, by Lemma 3.2, we may use \( MSS_{18} \) and \( MSS'_{18} \) as \( MSS_{26} \) and \( MSS'_{26} \), respectively. Namely, we may use \( MSS_{18} := MSS_{26} \) and \( MSS'_{18} := MSS'_{26} \) in this paper.

4. Commutative monoid of the set of \( k \)-isomorphism classes of simple closed \( k \)-surfaces in \( \mathbb{Z}^3 \)

In relation with the establishment of a digital version of a connected sum, we have used the following spaces \( MSC'_{8} := MSC'_{8} \cup \{ q \} \) and \( MSC'_{8} := MSC'_{8} \cup \{ x_1, x_2 \} \), come from \( MSC'_{8} \) and \( MSC_{8} \) in \( \mathbb{Z}^2 \) [9, 14]. \( MSC'_{8} \) has been used in establishing a digital connected sum. In this section we denote by \( SC_k \) the set of all simple closed \( k \)-surface \( X \subset \mathbb{Z}^3 \) in which each point \( x \in X \) has a subset \( N_k(x, 1) \subset X \) satisfying \( N_k(x, 1) \approx (k, 8) MSC'_{8} \).

In addition, we obtain the following:

(1) \( MSC''_{8} := MSC''_{8} \cup \text{Int}(MSC'_{8}) \), where \( MSC''_{8} \approx \{ w_0 = (0, 0), w_1 = (-1, 1), w_2 = (-2, 0), w_3 = (-1, -1) \} \).

(2) \( MSC''_{8} := MSC_{8} \cup \text{Int}(MSC_{8}) \approx (8, 4) N_4(p, 1) \subset \mathbb{Z}^2, p \in \mathbb{Z}^2 \), where \( MSC_{8} \approx \{ c_0 = (0, 0), c_1 = (1, 1), c_2 = (1, 2), c_3 = (0, 3), c_4 = (-1, 2), c_5 = (-1, 1) \} \).

Since a simple closed \( k \)-surface in \( SC_k \) has a subset \( A \subset X \) satisfying \( A \approx (k, 8) MSC''_{8} \), \( k \in \{18, 26\} \), hereafter, we may take a subset \( A \approx (k, 8) MSC''_{8} \) for the digital connected sum of Definition 6 below. Thus we can establish a commutative monoid structure of the set of \( k \)-isomorphism classes of simple closed \( k \)-surfaces in \( SC_k \) with an operation derived from the digital connected sum, \( k \in \{18, 26\} \). As a special case of the digital connected sum in [9], we introduce the following which is suitable for an establishment of a monoid of the set of \( k \)-isomorphism classes of simple closed \( k \)-surfaces in \( SC_k \).
Definition 6. Let $X$ and $Y$ be simple closed $k$-surfaces in $SC_k$, $k \in \{18, 26\}$. Consider $A' \subset A \subset X$ and take $A - A' \subset X$, where $A \cong (k,8) MSC'$ and $A' \cong (k,8) Int(MSC')$. Let $f : A \rightarrow f(A) \subset Y$ be a $k$-isomorphism. Remove $A'$ and $f(A')$ from $X$ and $Y$, respectively. Then, the disjoint union of $X'$ and $Y'$ induced from the identification $x$ with $f(x) \in Y'$ for all $x \in A - A'$ is taken, denoted by $X\sharp Y$, where $X' = X - A'$, $Y' = Y - f(A')$ and any two points $p \in X' \subset X\sharp Y$ and $q \in Y' \subset X\sharp Y$ with $p,q \notin X\sharp Y - f(A - A')$ are not 26-adjacent in $X\sharp Y$.

Remark 4.1. In relation with the conditions (a) and (b) of (1), and (b) of (2) in Definition 5, we need the statement that any two points $p \in X' \subset X\sharp Y$ and $q \in Y' \subset X\sharp Y$ with $p,q \notin X\sharp Y - f(A - A')$ are not 26-adjacent in $X\sharp Y$ of Definition 6.

In order to show that a digital connected sum is essentially used in establishing a monoid structure of the set of $k$-isomorphism classes of simple closed $k$-surfaces in $SC_k$, $k \in \{18, 26\}$, we use the following:

Example 4.2. (1) $MSS_{26} \sharp MSS'_{26} \cong_{26} MSS_{26}$.
(2) $MSS'_{26} \sharp MSS''_{26} \cong_{26} MSS''_{26}$.

Proof: (1) We can consider $MSS_{26} \sharp MSS'_{26}$ with 26-adjacency in $(Z^3, 26, 6, MSS_{26} \sharp MSS'_{26})$ so that $MSS_{26} \sharp MSS'_{26} \cong_{26} MSS_{26}$ [9]. Precisely, take two subsets, $\{p_0, p_1, p_3, p_5, p_7\} := A \subset MSS_{26}$ (see Figure 1) and $\{c_0, c_1, c_2, c_3, c_4\} := B \subset MSS'_{26}$ (see Figure 1) which are 26-isomorphic to each other. Then, consider a 26-isomorphism $f : A \rightarrow B$ such that

\[ f(p_0) = c_0, f(p_1) = c_1, f(p_3) = c_2, f(p_5) = c_3, f(p_7) = c_4 \]

and remove the two points $p_0 \in MSS_{26}$ and $c_0 = f(p_0) \in MSS'_{26}$.

Gluing the two remaining sets $MSS_{26} - \{p_0\}$ and $MSS'_{26} - \{c_0\}$, we obtain $MSS_{26} \sharp MSS'_{26}$ by using the map $f$ so that $MSS_{26} \sharp MSS'_{26}$ is still 26-isomorphic to the space $MSS_{26}$.

By the same method as above, we obtain $MSS_{18} \sharp MSS'_{18} \cong_{18} MSS_{18}$ is also established with 18-adjacency in $(Z^3, 18, 6, MSS_{18} \sharp MSS'_{18})$.

(2) By the same method as Example 4.2(1), the proof is completed.

By the same method as above, we obtain that $MSS_{26} \sharp MSS_{26}$ is another simple closed 26-surface. While there are many types of $MSS_{26} \sharp MSS_{26}$, those are 26-isomorphic to each other.

Consequently, we obtain the following:

Theorem 4.3. Let $X$ and $Y$ be simple closed $k$-surfaces in $SC_k$, $k \in \{18, 26\}$. Then $X\sharp Y$ is a simple closed $k$-surface in $SC_k$. 

Let $X$, $Y$, and $Z$ be simple closed $k$-surfaces in $SC_k$, $k \in \{18, 26\}$. Even though $X\# Y$ and $(X\# Y)\# Z$ need not be equal to $Y\# X$ and $X\# (Y\# Z)$, respectively, $X\# Y$ and $(X\# Y)\# Z$ are $k$-isomorphic to $Y\# X$ and $X\# (Y\# Z)$, respectively. Thus we observe that the set of $k$-isomorphism classes of simple closed $k$-surfaces in $SC_k$ forms a commutative monoid with an operation induced from the digital connected sum of Definition 6. For a simple closed $k$-surface $X$ in $SC_k$, consider the $k$-isomorphism class of $X$, $k \in \{18, 26\}$, i.e.,

$$[X] := \{X' | X \approx_k X'\}.$$

**Lemma 4.4.** Let $X, Y, Z, \text{ and } W$ be simple closed $k$-surfaces in $SC_k$, $k \in \{18, 26\}$. If $X \approx_k Y$ and $Z \approx_k W$, then $X\# Z \approx_k Y\# W$ in $SC_k$.

**Proof:** Let $h_1 : X \rightarrow Y$ be a $k$-isomorphism and let $h_2 : Z \rightarrow W$ be a $k$-isomorphism. Since each of $X, Y, Z, \text{ and } W$ has a subset $A \approx (k, 8)\text{ Int}(MSC_8')$, we obtain both $X\# Z$ and $Y\# W$ with $k$-adjacency, $k \in \{18, 26\}$.

For any $k$-isomorphism, its restriction map on any subset of the domain of the given $k$-isomorphism is also a $k$-isomorphism [14].

For $A \subset X$, consider $f : A \rightarrow f(A) \subset Z$ which is a $k$-isomorphism of Definition 6 related to $X\# Z$, and $i_1 : X - A' \rightarrow X\# Z$ which is an inclusion map, where $A' \approx (k, 8)\text{ Int}(MSC_8')$, and $A' \subset A$ and further, $i_2 : Z - f(A') \rightarrow X\# Z$ which is an inclusion map.

Besides, for $A \subset Y$ consider $g : A \rightarrow g(A) \subset W$ which is a $k$-isomorphism of Definition 6 related to $Y\# W$ and further, $j_1 : Y - A' \rightarrow Y\# W$ which is an inclusion map, and $j_2 : W - g(A') \rightarrow Y\# W$ which is an inclusion map.

Then we have a map $h : X\# Z \rightarrow Y\# W$ defined by

$$h(t) = \begin{cases} 
    j_1 \circ h_1|_{X - A'} \circ i_1^{-1}(t) & \text{if } t \in X - A' \subset X\# Z; \\
    j_2 \circ h_2|_{Z - f(A')} \circ i_2^{-1}(t) & \text{if } t \in Z - f(A') \subset X\# Z,
\end{cases}$$

where $A' \approx (k, 8)\text{ Int}(MSC_8')$ and $A' \subset A$. Then $h$ is a $k$-isomorphism, which means that $X\# Z \approx_k Y\# W$.

By Lemma 4.4 we obtain the following:

**Definition 7.** Let $X$ and $Y$ be simple closed $k$-surfaces in $SC_k$, $k \in \{18, 26\}$. Then we define $[X] \cdot [Y] = [X\# Y]$.

By Definition 7, Remark 3.1, and Lemma 4.4, we obtain the following:
**Theorem 4.5.** The set of \(k\)-isomorphism classes of simple closed \(k\)-surfaces in \(SC_k\) is a commutative monoid with the ‘\(\cdot\)’ operation in Definition 7, \(k \in \{18, 26\}\).

**Proof:** Let us prove the following: Let \(X, Y,\) and \(Z\) be simple closed \(k\)-surfaces in \(SC_k\), \(k \in \{18, 26\}\), then we suffice to prove the following:

1. \(((X \cdot [Y]) \cdot Z) = [X] \cdot ([Y] \cdot [Z]).\)
2. \([MSS'_k] \cdot [X] = [X]\) and \([X] \cdot [MSS'_k] = [X].\)
3. \([X] \cdot [Y] = [Y] \cdot [X].\)

Let us now prove (1). We suffice to prove that \((X\sharp Y)\sharp Z \approx_k X\sharp(Y\sharp Z),\)

\(k \in \{18, 26\}\). By Definition 6, consider a subset \(A \subset X, Y,\) and \(Z\) such that \(A \approx_{(k,8)} MSC^e_{8}\). While \((X\sharp Y)\sharp Z\) need not be equal to \(X\sharp(Y\sharp Z),\) they are \(k\)-isomorphic to each other by the similar method as that of Lemma 4.4, which proves the assertion (1).

2. Since \(MSS'_kX \approx_k X\approx_k X\sharp MSS'_k\) via \(A \subset MSS'_k \approx_{(k,8)} MSC^e_{8}\) in \(X, k \in \{18, 26\}\), which proves the assertion (2).

3. Obviously, by Definition 6, consider two \(k\)-isomorphisms \(f : X \to Y\) and \(f^{-1} : f(A) \to A.\) Then, \(X\sharp Y\) is \(k\)-isomorphism to \(Y\sharp X,\) which proves the assertion (3).

By Theorem 4.5 (2), it turns out that \([MSS'_k]\) acts the identity element under the operation ‘\(\cdot\)’ of Definition 6, \(k \in \{18, 26\}\).

5. **Commutative monoid of the set of \(k\)-homotopy classes of closed \(k\)-surfaces**

The notion of \(k\)-homotopy equivalence has been introduced in [11] and has been used in classifying discrete objects with a \(k\)-homotopy equivalence; however there are insufficient presentations of some topics in [11]. Thus, the paper [23] contains the corrected one.

**Definition 8.** [11] (see also [23]) For two discrete topological spaces with \(k\)-adjacency \((X, k)\) and \((Y, k)\) in \(Z^n\), if there are \(k\)-continuous maps \(h : X \to Y\) and \(l : Y \to X\) such that \(l \circ h \simeq_k 1_x\) and \(h \circ l \simeq_k 1_y,\) then the map \(h : X \to Y\) is called a (digital) \(k\)-homotopy equivalence. And we use the notation \(X \simeq_{k,h,e} Y.\)

In Section 5, we still need to take the subset \(A \approx_{(k,8)} MSC^e_{8}\) to establish a commutative monoid of the set of \(k\)-homotopy equivalence classes of closed \(k\)-surfaces in \(Z^3\) with an operation derived from a digital connected sum of Definition 6.

Unlike the digital connected sum of Definition 6, there are some difficulties in establishing a digital connected sum of two closed \(k\)-surfaces \(X\)
and $Y$ which are not simple because there may not be subsets $A$ in both $X$ and $Y$ such that $A$ is $(k, 8)$-isomorphism to $MSC^*_k$. Furthermore, we may also meet an obstacle to the establishment of $X \sharp Y \sharp Z$ for some closed $k$-surfaces $X, Y,$ and $Z$ in $\mathbb{Z}^3$. Thus, in this section we consider the set of only closed $k$-surfaces $X \subset \mathbb{Z}^3$ having a subset $A \subset X$ such that $A \equiv_{N_k(x, 1)} (k, 8) MSC^*_k$ and establishing the associativity of the commutative monoid of the set of $k$-homotopy equivalence classes of closed $k$-surfaces in $\mathbb{Z}^3$, $k \in \{18, 26\}$. Then we denote by $CS_k$ the above set. Some $k$-homotopic properties of $X \in CS_k$ are now investigated in relation with the digital connected sum of Definition 6.

In $CS_k$, for a closed $k$-surface $X$, consider the $k$-homotopy equivalence class of $X$ as follows.

$$[X] := \{X' | X \approx_{k, h, e} X'\}.$$  

Using both an argument similar to that given for the proof of Lemma 4.4, Remark 4.1, and a $k$-homotopy equivalence instead of a $k$-isomorphism of Lemma 4.4, we obtain the following:

**Lemma 5.1.** In $\mathbb{Z}^3$, let $X, Y, Z,$ and $W$ be spaces in $CS_k$. If $X \approx_{k, h, e} Y$ and $Z \approx_{k, h, e} W$, then $X \sharp Z \approx_{k, h, e} Y \sharp W$.

By Lemma 5.1 and Definitions 6 and 7, we obtain that for $X, Y \in CS_k$, we define $[X] \cdot [Y]$ to be $[X \sharp Y]$.

Obviously, for $X, Y,$ and $Z$ in $CS_k$, $X \sharp Y$ and $(X \sharp Y) \sharp Z$ need not be equal to $Y \sharp X$ and $X \sharp (Y \sharp Z)$, respectively. Meanwhile, we obtain the following:

**Theorem 5.2.** Let $X, Y,$ and $Z$ be closed $k$-surfaces in $CS_k, k \in \{18, 26\}$. Then we obtain the following:

(1) $([X] \cdot [Y]) \cdot [Z] = [X] \cdot ([Y] \cdot [Z]), k \in \{18, 26\}$.

(2) $[MSS^*_k] \cdot [X] = [X] \cdot [MSS^*_k] = [X]$.

(3) $[X] \cdot [Y] = [Y] \cdot [X]$.

**Proof:**

(1) Since $(X \sharp Y) \sharp Z$ is $k$-homotopy equivalent to $X \sharp (Y \sharp Z)$, $k \in \{18, 26\}$, the proof is completed.

(2) Since $MSS^*_k \sharp X \approx_{k, h, e} X \approx_{k, h, e} X \sharp MSS^*_k$ by using $A(\subset MSS^*_k) \approx (k, 8) MSC^*_k \subset X$, $k \in \{18, 26\}$, the proof is completed.

(3) Obviously, $X \sharp Y$ and $Y \sharp X$ are $k$-homotopy equivalent to each other, the proof is completed.

By Theorem 5.2(2), it turns out that $[MSS^*_k]$ is the identity element under the operation $\cdot$ in Definition 7, $k \in \{18, 26\}$.  


Remark 5.3. (Correcting) In [21], since the two objects $U_1$ of Figure 6 and $U_1$ of Figure 7 are misprinted at the point $(0,0) \in \mathbb{Z}^2$. Thus they can be corrected, as follows (see Figure 2). With the same criterion, the objects $E_1$ of Figure 1 in [20] should be corrected at the point $(0,0) \in \mathbb{Z}^2$ (motivated from Figure 4 of [12]).

\textbf{Figure 2. Correction of objects in [20, 21]}

References


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