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THE KRONECKER FUNCTION RING OF THE RING $D[X]_{N_*}$

Gyu Whan Chang

Abstract. Let D be an integrally closed domain with quotient field K, $*$ be a star operation on D, X, Y be indeterminates over D, $N_* = \{f \in D[X] \mid (c_D(f))^* = D\}$ and $R = D[X]_{N_*}$. Let b be the b-operation on R, and let $*_c$ be the star operation on D defined by $I^{*_c} = (ID[X]_{N_*})^b \cap K$. Finally, let $Kr(R, b)$ (resp., $Kr(D, *_c)$) be the Kronecker function ring of R (resp., D) with respect to Y (resp., X, Y). In this paper, we show that $Kr(R, b) \subseteq Kr(D, *_c)$ and $Kr(R, b)$ is a kfr with respect to $K(Y)$ and X in the notion of [2]. We also prove that $Kr(R, b) = Kr(D, *_c)$ if and only if D is a P$*$MD. As a corollary, we have that if D is not a P$*$MD, then $Kr(R, b)$ is an example of a kfr with respect to $K(Y)$ and X but not a Kronecker function ring with respect to $K(Y)$ and X.

1. Introduction

Let D be an integral domain with quotient field K. An *overring* of D means a ring between D and K. Let $\{X_\alpha\}$ be a nonempty set of indeterminates over D. We denote by $c_D(h)$ the fractional ideal of D generated by the coefficients of a polynomial $h \in K[\{X_\alpha\}]$. Let $*$ be an *e.a.b.* star operation on D (Definitions related to star operations are reviewed in the sequel). The *Kronecker function ring* (KFR) of D with respect to $*$ and $\{X_\alpha\}$ is the ring

$$Kr(D, *) = \left\{ \frac{f}{g} \mid f, g \in D[\{X_\alpha\}] \text{ and } c_D(f) \subseteq (c_D(g))^* \right\}.$$

It is well known that $Kr(D, *)$ is a Bezout domain, $Kr(D, *) \cap K = D$ and $IKr(D, *) \cap K = I^{*_f}$ for all nonzero ideals I of D [8, Theorem 37.2]. An integral domain S with quotient field $K(\{X_\alpha\})$ is called a KFR with respect to K and $\{X_\alpha\}$ if there are an integral domain D with quotient field K and an *e.a.b.* star operation $*$ on D such that $S = Kr(D, *)$.

Let X be an indeterminate over D. In [2], the authors studied when a Bezout domain is a KFR. In particular, they introduced the notion of kfr's, which is

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a proper subclass of Bezout domains containing KFR's. As in [2], we say that an integral domain S is a *kfr* (*with respect to* K *and* X) in case $S = Kr(R, *)$, where R is a subring of $K(X)$ having quotient field F, $K(X) = F(Y)$ for some indeterminate Y over F, $*$ is an *e.a.b.* star operation on R, and $Kr(R, *)$ is constructed with respect to Y. It is clear that the KFR $Kr(D, *)$ of D with respect to an *e.a.b.* star operation $*$ and X is a kfr with respect to K and X (by choosing $F = K$ and $Y = X$) and kfr's are Bezout domains. But, each of the implications is not reversible. For example, if X and Y are indeterminates over the field $\mathbb{Q}$ of rational numbers, then $\mathbb{Q}[X]$ is a Bezout domain but not a kfr [2, Proposition 2.3] and $\mathbb{Q}[X](Y)$ is a kfr but not a KFR [2, Example 2.2].

Let $*$ be a star operation on an integrally closed domain D, X, Y be indeterminates over D, $N_* = \{f \in D[X] \mid (c_D(f))^* = D\}$ and $R = D[X]_{N_*}$. Let b be the b-operation on R, and let $*_c$ be the star operation on D defined by $I^{*_c} = (ID[X]_{N_*})^b \cap K$. Finally, let $Kr(R, b)$ (resp., $Kr(D, *_c)$) be the KFR of R (resp., D) with respect to b and Y (resp., $*_c$ and X, Y). In this paper, we show that $Kr(R, b) \subseteq Kr(D, *_c)$ and $Kr(R, b)$ is a kfr with respect to $K(Y)$ and X. We prove that $Kr(R, b) = Kr(D, *_c)$ if and only if D is a P$*$MD. We also show that if D^{*_c} is the KFR of D with respect to $*_c$ and X, then $D^{*_c}(Y) = Kr(D, *_c)$. As a corollary, we have that if D is not a P$*$MD, then $Kr(R, b)$ is a kfr with respect to $K(Y)$ and X but not a KFR with respect to $K(Y)$ and X.

Definitions related to star operations

Let $\mathbf{F}(D)$ (resp., $\mathbf{f}(D)$) be the set of nonzero (resp., nonzero finitely generated) fractional ideals of D. A map $* : \mathbf{F}(D) \to \mathbf{F}(D)$, $I \mapsto I^*$, is called a *star operation on* D if the following three conditions are satisfied for all $0 \neq a \in K$ and $I, J \in \mathbf{F}(D)$: (i) $(aD)^* = aD$ and $(aI)^* = aI^*$, (ii) $I \subseteq I^*$ and if $I \subseteq J$ then $I^* \subseteq J^*$, and (iii) $(I^*)^* = I^*$. Given a star operation $*$ on D, we can construct a new star operation $*_f$ as follows: $I^{*_f} = \cup\{J^* \mid J \in \mathbf{f}(D) \text{ and } J \subseteq I\}$ for all $I \in \mathbf{F}(D)$. Clearly, $(*_f)_f = *_f$ and $I^* = I^{*_f}$ for all $I \in \mathbf{f}(D)$. Examples of the most well-known star operations include the v-, t-, and d-operations. The v-operation is defined by $I^v = (I^{-1})^{-1}$, where $I^{-1} = \{x \in K \mid xI \subseteq D\}$, and the t-operation is defined by $t = v_f$. The d-operation is just the identity function on $\mathbf{F}(D)$, i.e., $I^d = I$ for all $I \in \mathbf{F}(D)$; so $d_f = d$.

A star operation $*$ on D is said to be of *finite character* if $*_f = *$; hence $*_f$ is of finite character. An $I \in \mathbf{F}(D)$ is called a $*$*-ideal* if $I^* = I$. A $*$-ideal is called a *maximal* $*$*-ideal* if it is maximal among proper integral $*$-ideals. Let $*$-Max(D) be the set of maximal $*$-ideals of D. An $I \in \mathbf{F}(D)$ is said to be $*$-*invertible* if $(II^{-1})^* = D$, while D is a *Prüfer* $*$*-multiplication domain* (P$*$MD) if each nonzero finitely generated ideal of D is $*_f$-invertible; equivalently, D_P is a valuation domain for all $P \in *_f$-Max(D) [9, Theorem 1.1]. An overring R of D is said to be $*$*-linked over* D if $I^* = D$ implies $(IR)^v = R$ for any $I \in \mathbf{F}(D)$ [3, Proposition 3.2].

A star operation $*$ on D is said to be an *endlish arithmetisch brauchbar* (*e.a.b.*) star operation if $(AB)^* \subseteq (AC)^*$ implies $B^* \subseteq C^*$ for all $A, B, C \in \mathbf{f}(D)$. Clearly, $*$ is an *e.a.b.* star operation if and only if $*_f$ is an *e.a.b.* star operation. We know that if D admits an *e.a.b.* star operation, then D is integrally closed [8, Corollary 32.8]. Conversely, suppose that D is integrally closed, and define

$$I^b = \cap\{IV \mid V \text{ is a valuation overring of } D\}$$

for each $I \in \mathbf{F}(D)$. Then the mapping $b : \mathbf{F}(D) \to \mathbf{F}(D)$, given by $I \mapsto I^b$, is an *e.a.b.* star operation on D [8, Theorem 32.5]. More generally, Chang proved that, for a star operation $*$ on an integrally closed domain D, if we set

$$I^{*_c} = \cap\{IV \mid V \text{ is a } *\text{-linked valuation overring of } D\},$$

then $*_c$ is an *e.a.b.* star operation of finite character on D [4, Lemma 3.1] and $I^{*_c} = (ID[X]_{N_*})^b \cap K$ for all $I \in \mathbf{F}(D)$, where b is the b-operation on $D[X]_{N_*}$, [4, Corollary 3.4] (or see Lemma 1). For more on KFR's, see [8, §32-34] or [7].

2. Main results

Throughout D is an integral domain with quotient field K, $*$ is a star operation on D, X is an indeterminate over D, $N_* = \{f \in D[X] \mid (c_D(f))^* = D\}$ and $R = D[X]_{N_*}$. We say that $D[X]_{N_*}$ is the ($*$-)*Nagata ring of* D; in particular, if $* = d$, then $D[X]_{N_*}$ is denoted by $D(X)$ and called the Nagata ring of D. It is easy to see that if $*$ is *e.a.b.*, then $D[X]_{N_*} \subseteq Kr(D, *)$, and equality holds if and only if D is a P$*$MD (cf. [4, Theorem 2.2]).

Note that if D is integrally closed, then $D[X]_{N_*}$ is also integrally closed; hence the b-operation on $D[X]_{N_*}$ is well defined.

Lemma 1. *Suppose that D is integrally closed. Let b be the b-operation on $D[X]_{N_*}$, and define $I^{*_c} = (ID[X]_{N_*})^b \cap K$ for all $I \in \mathbf{F}(D)$.*

(1) *The mapping $*_c : \mathbf{F}(D) \to \mathbf{F}(D)$, given by $I \mapsto I^{*_c}$, is an e.a.b. star operation of finite character on D and $*_f$-$\mathrm{Max}(D) = *_c$-$\mathrm{Max}(D)$.*
(2) *$(ID[X]_{N_*})^b = I^{*_c}D[X]_{N_*}$ for all $I \in \mathbf{F}(D)$.*
(3) *$I^{*_c} = IKr(D[X]_{N_*}, b) \cap K$ for all $I \in \mathbf{F}(D)$.*

Proof. (1) [4, Lemma 3.1 and Corollary 3.4].

(2) Clearly, $I^{*_c}D[X]_{N_*} \subseteq (ID[X]_{N_*})^b$. Conversely, if $\frac{f}{g} \in (ID[X]_{N_*})^b$ with $f \in K[X]$ and $g \in N_*$, then $f \in (ID[X]_{N_*})^b = \cap\{IW \mid W$ is a valuation overring of $D[X]_{N_*}\}$. Hence $c_D(f) \subseteq (\cap IW) \cap K = \cap(IW \cap K) = \cap I(W \cap K) = I^{*_c}$ [4, Lemma 3.3]. So $f \in c_D(f)D[X]_{N_*} \subseteq I^{*_c}D[X]_{N_*}$, and hence $\frac{f}{g} \in I^{*_c}D[X]_{N_*}$. Thus $(ID[X]_{N_*})^b = I^{*_c}D[X]_{N_*}$.

(3) Note that $D[X]_{N_*} \subseteq Kr(D[X]_{N_*}, b)$; so $IKr(D[X]_{N_*}, b)$ is the extension of $ID[X]_{N_*}$. Hence $IKr(D[X]_{N_*}, b) \cap K(X) = (ID[X]_{N_*})^b$ [8, Theorem 32.7(c)]. Thus $IKr(D[X]_{N_*}, b) \cap K = (IKr(D[X]_{N_*}, b) \cap K(X)) \cap K = (ID[X]_{N_*})^b \cap K = I^{*_c}$ by definition. $\square$

We next give the main result of this paper.

Theorem 2. *Let X, Y be indeterminates over an integrally closed domain D. Let b be the b-operation on $R = D[X]_{N_*}$, and let $Kr(R, b)$ (resp., $Kr(D, *_c)$) be the KFR of R (resp., D) with respect to b and Y (resp., $*_c$ and X, Y).*

(1) *$Kr(R, b) \subseteq Kr(D, *_c)$.*
(2) *$Kr(R, b) \cap K(Y) = \{\frac{f}{g} \mid f, g \in D[Y], g \neq 0$ and $c_D(f) \subseteq c_D(g)^{*_c}\}$, which is the KFR of D with respect to $*_c$ and Y.*
(3) *$Kr(R, b)$ is a kfr with respect to $K(Y)$ and X.*

Proof. (1) Let $T = \{\frac{f}{g} \mid f, g \in D[X], g \neq 0$, and $c_D(f) \subseteq c_D(g)^{*_c}\}$, i.e., T is the KFR of D with respect to $*_c$ and X; so T is a Bezout domain. Clearly, $R \subseteq T$, and hence each valuation overring of T is an overring of R. If $\frac{f}{g} \in Kr(R, b)$, where $0 \neq f, g \in R[Y]$, then

$$\begin{aligned} c_R(f)^b \subseteq c_R(g)^b &\Leftrightarrow c_R(f)W \subseteq c_R(g)W \text{ for all valuation overrings } W \text{ of } R \\ &\Rightarrow c_T(f)W \subseteq c_T(g)W \text{ for all valuation overrings } W \text{ of } T \\ &\Leftrightarrow c_T(f) \subseteq c_T(g) \text{ (since } T \text{ is a Bezout domain)} \\ &\Leftrightarrow \frac{f}{g} \in T(Y). \end{aligned}$$

Thus $Kr(R, b) \subseteq T(Y)$. Next, we show that $T(Y) \subseteq Kr(D, *_c)$, and hence $Kr(R, b) \subseteq Kr(D, *_c)$. Clearly, $T[Y] \subseteq Kr(D, *_c)$; so it suffices to show that if $u \in T[Y]$ with $c_T(u) = T$, then $\frac{1}{u} \in Kr(D, *_c)$.

Let $u = \frac{g_0}{f_0} + \frac{g_1}{f_1}Y + \cdots + \frac{g_n}{f_n}Y^n$, where $g_i, f_i \in D[X]$ with $f_i \neq 0$ and $c_D(g_i) \subseteq c_D(f_i)^{*_c}$. Set $f = f_0 f_1 \cdots f_n$ and $\hat{f}_i = \frac{f}{f_i}$; so $u = \frac{1}{f}(g_0\hat{f}_0 + g_1\hat{f}_1 Y + \cdots + g_n\hat{f}_n Y^n)$. Let $h_i = g_i\hat{f}_i$, and note that $c_T(u) = (\frac{h_0}{f}, \frac{h_1}{f}, \ldots, \frac{h_n}{f})T$; so $(h_0, h_1, \ldots, h_n)T = fT$. Set

$$h = h_0 + h_1 X^{\deg h_0 + 1} + \cdots + h_n X^{\deg h_0 + \cdots + \deg h_n + n}.$$

Then $(h_0, h_1, \ldots, h_n)T = hT$ (see the proof of [8, Theorem 32.7(b)]). Hence $fT = hT$, and thus $c_D(f)^{*_c} = c_D(h)^{*_c}$. Note also that $c_D(h) = c_D(h_0) + c_D(h_1) + \cdots + c_D(h_n) = c_D(h_0 + h_1 Y + \cdots + h_n Y^n) = c_D(fu)$. Hence $c_D(f)^{*_c} = c_D(h)^{*_c} = c_D(fu)^{*_c}$, and thus $\frac{1}{u} = \frac{f}{fu} \in Kr(D, *_c)$.

(2) If $f, g \in D[Y]$ with $g \neq 0$, then $c_R(f) = c_D(f)R$ and $c_R(g) = c_D(g)R$. Thus the result is an immediate consequence of the definition of $*_c$.

(3) This just follows from the definition of kfr's. □

Let $\{X_\alpha\}$ be a nonempty set of indeterminates over D, and let $N = \{f \in D[\{X_\alpha\}] \mid c_D(f)^* = D\}$. Then $\{P[\{X_\alpha\}]_N \mid P \in *_f\text{-Max}(D)\}$ is the set of maximal ideals of $D[\{X_\alpha\}]_N$ [10, Proposition 2.1], and hence by [8, Theorem 4.10]

$$\begin{aligned} D[\{X_\alpha\}]_N &= \cap_{P \in *_f\text{-Max}(D)} (D[\{X_\alpha\}]_N)_{P[\{X_\alpha\}]_N} \\ &= \cap_{P \in *_f\text{-Max}(D)} D[\{X_\alpha\}]_{P[\{X_\alpha\}]}. \end{aligned}$$

Also, if D is integrally closed, then D is a P$*$MD if and only if $D[\{X_\alpha\}]_N$ is a Prüfer domain, if and only if $D[\{X_\alpha\}]_N = \{\frac{f}{g} \mid f, g \in D[\{X_\alpha\}], g \neq 0$, and $c_D(f) \subseteq c_D(g)^{*_c}\}$ [4, Theorems 2.2 and 3.7].

We next give a new characterization of P$*$MD's.

Corollary 3. *Let the notation be as in Theorem 2. Then* $Kr(R, b) = Kr(D, *_c)$ *if and only if* D *is a P$*$MD.*

Proof. Let $T = \{\frac{f}{g} \mid f, g \in D[X], g \neq 0$, and $c_D(f) \subseteq c_D(g)^{*_c}\}$. Then $Kr(D, *_c) \cap K(X) = T$. Hence if $Kr(R, b) = Kr(D, *_c)$, then $R = Kr(R, b) \cap K(X) = T$, and thus D is a P$*$MD.

Conversely, assume that D is a P$*$MD, and let $N = \{f \in D[X, Y] \mid c_D(f)^* = D\}$ (note that $N = \{f \in D[X, Y] \mid c_D(f)^{*_c} = D\}$ because $*_f$-Max$(D) = *_c$-Max(D) by Lemma 1). First, by the remark before Corollary 3, we have

$$\begin{aligned} Kr(R, b) = R(Y) &= (D[X]_{N_*})(Y) \\ &= \cap_{P \in *_f\text{-Max}(D)} (D[X]_{N_*})[Y]_{(P[X]_{N_*})[Y]} \\ &= \cap_{P \in *_f\text{-Max}(D)} D[X, Y]_{P[X,Y]} \\ &= D[X, Y]_N. \end{aligned}$$

Next, we show that $D[X, Y]_N = Kr(D, *_c)$, and thus $Kr(R, b) = Kr(D, *_c)$. Clearly, $D[X, Y]_N \subseteq Kr(D, *_c)$. For the reverse containment, let $\frac{f}{g} \in Kr(D, *_c)$, where $f, g \in D[X, Y]$ with $c_D(f) \subseteq c_D(g)^{*_c}$. Since D is a P$*$MD, $c_D(g)$ is $*_f$-invertible, and hence $c_D(g)$ is $*_c$-invertible by Lemma 1(1). So there exists a $0 \neq h \in K[X, Y]$ such that $c_D(g)^{-1} = c_D(h)^v = c_D(h)^{*_c}$ (cf. [6, Lemma 2.1]). Hence $D = (c_D(g)c_D(h))^{*_c} = c_D(gh)^{*_c}$ by the Dedekind-Mertens Lemma [8, Theorem 28.1], and so $gh \in N$. Thus $\frac{f}{g} = \frac{fh}{gh} \in D[X, Y]_N$. $\square$

Let $Kr(D, *)$ be the KFR of D with respect to $*$ and X. If T is an overring of $Kr(D, *)$, then T is also a KFR. In fact, if we set $I^{*_1} = IT \cap K$ for all $I \in \mathbf{F}(T \cap K)$, then $*_1$ is an *e.a.b.* star-operation on $T \cap K$ [5, Lemma 1(2)] and $T = Kr(T \cap K, *_1)$ (see the proof of [5, Theorem 3]).

Corollary 4. *Let the notation be as in Theorem 2. Then each overring of* $Kr(R, b)$ *is a kfr with respect to* $K(Y)$ *and* X.

Proof. Let T be an overring of $Kr(R, b)$, and let $A^\star = AT \cap K(X)$ for all $A \in \mathbf{F}(T \cap K(X))$. Then $\star$ is an *e.a.b.* star-operation on $T \cap K(X)$ and $T = Kr(T \cap K(X), \star)$. Clearly, $T \cap K(X)$ is a subring of $K(Y)(X)$, $K(X)$ is the quotient field of $T \cap K(X)$, and $K(X)(Y) = K(Y)(X)$. Thus T is a kfr with respect to $K(Y)$ and X. $\square$

The next result was proved by Anderson [1, Lemma] when $* = d$ and by Kang [10, Theorem 2.10] when $* = v$.

Proposition 5. *Let* $\{X_\alpha\}$ *and* $\{Y_\beta\}$ *be two disjoint nonempty sets of indeterminates over* D, *and let* $N_1 = \{f \in D[\{X_\alpha\}] \mid c_D(f)^* = D\}$ *and* $N_2 =$

$\{f \in D[\{X_\alpha\} \cup \{Y_\beta\}] \mid c_D(f)^* = D\}$. Then $(D[\{X_\alpha\}]_{N_1})(\{Y_\beta\}) = D[\{X_\alpha\} \cup \{Y_\beta\}]_{N_2}$.

Proof. Let $\text{Max}(D[\{X_\alpha\}]_{N_1})$ be the set of maximal ideals of $D[\{X_\alpha\}]_{N_1}$. Then

$$\begin{aligned}(D[\{X_\alpha\}]_{N_1})(\{Y_\beta\}) &= \cap_{M \in \text{Max}(D[\{X_\alpha\}]_{N_1})}((D[\{X_\alpha\}]_{N_1})[\{Y_\beta\}])_{M[\{Y_\beta\}]} \\ &= \cap_{P \in *_f\text{-Max}(D)}((D[\{X_\alpha\}]_{N_1})[\{Y_\beta\}])_{(P[\{X_\alpha\}]_{N_1})[\{Y_\beta\}]} \\ &= \cap_{P \in *_f\text{-Max}(D)} D[\{X_\alpha\} \cup \{Y_\beta\}]_{P[\{X_\alpha\} \cup \{Y_\beta\}]} \\ &= \cap_{P \in *_f\text{-Max}(D)} (D[\{X_\alpha\} \cup \{Y_\beta\}]_{N_2})_{P[\{X_\alpha\} \cup \{Y_\beta\}]_{N_2}} \\ &= D[\{X_\alpha\} \cup \{Y_\beta\}]_{N_2}\end{aligned}$$

by the remark before Corollary 3. □

We next give a KFR analog of Proposition 5.

Corollary 6. *Let $\{X_\alpha\}$ and $\{Y_\beta\}$ be two disjoint nonempty sets of indeterminates over an integrally closed domain D, D^{*_c} be the KFR of D with respect to $*_c$ and $\{X_\alpha\}$, and $Kr(D, *_c)$ be the KFR with respect to $*_c$ and $\{X_\alpha\} \cup \{Y_\beta\}$. Then $D^{*_c}(\{Y_\beta\}) = Kr(D, *_c)$.*

Proof. Put $X = \{X_\alpha\}$ and $Y = \{Y_\beta\}$. Let $\{V_\lambda\}$ be the set of $*$-linked valuation overrings of D; then $D^{*_c} = \cap_\lambda V_\lambda(X)$ and $Kr(D, *_c) = \cap_\lambda V_\lambda(X, Y)$ [8, Theorem 32.11]. Next, note that D^{*_c} is a Prüfer domain; so $D^{*_c}(Y)$ is the KFR of D^{*_c} with respect to Y and the b-operation on D^{*_c}. Also, note that $\{V_\lambda(X)\}$ is the set of valuation overrings of D^{*_c} [8, Theorem 32.10]. Thus $D^{*_c}(Y) = \cap_\lambda V_\lambda(X)(Y) = \cap_\lambda V_\lambda(X, Y) = Kr(D, *_c)$ by [8, Theorem 32.11] and Proposition 5. □

Corollary 7. *Let the notation be as in Theorem 2. If D is not a P$*$MD, then $Kr(R, b)$ is not a KFR with respect to $K(Y)$ and X.*

Proof. Assume to the contrary that $Kr(R, b)$ is a KFR with respect to $K(Y)$ and X. Then there are an integral domain T with quotient field $K(Y)$ and an *e.a.b.* star operation $\star$ on T such that $Kr(R, b) = Kr(T, \star)$ (with respect to X). Hence if we let D^{*_c} be the KFR with respect to $*_c$ and Y, then $D^{*_c} = Kr(R, b) \cap K(Y) = Kr(T, \star) \cap K(Y) = T$ by Theorem 2(2). Note that D^{*_c} is a Bezout domain; so the d-operation d on T is an *e.a.b.* star operation such that $d = \star_f$ and $Kr(T, \star) = T(X) = D^{*_c}(X)$ [8, Theorem 33.4]. But, $D^{*_c}(X) = Kr(D, *_c)$ by Corollary 6, and hence $Kr(R, b) = Kr(D, *_c)$, which implies that D is a P$*$MD by Corollary 3. □

Let b be the b-operation on an integrally closed domain D, and let $\{X_i\}_{i=1}^\infty$ be a set of indeterminates over D, It is well known that if $Kr(D, b)$ is the KFR of D with respect to b and $\{X_i\}_{i=1}^\infty$, then each KFR with respect to K and $\{X_i\}_{i=1}^\infty$ contains $Kr(D, b)$ [8, Corollary 32.14]. We end this paper by showing that if D is not a Prüfer domain then $Kr(D, b)$ contains infinitely many Bezout domains that are not KFR's with respect to K and $\{X_i\}_{i=1}^\infty$.

Example 8. Let D be an integrally closed domain that is not a Prïfer domain, and let $\{X_i\}_{i=1}^{\infty}$ be a set of indeterminates over D. Set $N_k = \{f \in D[X_1, \ldots, X_k] \mid c_D(f) = D\}$, $R_k = D[X_1, \ldots, X_k]_{N_k}$, and let $D(\{X_i\}_{i=1}^{\infty})$ be the Nagata ring of D. Let b_k be the b-operation on R_k, and let $Kr(R_k, b_k)$ be the KFR of R_k with respect to b_k and $\{X_i\}_{i=k+1}^{\infty}$ (here $R_0 = D$ and b_0 is the b-operation on D). Then $D(\{X_i\}_{i=1}^{\infty}) \subsetneq Kr(R_{k+1}, b_{k+1}) \subsetneq Kr(R_k, b_k) \subsetneq Kr(D, b)$ for all $k \geq 1$.

Proof. First, note that for any $k \geq 1$,

$$R_k(\{X_i\}_{i=k+1}^{\infty}) = D(\{X_i\}_{i=1}^{k})(\{X_i\}_{i=k+1}^{\infty}) = D(\{X_i\}_{i=1}^{\infty})$$

by Proposition 5; hence $D(\{X_i\}_{i=1}^{\infty}) \subseteq Kr(R_k, b_k)$. Also, since D is not a Prüfer domain, $D(\{X_i\}_{i=1}^{\infty}) \subsetneq Kr(R_k, b_k)$ (cf. [8, Theorem 33.4]).

Let T_{k+1} be the KFR of R_k with respect to b_k and X_{k+1} and $N = \{f \in R_k[X_{k+1}] \mid c_{R_k}(f) = R_k\}$. Then $R_{k+1} = R_k[X_{k+1}]_N$ by Proposition 5; so $R_{k+1} \subsetneq T_{k+1}$ by Corollary 3, and hence $Kr(R_{k+1}, b_{k+1}) \subsetneq T_{k+1}(\{X_i\}_{i=k+2}^{\infty})$ [8, Theorem 32.15]. By Corollary 6, $T_{k+1}(\{X_i\}_{i=k+2}^{\infty}) = Kr(R_k, b_k)$, and thus $Kr(R_{k+1}, b_{k+1}) \subsetneq Kr(R_k, b_k)$. □

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References

[1] D. D. Anderson, *Some remarks on the ring $R(X)$*, Comment. Math. Univ. St. Paul. **26** (1977/78), no. 2, 137–140.

[2] D. F. Anderson, D. E. Dobbs, and M. Fontana, *Characterizing Kronecker function rings*, Ann. Univ. Ferrara Sez. VII (N.S.) **36** (1990), 1–13.

[3] G. W. Chang, *$*$-Noetherian domains and the ring $D[X]_{N_*}$*, J. Algebra **297** (2006), no. 1, 216–233.

[4] ———, *Prüfer $*$-multiplication domains, Nagata rings, and Kronecker function rings*, J. Algebra **319** (2008), no. 1, 309–319.

[5] ———, *Overrings of the Kronecker function ring $Kr(D, *)$ of a Prufer $*$-multiplication domain D*, Bull. Korean Math. Soc. **46** (2009), no. 5, 1013–1018.

[6] G. W. Chang and J. Park, *Star-invertible ideals of integral domains*, Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8) **6** (2003), no. 1, 141–150.

[7] M. Fontana and K. A. Loper, *An historical overview of Kronecker function rings, Nagata rings, and related star and semistar operations*, Multiplicative ideal theory in commutative algebra, 169–187, Springer, New York, 2006.

[8] R. Gilmer, *Multiplicative Ideal Theory*, Marcel Dekker, Inc., New York, 1972.

[9] E. G. Houston, S. B. Malik, and J. L. Mott, *Characterizations of *-multiplication domains*, Canad. Math. Bull. **27** (1984), no. 1, 48–52.

[10] B. G. Kang, *Prüfer v-multiplication domains and the ring $R[X]_{N_v}$*, J. Algebra **123** (1989), no. 1, 151–170.

Department of Mathematics
University of Incheon
Incheon 402-749, Korea
E-mail address: whan@incheon.ac.kr