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EXTENSION PROBLEM OF SEVERAL CONTINUITIES IN COMPUTER TOPOLOGY

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Abstract. The goal of this paper is to study extension problems of several continuities in computer topology. To be specific, for a set $X \subset \mathbf{Z}^n$ take a subspace (X, T^n_X) induced from the Khalimsky nD space $(\mathbf{Z}^n, T^n)$. Considering (X, T^n_X) with one of the k-adjacency relations of $\mathbf{Z}^n$, we call it a computer topological space (or a space if not confused) denoted by $X_{n,k}$. In addition, we introduce several kinds of k-retracts of $X_{n,k}$, investigate their properties related to several continuities and homeomorphisms in computer topology and study extension problems of these continuities in relation with these k-retracts.

1. Introduction

Let $\mathbf{N}$, $\mathbf{Z}$, $\mathbf{R}$ and $\mathbf{Z}^n$ be the sets of natural numbers, integers, real numbers and points in the Euclidean nD space with integer coordinates, respectively. The *extension problem* well known is also very important in applied topology including digital topology [20], Scott topology [5] and computer topology [7, 10, 15] and further, we can expand the problem with various continuities in [10]. More precisely, for sets $X \subset \mathbf{Z}^{n_0}$ and $Y \subset \mathbf{Z}^{n_1}$ we remind that for a map $f : X \to Y$ a map $F : X' \to Y$ is an extension of f if the restriction map F on $X \subset X'$, *briefly* $F|_X$, is equal to f.

The paper begins with a brief review of some digital topological results of an extension problem. Motivated by the extension problem in [2], the paper [3] established a digital k-retract for studying an extension problem of digital continuity. In digital topology we have usually studied a set $X \subset \mathbf{Z}^n$ with one of the k-adjacency relations of $\mathbf{Z}^n$ (briefly, a digital space) denoted by (X, k).

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For a map $f : (X, k_0) \to (Y, k_1)$, since a preservation of the k_0-connectivity of (X, k_0) into the k_1-one of (Y, k_1) can strongly contribute to the study of a digital space, digital (k_0, k_1)-continuity has been used in studying digital spaces [3, 7, 9, 11, 20].

As another digital topological approach, we have often considered a set $X \subset \mathbf{Z}^n$ to be a subspace induced from the Khalimsky nD space $(\mathbf{Z}^n, T^n)$ denoted by (X, T^n_X). The paper [19] studied an extension problem of a continuous function from a subspace of the Khalimsky nD space to the Khalimsky line. When studying a Khalimsky topological space, a preservation of a digital connectivity is also important. However, if a Khalimsky topological space (X, T^n_X) is not connected from the view point of Khalimsky topology, then a Khalimsky continuous map $f : (X, T^n_X) \to (Y, T^n_Y)$ need not preserve digital connectivity [10]. This is one of the reasons why we go together a Khalimsky topological structure and digital connectivity (see Example 3.3 of the current paper). Thus the recent papers [10, 14, 15] studied a Khalimsky topological space (X, T^n_X) with one of the k-adjacency relations of $\mathbf{Z}^n$. Further, we have developed special kinds of continuities related to the preservation of digital connectivity as well as Khalimsky continuity such as KD-(k_0, k_1)-, (k_0, k_1)- and K-(k_0, k_1)-continuities [10], and KD-(k_0, k_1)-, (k_0, k_1)- and K-(k_0, k_1)-homeomorphisms [10].

By computer topology is now meant the mathematical recognition of a set in $\mathbf{Z}^n$ with some reasonable topological structure and k-adjacency relations of $\mathbf{Z}^n$, *e.g.* a development of tools implementing topological concepts for use in science and engineering.

In this paper, by using several computer topological continuities between Khalimsky n_0D- and n_1D-subspaces, we can study various properties of several kinds of k-retracts in computer topology and further, study extension problems of these continuous maps in relation with several kinds of k-retracts.

This paper is organized as follows. In Section 2 we provide some basic notions. In Section 3 we establish several computer topological continuities and investigate their various properties. In Section 4 several kinds of k-retracts are introduced and studied with several sorts of computer topological homeomorphisms. In Section 5 we study extension problems of these several continuities in relation with these k-retracts. Finally, in Section 6 we conclude the paper with a summary.

2. Preliminaries

In digital topology a set $X \subset \mathbf{Z}^n$ with k-adjacency, *briefly* (X, k), has been usually considered in a quadruple $(\mathbf{Z}^n, k, \bar{k}, X)$, where $n \in \mathbf{N}$, k represents an adjacency relation for X and $\bar{k}$ represents an adjacency relation for $\mathbf{Z}^n - X$ [17, 18, 20]. In this paper a (binary) set $X \subset \mathbf{Z}^n$ can be assumed in $(\mathbf{Z}^n, k, 2n, X)$ or $(\mathbf{Z}^n, 2n, 3^n - 1, X)$ with $k \neq 2n$ except for $(\mathbf{Z}, 2, 2, X)$ because of the *digital connectivity paradox* related to *digital Jordan curve theorem* [18].

In order to study a multi-dimensional set $X \subset \mathbf{Z}^n$ with k-adjacency denoted by (X, k), we need to recall the k-adjacency relations of $\mathbf{Z}^n$ as follows. Motivated by 4- and 8-adjacencies of a 2D digital space, and 6-, 18- and 26-adjacencies of a 3D digital space [18, 20], the k-adjacency relations of $\mathbf{Z}^n$ are induced from the following operator [6] (see also [10]):

For a natural number m with $1 \leq m \leq n$, two distinct points

$$p = (p_1, p_2, \ldots, p_n) \quad \text{and} \quad q = (q_1, q_2, \ldots, q_n) \in \mathbf{Z}^n$$

are $k(m, n)$-(*briefly*, k- or k_m-)adjacent if

- there are at most m indices i such that $|p_i - q_i| = 1$ and
- for all other indices i such that $|p_i - q_i| \neq 1, p_i = q_i$.

In this operator $k := k(m, n)$ is the number of points q that are k-adjacent to a given point p according to the numbers m and n in $\mathbf{N}$, where " :=" means *equal* by definition. Indeed, this $k(m, n)$-adjacency is another presentation of the k-adjacency of [6] and a generalization of the k-adjacency of 2D and 3D digital images in [20]. Consequently, this operator leads to the k-adjacency relations of $\mathbf{Z}^n$ [8] (for more details, see [14]):

Proposition 2.1 ([12])**.** *There are k-adjacency relations of $\mathbf{Z}^n$:*

$$k := k(m, n) = \sum_{i=n-m}^{n-1} 2^{n-i} C_i^n, \ \textit{where}\ C_i^n = \frac{n!}{(n-i)!\ i!}.$$

In this paper each set $X \subset \mathbf{Z}^n$ is considered to be a subspace (X, T_X^n) induced from the Khalimsky nD space $(\mathbf{Z}^n, T^n)$. Indeed, motivated by the Alexandroff topological space [1], *Khalimsky line topology* on $\mathbf{Z}$ is induced from the family of the set $\{\{2n+1\}, \{2m-1, 2m, 2m+1\} \mid m, n \in \mathbf{Z}\}$, which is a basis of the Khalimsky line topology on $\mathbf{Z}$. For $\{a, b\} \subset \mathbf{Z}$ with $a \lneq b$, $[a, b]_{\mathbf{Z}} = \{a \leq n \leq b \mid n \in \mathbf{Z}\}$ with 2-adjacency is considered to be a subspace of $(\mathbf{Z}, T)$ or a set related to the presentation of a sequence in Section 3. Further, the *product topology* on $\mathbf{Z}^n$ is derived from the Khalimsky line topology on $\mathbf{Z}, n \geq 2$. Then the topology on $\mathbf{Z}^n$ is called the *product Khalimsky topology* on $\mathbf{Z}^n$, and we use the notation $(\mathbf{Z}^n, T^n)$. A topological space is said to be a $T_{\frac{1}{2}}$-space if each singleton is either open or closed [4] and further, it turns out that $(\mathbf{Z}, T)$ is a $T_{\frac{1}{2}}$-space. If $n \geq 2$, then the Khalimsky nD space $(\mathbf{Z}^n, T^n)$ is a T_0-space instead of a $T_{\frac{1}{2}}$-space [17].

For the *Khalimsky nD space* $(\mathbf{Z}^n, T^n)$ let us recall that a point $x = (x_1, x_2, \ldots, x_n) \in \mathbf{Z}^n$ is *open* if all coordinates are odd, and *closed* if each of the coordinates is even [17]. These points are called *pure* and the other points in $\mathbf{Z}^n$ is called *mixed*. In all subspaces of $(\mathbf{Z}^n, T^n), n \geq 2$, of Figures 1, 2, 3, 4, 5, 6, 7 and 8, the symbols ■ and • mean a pure closed point and a mixed point, respectively. Besides, a jumbo dot stands for a pure open point.

3. Several continuities in computer topology

Let us begin with a brief review of various continuities and homeomorphisms for studying computer topological spaces of Definition 1. We now recall the following basic terminology. We say that a set of lattice points is *k-connected* if it is not a union of two disjoint non-empty sets that are not k-adjacent to each other [18]. For a digital space (X, k), two points $x, y \in X$ are called *k-connected* if there is a sequence $(x_i)_{i\in[0,m]_{\mathbf{Z}}} \subset X$ such that $x_0 = x$, $x_m = y$ and further, x_i and x_{i+1} are k-adjacent, $i \in [0, m-1]_{\mathbf{Z}}, m \geq 1$ [20]. The number m is called the *length* of this k-path [20]. For an adjacency relation k, a *simple k-path* in X is a sequence $(x_i)_{i\in[0,m]_{\mathbf{Z}}} \subset X$ such that x_i and x_j are k-adjacent if and only if either $i = j \pm 1$ [18]. For (X, k), a point $x \in X$ is called *isolated* if it is not k-connected with any point in X [18]. A *simple closed k-curve with l elements* in $\mathbf{Z}^n$ is said to be a sequence $(w_i)_{i\in[0,l-1]_{\mathbf{Z}}} \subset \mathbf{Z}^n$ such that w_i and w_j are k-adjacent if and only if either $j = i + 1 \pmod l$ or $i = j + 1 \pmod l$ [18] and is denoted by $SC_k^{n,l}$ [8].

For a set $X \subset \mathbf{Z}^n$, let us consider the subspace induced from the Khalimsky nD space $(\mathbf{Z}^n, T^n)$ denoted by (X, T_X^n). Further, we recall the following presentation.

Definition 1 ([10])**.** A space (X, T_X^n) with k-adjacency is called a computer topological space (briefly, a space if not confused) and is denoted by $X_{n,k} := (X, k, T_X^n)$.

Obviously, for a set $Y \subset \mathbf{Z}$, the subspace $(Y, T_Y) \subset (\mathbf{Z}, T)$ is always assumed to have 2-adjacency. For a space $X_{n,k}$ we need two kinds of k-neighborhoods for establishing several continuities in computer topology (see Definitions 2 and 3).

Definition 2 ([7] (see also [10]))**.** Let (X, k) be a digital space, $x, y \in X$, and $\varepsilon \in \mathbf{N}$.

(1) By $l_k(x, y)$ we denote the length of a shortest simple k-path from x to y in X. Further, we define $l_k(x, y) = \infty$ if there is no k-path from x to y.

(2) By $N_k(x, \varepsilon)$ we denote the set $\{y \in X : l_k(x, y) \leq \varepsilon\} \cup \{x\}$. This $N_k(x, \varepsilon)$ is called a digital k-neighborhood of x with radius ε [7] (see also [8]).

According to Definition 2, for any $\varepsilon \in \mathbf{N}$ we observe that $N_k(x, \varepsilon) = \{x\}$ if the point x is isolated.

Let us now consider another k-neighborhood from the view point of computer topology.

Definition 3 ([7] (see also [10, 15]))**.** For $X_{n,k}$ we define the following notion.

(1) A subset V of X is called a topological neighborhood of x if there exists $O_x \in T_X^n$ such that $x \in O_x \subseteq V$ as usual.

(2) If the set $N_k(x, \varepsilon)$ in Definition 2 is a (Khalimsky) topological neighborhood of x in (X, T_X^n), then this set is called a (Khalimsky topological) k-neighborhood of x with radius ε and is denoted by $N_k^*(x, \varepsilon)$.

Remark 3.1. The current k-neighborhood $N_k^*(x,\varepsilon) \subset X_{n,k}$ is different from the digital k-neighborhood $N_k(x,\varepsilon)$ of Definition 2 and further, $N_k^*(x,\varepsilon)$ absolutely depends on the computer topological structure of $X_{n,k}$.

Example 3.2. Consider the space $A_{2,k}, k \in \{4,8\}$, where $A = \{x_i \mid i \in [0,11]_{\mathbf{Z}}\}$ in Figure 1. Then some computer topological k-neighborhoods can be considered, as follows.

Since the smallest open set containing the point x_8 is the set $\{x_8, x_9\}$, we obtain the following:

$$N_8^*(x_8,1) = \{x_7,x_8,x_9,x_{10}\}, N_8^*(x_8,2) = \{x_6,x_7,x_8,x_9,x_{10},x_{11}\}, \text{ and}$$
$$N_4^*(x_8,1) = \{x_8,x_9\}.$$

Meanwhile, since the singleton $\{x_0\}$ is an open set and the point x_0 is not 4-connected with any point in A, we obtain $N_4^*(x_0,\varepsilon) = \{x_0\}, \varepsilon \in \mathbf{N}$.

But we obtain $N_4(x_5,1) = \{x_3,x_5\}$ and $N_4(x_5,2) = \{x_2,x_3,x_4,x_5\}$. Since the smallest open set containing the point x_5 is the set $\{x_2,x_3,x_4,x_5,x_6\}$, for any $\varepsilon \in \mathbf{N}$, no $N_4^*(x_5,\varepsilon)$ exists because there is no open set $O_{x_5} \in T_X^2$ such that $O_{x_5} \subset N_4(x_5,\varepsilon)$, where O_{x_5} means an open set including the point x_5.

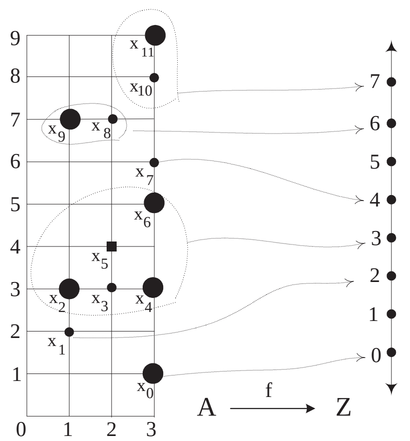

FIGURE 1. Comparison between Khalimsky continuity and digital-$(8,2)$-continuity.

In Khalimsky topology, we can observe that a Khalimsky continuous map $f : X_{n_0,k_0} \to Y_{n_1,k_1}$ need not preserve the k_0-connectivity into the k_1-connectivity [10]. To be specific, as discussed in [10, 15], consider the space $A_{2,8}$ and the map $f : A_{2,8} \to (\mathbf{Z},T)$ in Figure 1. While f is Khalimsky continuous at the point x_8, it cannot preserve the 8-connectedness of $A_{2,8}$ into the 2-one of $(\mathbf{Z},T)$. This is one of the reasons why we study a set $X \subset \mathbf{Z}^n$ with special kinds of continuities comprising both Khalimsky continuity and digital continuity. Indeed, in both digital topology and computer topology, for a map $f : X_{n_0,k_0} \to Y_{n_1,k_1}$ the preservation of the k_0-connectivity into the k_1-one by the map f should be required to study digital spaces and computer topological spaces. Thus various properties of digital continuity have been studied [3, 11, 16, 20] and further, several kinds of computer topological (k_0, k_1)-continuities were introduced in [10]. Let us now recall several continuities in computer topology.

Definition 4 ([10])**.** A function $f : X_{n_0,k_0} \to Y_{n_1,k_1}$ is said to be Khalimsky continuous with digital (k_0, k_1)-continuity (briefly, KD-(k_0, k_1)-continuous) at a point $x \in X_{n_0,k_0}$ if

(1) f is Khalimsky continuous at the point x and

(2) $f(N_{k_0}(x, 1)) \subset N_{k_1}(f(x), 1)$.

Further, we say that a map $f : X_{n_0,k_0} \to Y_{n_1,k_1}$ is KD-(k_0, k_1)-continuous if it is KD-(k_0, k_1)-continuous at every point $x \in X_{n_0,k_0}$.

Motivated by digital continuity of [3, 20], Definition 4(2) is established and called digital (k_0, k_1)-continuity in [6] (see also [8, 11]). It can be used without any limitation of studying a multi-dimensional digital space in $\mathbf{Z}^n, n \in \mathbf{N}$. Further, this digital (k_0, k_1)-continuity can be essentially used for studying the preservation of the *local k_0-property* into the *local k_1-property* [13] as well as the *almost pasting property* [16]. In addition, we can observe that none of the conditions (1) and (2) of Definition 4 implies the other (see Example 3.3) if $n_0, n_1 \in \mathbf{N} - \{1\}$.

Example 3.3. Consider $A_{2,8}$ in Figure 1, where $A = \{x_i\}_{i\in[0,11]_{\mathbf{Z}}} \subset \mathbf{Z}^2$ in Figure 1. Let us consider the map $f : A \to \mathbf{Z}$ (see Figure 1) for which $f(x_0) = 0, f(x_1) = 2, f(x_7) = 4, f(x_8) = f(x_9) = 6, f(A_1) = \{3\}$ and $f(A_2) = \{7\}$, where $A_1 = \{x_2, x_3, x_4, x_5, x_6\}$ and $A_2 = \{x_{10}, x_{11}\}$.

While the map f is Khalimsky continuous, it cannot be digitally $(8, 2)$-continuous at the point $x_8 \in A$ contrary to Definition 4(2) because we obtain $f(N_8(x_8, 1)) \nsubseteq N_2(6, 1)$.

Although KD-(k_0, k_1)-continuity is so helpful to study $X_{n,k}$, it need not be sufficient for the study of a space $X_{n,k}$ in computer topology. Thus, by using a computer topological k-neighborhood, we need to establish another continuity that is different from KD-(k_0, k_1)-continuity as follows.

Definition 5 ([10] (see also [15]))**.** A function $f : X_{n_0,k_0} \to Y_{n_1,k_1}$ is said to be (k_0, k_1)-continuous at a point $x \in X$ if for any $N^*_{k_1}(f(x), \varepsilon) \subset Y$ there is $\delta \in \mathbf{N}$ and $N^*_{k_0}(x, \delta) \subset X$ such that

$$f(N^*_{k_0}(x, \delta)) \subset N^*_{k_1}(f(x), \varepsilon),$$

where for some $\varepsilon \in \mathbf{N}$, $N^*_{k_1}(f(x), \varepsilon)$ is assumed to be existed. Further, we say that a map $f : X_{n_0,k_0} \to Y_{n_1,k_1}$ is (k_0, k_1)-continuous if it is (k_0, k_1)-continuous at every point $x \in X_{n_0,k_0}$.

In Definition 5 if such a neighborhood $N^*_{k_1}(f(x), \varepsilon)$ does not exist in Y_{n_1,k_1}, then we obviously say that the map f cannot be (k_0, k_1)-continuous at the point $x \in X_{n_0,k_0}$. The (k_0, k_1)-continuity of Definition 5 is equivalent to the following presentation [10] (see also [15]):

$$f(N^*_{k_0}(x,r)) \subset N^*_{k_1}(f(x), s),$$

where the number r is the least element of $\mathbf{N}$ such that $N^*_{k_0}(x,r)$ contains an open set including the point x (so $N^*_{k_0}(x,r) = N_{k_0}(x,r)$) and s is the least element of $\mathbf{N}$ such that $N^*_{k_1}(f(x), s)$ contains an open set including the point $f(x)$ (so $N^*_{k_1}(f(x), s) = N_{k_1}(f(x), s)$).

Indeed, the current (k_0, k_1)-continuity of Definition 5 is different from the digital (k_0, k_1)-continuity of Definition 4(2) and can be used for studying a computer topological space $X_{n,k}$. Although both KD-(k_0, k_1)- and (k_0, k_1)-continuities can contribute to the study of computer topological space, they need not be sufficient for studying $X_{n,k}$, either. Thus, the following notion can be used in studying spaces $X_{n,k}$.

Definition 6 ([10]). A function $f : X_{n_0,k_0} \to Y_{n_1,k_1}$ is said to be Khalimsky (k_0, k_1)-continuous (briefly, K-(k_0, k_1)-continuous) at a point $x \in X_{n_0,k_0}$ if

(1) $f : (X, T^{n_0}_X) \to (Y, T^{n_1}_Y)$ is Khalimsky continuous at the point x, and

(2) the map f satisfies Definition 5 at the point x.

Further, we say that a map $f : X_{n_0,k_0} \to Y_{n_1,k_1}$ is K-(k_0, k_1)-continuous if the map f is K-(k_0, k_1)-continuous at every point $x \in X_{n_0,k_0}$.

We observe that none of the conditions (1) and (2) of Definition 6 implies the other if $n_0, n_1 \in \mathbf{N} - \{1\}$ (see Remark 3.4). In addition, since Definition 4(2) and Definition 6(2) are so different from each other, we clearly observe that KD-(k_0, k_1)-continuity and KD-(k_0, k_1)-continuity cannot be equivalent to each other.

Remark 3.4. (1) We now show that Definition 6(2) does not imply Definition 6(1). Precisely, consider $X := (a_i)_{i\in[0,5]_{\mathbf{Z}}}$ with 26-adjacency and the map $f : X \to \mathbf{Z}$ in Figure 2(a) for which $f(A) = \{3\}$, $f(a_3) = 2$, $f(a_5) = 4$, where $A = \{a_0, a_1, a_2, a_4\}$. While the map f satisfies Definition 6(2), it cannot be Khalimsky continuous at the point a_4. To be specific, let us now examine K-$(26, 2)$-continuity of the map f at the point a_4 and let us remind that $N^*_{26}(a_4, 1)$ is the total set X. While for $N^*_2(3, 1)$ we obtain $f(N^*_{26}(a_4, 1)) \subset N^*_2(3, 1)$, which implies Definition 6(2) at the point a_4, f cannot be Khalimsky continuous at the point a_4 because Khalimsky continuity of f at the point a_4 should require $f(N^*_{26}(a_4, 1)) \subset \{3\} \in (\mathbf{Z}, T)$.

(2) We now show that Definition 6(1) need not imply Definition 6(2):

(Case 1) Let us consider Y_{n_1,k_1} of Definition 6 to be $(\mathbf{Z}, T)$.
Consider the map f in Example 3.3. Then the map $f : A_{2,8} \to (\mathbf{Z}, T)$ in Figure 1 cannot be K-$(8, 2)$-continuous at the point $x_8 \in A$ contrary to Definition 6(2). More precisely, for $N^*_8(x_8, 1) = \{x_7, x_8, x_9, x_{10}\}$ we can observe $f(N^*_8(x_8, 1)) \nsubseteq N^*_2(6, 1)$.

(Case 2) Let us consider $Y_{2,8}$ or $Y_{2,4}$ according to Definition 6.
Assume that $A = \{x_i\}_{i\in[1,11]_{\mathbf{Z}}} \subset \mathbf{Z}^2$ and $B = \{y_j\}_{j\in[1,6]_{\mathbf{Z}}} \subset \mathbf{Z}^2$ (see Figure

2(b)). Next, consider the map $f : A \to B$ for which $f(x_1) = y_1, f(x_2) = f(x_3) = f(x_4) = y_2, f(x_5) = f(x_6) = y_3, f(x_7) = y_4$, $f(x_8) = y_5$, and $f(x_9) = f(x_{10}) = f(x_{11}) = y_6$ (see Figure 2(b)).

Assume $A_{2,8}$ or $A_{2,4}$, and $B_{2,8}$ or $B_{2,4}$. Then, while the map f is Khalimsky continuous, f cannot be K-$(8,8)$-, K-$(4,4)$-, $(8,4)$-, $(4,8)$-continuous. To be specific, although x_1 and x_2 are 4-adjacent, the images $f(x_1)$ and $f(x_2)$, which is not 4-adjacent, are obviously 8-adjacent. While x_7 and x_8 are 8-adjacent, both their images of f are not 8-adjacent. Thus f cannot be K-$(8,8)$-continuous at the point x_7.

Similarly, we can observe that the map f cannot be K-$(4,4)$-, $(8,4)$-, $(4,8)$-continuous, either.

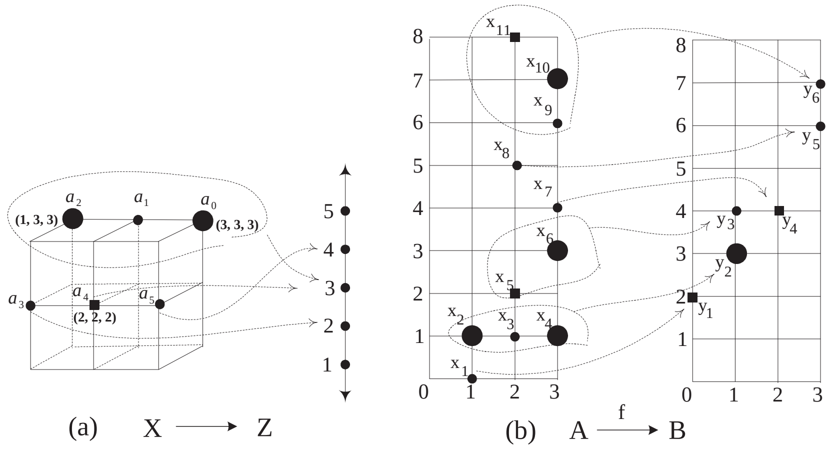

FIGURE 2. Non-existence of K-(k_0, k_1)-continuity, $k_0, k_1 \in \{4, 8\}$.

As referred above, for a map $f : X_{n_0,k_0} \to Y_{n_1,k_1}$ if either of $N^*_{k_0}(x,1) \subset X_{n_0,k_0}$ and $N^*_{k_1}(f(x),1) \subset Y_{n_1,k_1}$ does not exist, then we may have some difficulty in defining both K-(k_0,k_1)-continuity and (k_0,k_1)-continuity of the map f at the point x. Thus we observe that K-(k_0,k_1)-continuity is different from KD-(k_0,k_1)-continuity (see Remark 3.5).

Remark 3.5. If a space $X_{n,k}$ is not k-connected, then for some point $x \in X$ and any $\varepsilon \in \mathbf{N}$ we may not have $N^*_k(x,\varepsilon)$. For instance, consider $A_{2,4}$ instead of $A_{2,8}$ in Figure 2(b). Then for any $\varepsilon \in \mathbf{N}$, no $N^*_4(x_5,\varepsilon)$ exists. While $f : A_{2,4} \to B_{2,8}$ is Khalimsky continuous, it cannot satisfy Definition 6(2) at the point x_5, which means that the map f cannot be a K-$(4,8)$-continuous map. Meanwhile, we observe that f is KD-$(4,8)$-continuous at the point x_5.

With some hypothesis, we can compare the above-mentioned computer topological continuities, as follows.

Theorem 3.6. *For* (X, T^n_X) *and the Khalimsky line* $(\mathbf{Z}, T)$ *if a map* $f : X \to \mathbf{Z}$ *is Khalimsky continuous at a pure closed point* $x \in X$*, then* f *is also KD-*$(3^n - 1, 2)$*-,* $(3^n - 1, 2)$*-, and K-*$(3^n - 1, 2)$*-continuous at the point.*

Proof. If a point x is a pure closed point, then the smallest open neighborhood of x is $N^*_{3^n-1}(x, 1)$. Thus for any point $x_1 \in N^*_{3^n-1}(x, 1)$, we obtain $f(x_1) \in N^*_2(f(x), 1)$ so that $f(N^*_{3^n-1}(x, 1)) \subset N^*_2(f(x), 1)$ by the Khalimsky continuity of f (see the point x_5 in Figure 1). Consequently, the map f is also K-$(3^n-1, 2)$-continuous at the point.

Similarly, the proofs of $(3^n - 1, 2)$-, and K-$(3^n - 1, 2)$-continuities of the map f can be done by using the same method used to prove the above KD-$(3^n - 1, 2)$-continuity of f. □

Due to the above-mentioned several continuities in computer topology, we obtain the following several categories in computer and digital topologies [10].

(1) Let us consider the *Khalimsky topological category*, denoted by KTC, consisting of two things:

- A class of objects (X, T^n);
- KTC has Khalimsky continuous maps as morphisms.

(2) Let us consider the *KD-topological category*, denoted by KDTC, consisting of two things:

- A class of objects $X_{n,k}$;
- KDTC has KD-(k_0, k_1)-continuous maps as morphisms.

(3) Let us consider the *computer topological category*, denoted by CTC, consisting of two things:

- A class of objects $X_{n,k}$;
- CTC has (k_0, k_1)-continuous maps as morphisms.

(4) Let us consider the *Khalimsky computer topological category*, denoted by KCTC, consisting of two things:

- A class of objects $X_{n,k}$;
- KCTC has K-(k_0, k_1)-continuous maps as morphisms.

(5) Finally, let us consider the *digital-topological category*, denoted by DTC, consisting of two things:

- A class of objects (X, k) in $\mathbf{Z}^n$;
- DTC has digitally (k_0, k_1)-continuous maps as morphisms.

In Sections 4 and 5, by using these computer or digital topological categories, we will establish several kinds of k-retracts and study extension problems of the just above continuities from (1) to (5) in relation with several k-retracts in Section 4.

4. Various k-retracts and their properties in relation with computer topological homeomorphisms

For a subset $X \subset \mathbf{Z}^n$ or $\mathbf{Z}$, by Theorem 3.6, an extension problem of a K-$(3^n - 1, 2)$-continuous map $f : X_{n,3^n-1} \to (\mathbf{Z}, T)$ at a pure point can be easily studied. We now investigate extension problems of the other cases $f : X_{n,k} \to Y_{n_1,k_1}$.

Motivated by the notion of retract in [2], the paper [3] develops the notion of digital k-retract for the study of an extension problem of digital continuity:

Definition 7 ([3])**.** In DTC, we say that a digitally k-continuous map $r : (X', k) \to (X, k)$ is a digital k-retraction if
(1) $X \subset X'$, and
(2) $r(x) = x$ for all $x \in X$.
Then we say that X is a digital k-retract of X'.

In Definition 7, a point $x \in X' - X$ is said to be *digitally k-retractable.*

Similarly, in KTC, the Khalimsky topological version of the retraction of Definition 7 can be established, as follows.

In KTC, we say that a Khalimsky continuous map $r : (X', T^n_{X'}) \to X, T^n_X)$ is a *Khalimsky retraction* if
(1) (X, T^n_X) is a Khalimsky topological subspace of $(X', T^n_{X'})$, and
(2) $r(x) = x$ for all $x \in X$.
Then we call X a *Khalimsky retract* of X'.

Motivated by the digital k-retract of Definition 7 and the above-mentioned Khalimsky retraction, we now introduce several kinds of k-retracts for studying spaces $X_{n,k}$ (see Definitions 8, 10 and 12) and study their properties in relation with several homeomorphisms in Definitions 9, 11 and 13, as follows.

Definition 8 ([14])**.** In KDTC, we say that a KD-k-continuous map $r : X'_{n,k} \to X_{n,k}$ is a KD-k-retraction if
(1) $X_{n,k} \subset X'_{n,k}$, and
(2) $r(x) = x$ for all $x \in X_{n,k}$.
Then we say that $X_{n,k}$ is a KD-k-retract of $X'_{n,k}$. Further, we say that a point $x \in X'_{n,k} - X_{n,k}$ is KD-k-retractable.

Example 4.1. Consider a space $X_{2,4}$ in Figure 3(a), where $X = \{x_i\}_{i\in[1,11]_{\mathbf{Z}}}$. Assume that $X' = X \cup \{a_1, a_2\}$ in Figure 3(a). Let us now consider a space $Y_{2,4}$ in Figure 3(b), where $Y = \{y_i\}_{i\in[1,13]_{\mathbf{Z}}}$ and assume $Y' = Y \cup \{b_1, b_2, b_3\}$ in Figure 3(b). Then $X_{2,8}$ is a KD-8-retract of $X'_{2,8}$ and $Y_{2,4}$ is a KD-4-retract of $Y'_{2,4}$ in such a way that: $a_1 \to x_8$, $a_2 \to x_{11}$, and the other points in $X_{2,8}$ are remained.

Further, $b_1 \to y_4$, $b_2 \to y_6$, $b_3 \to y_{12}$, and the other points in Y are also remained.

Definition 9 ([10])**.** In KDTC, a function $f : X_{n_0,k_0} \to Y_{n_1,k_1}$ is said to be a KD-(k_0, k_1)-homeomorphism if
(1) the map f is bijective, and
(2) the map f is KD-(k_0, k_1)-continuous and further, f^{-1} is KD-(k_1, k_0)-continuous. Then we say that X_{n_0,k_0} is KD-(k_0, k_1)-homeomorphic to Y_{n_1,k_1}.

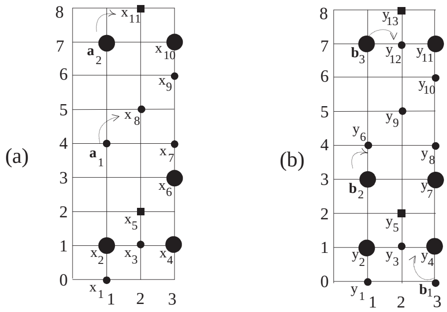

FIGURE 3. (a) KD-8-retract (b) KD-4-retract.

Example 4.2. Consider the simple closed 8-curves with eight elements $SC_8^{2,8} := (w_i)_{i\in[0,7]_{\mathbf{Z}}}$, $T := (e_i)_{i\in[0,7]_{\mathbf{Z}}}$ (see Figure 4). The map $g : SC_8^{2,8} \to T_{2,8}$ given by $g(w_i) = e_i$ is a KD-$(8,8)$-homeomorphism because the singletons $\{e_2\}$ and $\{e_6\}$ are also open sets in $T_{2,8}$.

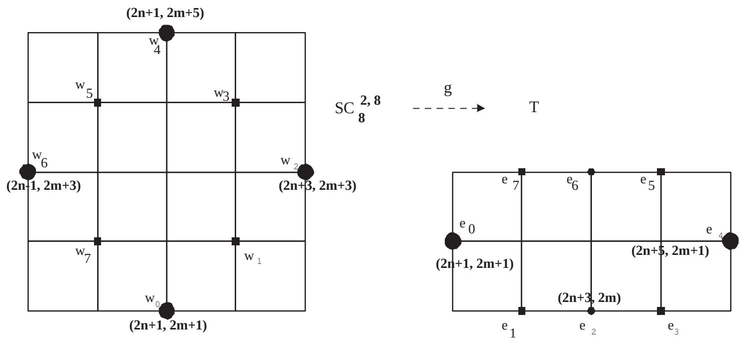

FIGURE 4. KD-$(8,8)$-homeomorphism.

Motivated by the digital isomorphic property of the digital k-retract in [3], we obtain the following property of a KD-(k_0, k_1)-homeomorphism.

Theorem 4.3. *Let $X_{n,k}$ be a KD-k-retract of $X'_{n,k}$ and let $h : X'_{n,k} \to Y_{n_1,k_1}$ be a KD-(k, k_1)-homeomorphism. Then $h(X_{n,k})$ is a KD-k_1-retract of Y_{n_1,k_1}.*

Proof. Let $r : X'_{n,k} \to X_{n,k}$ be a KD-k-retraction. Then $h \circ r \circ h^{-1} : Y_{n_1,k_1} \to h(X_{n,k})$ is a KD-k_1-retraction because the composition of KD-(k_1, k)-, KD-(k, k)- and KD-(k, k_1)-continuous maps is also KD-(k_1, k_1)-continuous. □

By using the similar method as Definition 8, in CTC we obtain the following:

Definition 10. In CTC, we say that a k-continuous map $r : X'_{n,k} \to X_{n,k}$ is a k-retraction if
(1) $X_{n,k} \subset X'_{n,k}$, and
(2) $r(x) = x$ for all $x \in X_{n,k}$.
Then we say that $X_{n,k}$ is a k-retract of $X'_{n,k}$. Further, we say that a point $x \in X'_{n,k} - X_{n,k}$ is k-retractable.

Definition 11 ([10])**.** In CTC, a function $f : X_{n_0,k_0} \to Y_{n_1,k_1}$ is said to be a (k_0, k_1)-homeomorphism if
(1) the map f is bijective, and
(2) the map f is a (k_0, k_1)-continuous map and further, f^{-1} is a (k_1, k_0)-continuous map. Then we say that X_{n_0,k_0} is (k_0, k_1)-homeomorphic to Y_{n_1,k_1}.

By using the same method used to prove Theorem 4.3, we obtain the following:

Theorem 4.4. *In CTC, let $X_{n,k}$ be a k-retract of $X'_{n,k}$ and let $h : X'_{n,k} \to Y_{n_1,k_1}$ be a (k, k_1)-homeomorphism. Then $h(X_{n,k})$ is a k_1-retract of Y_{n_1,k_1}.*

In KCTC, by using the same method as Definitions 8 and 10, we can also establish the notion of K-k-retraction:

Definition 12. In KCTC, we say that a K-(k, k)-continuous map $r : X'_{n,k} \to X_{n,k}$ is a K-k-retraction if
(1) $X_{n,k} \subset X'_{n,k}$, and
(2) $r(a) = a$ for all $a \in X_{n,k}$.
Then we say that $X_{n,k}$ is a K-k-retract of $X'_{n,k}$. Further, we say that the point $a \in X'_{n,k} - X_{n,k}$ is K-k-retractable.

Example 4.5. Consider the two spaces in Figure 3. Then $X_{2,8}$ is a K-8-retract of $X'_{n,k}$ and $Y_{2,4}$ is a K-4-retract of $Y'_{2,4}$ in the same way as Example 4.1.

Definition 13 ([10])**.** In KCTC, a function $f : X_{n_0,k_0} \to Y_{n_1,k_1}$ is said to be a K-(k_0, k_1)-homeomorphism if
(1) the map f is bijective, and
(2) the map f is a K-(k_0, k_1)-continuous map and further, f^{-1} is a K-(k_1, k_0)-continuous map. Then we say that the space X_{n_0,k_0} is K-(k_0, k_1)-homeomorphic to Y_{n_1,k_1}.

By using the same method used to prove the above KD-$(8, 4)$-homeomorphism of f in Example 4.2, we can observe that $SC_8^{2,8}$ is K-$(8, 8)$- and $(8, 8)$-homeomorphic to $T_{2,8}$. By using the same method used to prove Theorem 4.3, we obtain the following:

Theorem 4.6. *Let X_{n_0,k_0} be a K-k_0-retract of X'_{n_0,k_0} and let $h : X_{n_0,k_0} \to Y_{n_1,k_1}$ be a K-(k_0, k_1)-homeomorphism. Then $h(X_{n_0,k_0})$ is a K-k_1-retract of Y_{n_1,k_1}.*

Example 4.7. Put $X = X' - \{x_1, x_5, x_9\}$ (see Figure 5). Then we observe that $X_{2,8}$ is a K-8-retract of $X'_{2,8}$. Consider a K-$(8,8)$-homeomorphism $h : X_{2,8} \to Y_{2,8}$ defined by $f(x_i) = y_i, i \in [1,9]_{\mathbf{Z}}$. Then $h(X)_{2,8}$ is obviously a K-8-retract of $Y_{2,8}$, where $h(X) = Y - \{y_1, y_5, y_9\}$.

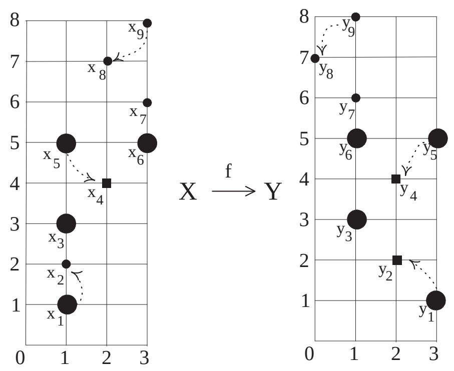

FIGURE 5. Computer topological homeomorphism in relation with K-8-retract.

In addition, the map f in Figure 5 also guarantees Theorems 4.3 and 4.6. We now say that properties that when possessed by a space in $\mathbf{Z}^n$ are also possessed by KD-(k_0, k_1)-, (k_0, k_1)-, and K-(k_0, k_1)-homeomorphisms are called *KD-(k_0, k_1)-, (k_0, k_1)-* and *KD-(k_0, k_1)-homeomorphic properties*, respectively. Obviously, k-connectedness is KD-k-homeomorphic property.

Theorem 4.8. *Each of KD-k-, k- and K-k-retracts holds the reflexivity and the transitivity.*

5. Extension problems of several continuities in computer topology

In classical topology [2], we recall the following: Let (X', T') be a topological space and (X, T) a subspace of (X', T'). Then (X, T) is a retract of (X', T') if and only if every continuous map $f : (X, T) \to (Y, T_1)$ has a continuous map $F : X' \to Y$ such that $F|_X = f$ for any (Y, T_1) [2]. Consequently, in KTC instead of KDTC, CTC and KCTC, every Khalimsky continuous map $f : (X, T_X^2) \to (Y, T_Y^{n_1})$ has an extension $F : (X', T_{X'}^2) \to (Y, T_Y^{n_1})$ such that $F|_X = f$ for any $(Y, T_Y^{n_1})$ if and only if X is a Khalimsky retraction of X'.

Unlike the extension property, in view of Remarks 3.4 and 3.5, the extension problems of the several computer topological continuities in Section 3 have their intrinsic features (see Theorems 5.1 and 5.3, and Corollary 5.4). Thus we need to study extension problems of the computer topological continuities in relation with the computer topological k-retracts in Section 4. We now establish a criterion of a subspace $X_{n,k}$ that is equivalent to the condition that for each of

KD-(k,k_1)-, (k,k_1)-, and K-(k,k_1)-continuous maps $f : X_{n,k} \to Y_{n_1,k_1}$, there is a space $X'_{n,k}$ containing the space $X_{n,k}$ and its corresponding KD-(k,k_1)-, (k,k_1)-, and K-(k,k_1)-continuous map $F : X'_{n,k} \to Y_{n_1,k_1}$ such that $F|_{X_{n,k}} = f$ for any Y_{n_1,k_1}.

Theorem 5.1. *In KDTC, $X_{n,k}$ is a KD-k-retract of $X'_{n,k}$ if and only if every KD-(k,k_1)-continuous map $f : X_{n,k} \to Y_{n_1,k_1}$ has a KD-(k,k_1)-continuous map $F : X'_{n,k} \to Y_{n_1,k_1}$ such that $F|_{X_{n,k}} = f$ for any Y_{n_1,k_1}.*

Proof. Let $r : X'_{n,k} \to X_{n,k}$ be a KD-k-retraction and $f : X_{n,k} \to Y_{n_1,k_1}$ a KD-(k,k_1)-continuous function. Then the composition $F := f \circ r : X'_{n,k} \to Y_{n_1,k_1}$ is a KD-(k,k_1)-continuous map that is an extension of f.

Conversely, suppose that every KD-(k,k_1)-continuous map $f : X_{n,k} \to Y_{n_1k_1}$ has a KD-(k,k_1)-continuous extension $F : X'_{n,k} \to Y_{n_1k_1}$ for every $Y_{n_1k_1}$. Then the identity map $1_{X_{n,k}}$ has a KD-(k,k_0)-continuous extension $r : X'_{n,k} \to X_{n,k}$. Thus $X_{n,k}$ is a KD-k-retract of $X'_{n,k}$. □

To guarantee Theorem 5.1, let us consider the space $X_{2,8}$ in Figure 6. In addition, consider another space $Y_{2,k}, k \in \{4,8\}$ in Figure 6. Since $X_{2,8}$ cannot be k-retract of $Y_{2,k}, k \in \{4,8\}$, we observe that there is no KD-$(8,8)$-continuous extension of $1_{X_{2,8}}$ into the map $f : Y_{2,k} \to X_{2,8}$ in Figure 6.

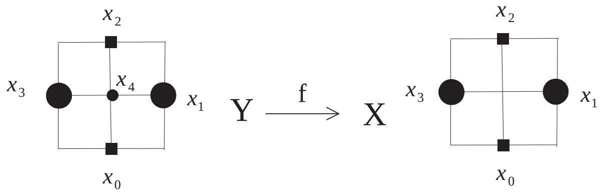

FIGURE 6. Non-existence of an extension of KD-$(8,8)$-continuity.

Example 5.2. In Figure 7(a), for two sets $X = \{x_i\}_{i\in[0,8]_{\mathbf{Z}}}$ and $Y = \{y_i\}_{i\in[0,3]_{\mathbf{Z}}}$, consider a map $g : X_{2,4} \to Y_{2,8}$ for which $g(\{x_0,x_1,x_2,x_3,x_6\}) = \{y_0\}$, $g(x_4) = y_2$, $g(x_5) = y_1$, and $g(\{x_7,x_8\}) = \{y_3\}$. Then g is a KD-$(4,8)$-continuous map.

Assume $X' = X \cup \{c_1\}$ and consider $X'_{2,4}$. Then $X_{2,4}$ is a KD-4-retract of $X'_{2,4}$ in such a way that $c_1 \to x_4$ and the other points of $X'_{2,4}$ are remained, where $c_1 = (1,0)$. Then there is a map $G : X'_{2,4} \to Y_{2,8}$ such that $G|_{X_{2,4}}$ and $G(\{c_1\}) = \{y_2\}$, *i.e.,* G is a KD-$(4,4)$-continuous extension of g.

Meanwhile, consider the space in Figure 7(b). Assume $D_{2,4}$ and the map $f : X_{2,4} \to \mathbf{Z}$ for which $f(X_1) = \{3\}$ and $f(\{x_7,x_8\}) = \{-1\}$, where $D := X \cup \{d_1\}$ and $X = \{x_i\}_{i\in[0,8]_{\mathbf{Z}}}$ and $d_1 = (2,4)$ in Figure 7(b). Then, in KDTC, there is no extension of f on $D_{2,4}$ because $X_{2,4}$ cannot be a KD-4-retract of $D_{2,4}$. To be specific, in Figure 7(b), consider the set $D := X \cup \{d_1\}$ and assume

$D_{2,4}$. Then $X_{2,4}$ cannot be a 4-retract of $D_{2,4}$ because the smallest open set containing the point d_1 is the set $\{x_2, x_3, x_6, d_1, x_7, x_8\}$. Thus, in KDTC there is no extended map $F : D_{2,4} \to \mathbf{Z}$ such that $F|_{X_{2,4}} = f$ (see Figure 7(b)) owing to the non-existence of a KD-4-retraction from $D_{2,4}$ onto $X_{2,4}$ at the point d_1.

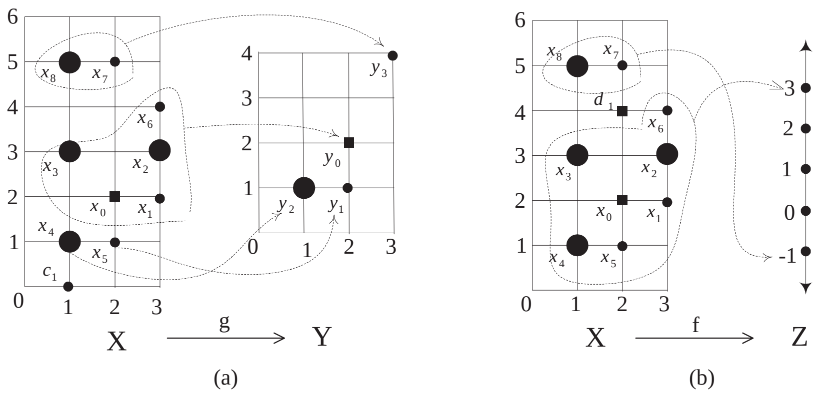

FIGURE 7. (a) Existence of KD-$(8,8)$- and K-$(8,8)$-extension maps; (b) Non-existence of KD-$(8,8)$- and K-$(8,8)$-extension maps .

By using the same method used to prove Theorem 5.1, in CTC we obtain the following:

Theorem 5.3. *In CTC, let* $X'_{n,k}$ *and* Y_{n_1,k_1} *be* k*- and* k_1*-connected spaces, respectively. Let* $X_{n,k}$ *be a* k*-connected subspace of* $X'_{n,k}$*. Then* $X_{n,k}$ *is a* k*-retract of* $X'_{n,k}$ *if and only if every* (k, k_1)*-continuous map* $f : X_{n,k} \to Y_{n_1,k_1}$ *has a* (k, k_1)*-continuous map* $F : X'_{n,k} \to Y_{n_1,k_1}$ *such that* $F|_{X_{n,k}} = f$ *for any* Y_{n_1,k_1}.

Before we prove Theorem 5.3, we need to discuss the assumption of Theorem 5.3 on the k-connectedness of $X_{n,k}$ and the k_1-connectedness of Y_{n_1,k_1}. If the hypothesis is omitted, we may not have some computer topological k- or k_1-neighborhood of some points in $X'_{n,k}$ and Y_{n_1,k_1} (see Remark 3.5). Hereafter, we always assume that each space $X_{n,k}$ in CTC and KCTC is k-connected.

Proof of Theorem 5.3. The proof can be done by the same method as that of Theorem 5.1 by replacing both a KD-(k, k_1)-continuous function and a KD-k-retraction in KDTC into both a (k, k_1)-continuous function and a k-retraction in CTC, respectively. □

By combining Theorems 5.1 and 5.3, in KCTC we obtain the following:

Corollary 5.4. *In KCTC, let A'_{n_0,k_0} and B_{n_1,k_1} be k_0- and k_1-connected, respectively. For a k_0-connected space $A_{n_0,k_0} \subset A'_{n_0,k_0}$, a K-$(k_0, k_1)$-continuous map $f : A_{n_0,k_0} \to B_{n_1,k_1}$ has an extended map $F : A'_{n_0,k_0} \to B_{n_1,k_1}$ for any B_{n_1,k_1} that is also K-(k_0, k_1)-continuous if and only if A_{n_0,k_0} is a K-k_0-retract of A'_{n_0,k_0}.*

Let us consider the domain of the map f in Corollary 5.4 to be a subspace in $(\mathbf{Z}^2, T^2)$, as follows.

Example 5.5. In Figure 5, assume $A = X - \{x_1, x_5, x_9\}$, $B = Y - \{y_1, y_5, y_9\}$. Consider $f : A_{2,8} \to B_{2,8}$ given by $f(x_i) = y_i$, $i \in \{2, 3, 4, 6, 7, 8\}$. Then f is a K-$(8, 8)$-continuous map. Further, the space $A_{2,8}$ is a K-8-retract of $X_{2,8}$. Thus, there is an extension map $F : X_{2,8} \to B_{2,8}$ defined by $F(x_1) = y_2$, $F(x_5) = y_4$, $F(x_9) = y_8$, and $F|_{X_{2,8}} = f$, which is a K-$(8, 8)$-continuous extension of f.

Let us consider the domain of the map f in Corollary 5.4 as a subspace in $(\mathbf{Z}^3, T^3)$, as follows.

Example 5.6. Assume the space $X = A \cup \{(2, 2, 2)\}$ (see Figure 8), where $A = \{(1, 2, 2), (1, 3, 3), (2, 3, 3), (3, 3, 3), (3, 2, 2)\}$. In addition, consider the map $f : A_{3,18} \to (\mathbf{Z}, T)$ for which $f((1, 2, 2)) = 2, f((1, 3, 3)) = 3, f((2, 3, 3)) = 3, f((3, 3, 3)) = 3$ and $f((3, 2, 2)) = 4$ so that f is a K-$(18, 2)$-continuous map. Assume $X = A \cup \{(2, 2, 2)\}$. Then there is no extended K-$(18, 2)$-continuous map $F : X_{3,18} \to (\mathbf{Z}, T)$ because the point $(2, 2, 2) \in X$ cannot be K-18-retractable.

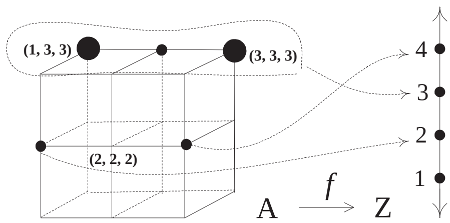

Figure 8. Non-existence of KD-$(18, 2)$-, $(18, 2)$-, and K-$(18, 2)$-continuities.

In KDTC and CTC, let us now consider the map f in Figure 8. Obviously, it is also a KD-$(18, 2)$- as well as an $(18, 2)$- continuous map. By using the same method as Example 5.6, by Theorems 5.1 and 5.3, we can observe that the map f cannot be extended to the above F that is KD-$(18, 2)$- or $(18, 2)$- continuous because the point $(2, 2, 2) \in X$ is neither KD-18- nor 18-retractable.

6. Summary

All results in this paper can be presented from the view point of digital topology in terms of the forgetful functor from KDTC, CTC, and KCTC into DTC: Assume forgetful functors from KDTC, CTC, and KCTC into DTC, denoted by F^*: KDTC, CTC, and KCTC $\to$ DTC. To be specific, for a space $X_{n,k}$ in KDTC, CTC, and KCTC, F^* is defined in such a way that $F^*(X_{n,k}) = (X, k) \in$ DTC. Besides, for a KD-(k_0, k_1)-, (k_0, k_1)-, or a K-(k_0, k_1)-continuous map f, we obtain $F^*(f) = f \in$ DTC. This can be essentially applied to digital geometry and computer science. The results of Sections 4 and 5 are computer topological versions in KDTC, CTC, and KCTC of digital k-retract properties in DTC [3].

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