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EXOTIC SMOOTH STRUCTURES ON $(2n+2l-1)\mathbb{CP}^2\sharp(2n+4l-1)\overline{\mathbb{CP}}^2$

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Abstract. As an application of 'reverse engineering' technique introduced by R. Fintushel, D. Park and R. Stern [9], we present a simple way to construct an infinite family of exotic $(2n+2l-1)\mathbb{CP}^2\sharp(2n+4l-1)\overline{\mathbb{CP}}^2$'s for all $n \geq 0$, $l \geq 1$.

1. Introduction

One of the fundamental problems in smooth 4-manifolds is to find an exotic smooth structure on a given smooth 4-manifold. We say that a smooth 4-manifold X has an *exotic smooth structure* if X admits more than one distinct smooth structure, i.e., there exists a smooth 4-manifold X' which is homeomorphic to X, but not diffeomorphic to X. Though the complete answer for this problem is still far from reach, gauge theory has made topologists to get many striking results on smooth 4-manifolds ([7, 8, 10, 11, 13, 14, 15]).

Recently, R. Fintushel, D. Park, and R. Stern introduced a new surgery technique, called 'reverse engineering', and they constructed a family of homology $(2n-1)(S^2\times S^2)$'s by performing Luttinger surgeries on $\Sigma_2\times\Sigma_{n+1}$ for any $n\geq 1$ [9]. Here $(2n-1)(S^2\times S^2)$ means the connected sum of $2n-1$ copies of a 4-manifold $S^2\times S^2$ and Σ_g means a Riemann surface of genus g.

In this article we present an easy way to produce an infinite family of exotic $(2n+2l-1)\mathbb{CP}^2\sharp(2n+4l-1)\overline{\mathbb{CP}}^2$'s for all $n\geq 0$, $l\geq 1$. The main idea is very simple: For each $n\geq 0$ and $l\geq 1$, we first take a symplectic fiber sum with l copies of $\mathrm{Sym}^2(\Sigma_3)$ and n copies of $\Sigma_2\times\Sigma_2$ along an essential Lagrangian torus, where $\mathrm{Sym}^2(\Sigma_3)$ is the 2-fold symmetric product of genus 3 Riemann surface Σ_3. And then we perform $5l+7n$ times ± 1-Luttinger surgeries and one more k-surgery on the fiber sum 4-manifold $l(\mathrm{Sym}^2(\Sigma_3))\sharp n(\Sigma_2\times\Sigma_2)$. Next we show that the resultant 4-manifolds are simply connected, so that they are all homeomorphic to each other. Finally, by applying the same technique in [9]

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for the computation of Seiberg-Witten invariants, we get the following main result.

Theorem 1.1. *For each integer $n \geq 0$ and $l \geq 1$, there are infinitely many exotic smooth structures on the connected sum 4-manifold* $(2n+2l-1)\mathbb{CP}^2\sharp(2n+4l-1)\overline{\mathbb{CP}}^2$.

Remarks. 1. As a special case of Theorem 1.1 above, we have a new family of exotic $3\mathbb{CP}^2\sharp5\overline{\mathbb{CP}}^2$'s and $3\mathbb{CP}^2\sharp7\overline{\mathbb{CP}}^2$'s. It is unclear whether these are diffeomorphic to the exotics in [1, 2, 3, 5].

2. S. Baldridge and P. Kirk also constructed independently a family of exotic smooth structures on $(2n+2m-1)\mathbb{CP}^2\sharp(6n+4m-1)\overline{\mathbb{CP}}^2$'s ([5, Corollary 19]).

Furthermore, for each integer $n \geq 0$, $l \geq 1$ and $g \geq 3$, we can also perform the same procedure as above on the fiber sum 4-manifold $l(\mathrm{Sym}^2(\Sigma_g))\sharp n(\Sigma_2 \times \Sigma_2)$. Then we have:

Corollary 1.1. *For each integer $n \geq 0$, $l \geq 1$ and $g \geq 3$, there are infinitely many exotic smooth structures on the connected sum 4-manifold* $(2n + l(g-1)(g-2)-1)\mathbb{CP}^2\sharp(2n + l(g-1)^2-1)\overline{\mathbb{CP}}^2$.

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2. Preliminaries

In the section we first briefly review a Luttinger surgery and a process, so called *reverse engineering*, for constructing infinite families of distinct smooth structures on a given simply connected 4-manifold introduced by R. Fintushel, D. Park, and R. Stern [9]. And then we will review two basic building blocks, $\Sigma_2 \times \Sigma_2$ and $\mathrm{Sym}^2(\Sigma_3)$ ([4, 5, 9] for details).

2.1. Luttinger surgery

Let X be a smooth 4-manifold with an embedded torus T of self-intersection 0. Let N_T be its tubular neighborhood and let α, β be generators of $\pi_1(T)$ and let S^1_α, S^1_β be loops on $T^3 = \partial N_T$ such that S^1_α is homologous to α and S^1_β is homologous to β in N_T. Let μ_T denote a meridional circle to T in X.

Definition 2.1. A p/q-surgery on T with respect to β, denoted by $(T, \beta, p/q)$, is the process constructing a new smooth 4-manifold, denoted by $X_{T,\beta}(p/q)$, from X

$$X_{T,\beta}(p/q) = (X - N_T) \cup_\varphi (S^1 \times S^1 \times D^2)$$

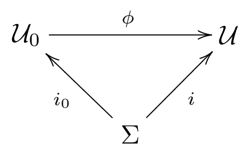
where the gluing map $\varphi : S^1 \times S^1 \times \partial D^2 \to \partial(X - N_T)$ is chosen so that it satisfies

$$\varphi_*([\partial D^2]) = q[S^1_\beta] + p[\mu_T]$$

in $H_1(\partial(X - N_T); \mathbb{Z})$.

In the case that a smooth 4-manifold X admits a symplectic structure ω and a torus T is a Lagrangian submanifold in X, we can use the following Moser-Weinstein tubular neighborhood theorem.

Theorem 2.1 ([6])**.** *Suppose that (X, ω) is a symplectic 4-manifold which contains a Lagrangian submanifold Σ. Let ω_0 be the canonical symplectic form on $T^*\Sigma$, $i_0 : \Sigma \hookrightarrow T^*\Sigma$ the Lagrangian embedding as the zero section, and $i : \Sigma \hookrightarrow X$ the Lagrangian embedding given by embedding. Then there are neighborhoods $\mathcal{U}_0$ of Σ in $T^*\Sigma$, $\mathcal{U}$ of Σ in X and a diffeomorphism $\phi : \mathcal{U}_0 \to \mathcal{U}$ such that the following diagram commutes and $\phi^*\omega = \omega_0$.*



Therefore we may assume that each $T \times \{b\}$ $(b \in D^2 \setminus \{0\})$ in $T \times D^2 \cong N_T \subset X$ is a Lagrangian submanifold and it is called Lagrangian push off or Lagrangian framing. For any embedded curve $\gamma \subset T$, its image γ' in $T \times \{b\}$ is called Lagrangian push off of γ and its homology class $[\gamma']$ in $H_1(X \setminus T; \mathbb{Z})$ is independent of the choice of $b \in D^2 \setminus \{0\}$. Therefore we can define a surgery operation by using this Lagrangian framing.

Definition 2.2. Let (X, ω) be a symplectic 4-manifold, $T \subset X$ be a Lagrangian submanifold of X with self-intersection 0 and let $\gamma \subset T$ be a co-oriented simple closed curve. Then a $1/n$-surgery $(T, \gamma, 1/n)$ corresponding to the Lagrangian push off γ', i.e., together with a gluing map $\varphi : S^1 \times S^1 \times \partial D^2 \to \partial(X - N_T)$ which satisfies

$$\varphi_*([\partial D^2]) = n[\gamma'] + [\mu_T]$$

is called a $1/n$-Luttinger surgery on T along γ.

Some of well-known properties of a $1/n$-Luttinger surgery on T along γ are the following:

Theorem 2.2 ([4, 5])**.** *Let X be a smooth 4-manifold, $T \subset X$ a submanifold of X with self-intersection 0 and let $\gamma \subset T$ be a simple closed curve. Suppose that $X_{T,\gamma}(p/q)$ is a smooth 4-manifold obtained by performing p/q-surgery on T with respect to γ. Then we have*

(1) $\pi_1(X_{T,\gamma}(p/q)) = \pi_1(X - T)/N(\mu_T^p \gamma'^q)$.
(2) $\sigma(X) = \sigma(X_{T,\gamma}(p/q))$ *and* $e(X) = e(X_{T,\gamma}(p/q))$.

(3) *Moreover, if* (X, ω) *is a symplectic* 4*-manifold and* T *is a Lagrangian submanifold of* X*, then* $X_{T,\gamma}(1/n)$ *which is obtained from* X *by performing* $1/n$*-Luttinger surgery on* T *along* γ *is a symplectic* 4*-manifold.*

2.2. Reverse engineering

As an application of Seiberg-Witten theory, various surgery techniques have been developed to produce exotic smooth 4-manifolds. Among them, R. Fintushel, B. Park, and R. Stern recently introduced a new surgery technique, so called *reverse engineering*, for constructing infinite families of distinct smooth structures on a given simply connected 4-manifold. Here is a brief review of the process ([9] for details).

In order to construct exotic smooth structures on a simply connected smooth 4-manifold X with Euler number e and signature σ, we start to choose a model 4-manifold Y with $e(Y) = e$, $\sigma(Y) = \sigma$ and $b_1(Y) > 0$ which contains a $b_1(Y)$ disjoint pairs of embedded essential tori

$$\{(T_{1,1}, T_{1,2}), (T_{2,1}, T_{2,2}), \ldots, (T_{b_1(Y),1}, T_{b_1(Y),2})\}$$

such that $\{T_{1,1}, \ldots, T_{b_1(Y),1}\}$ carries the generators of $H_1(Y; \mathbb{Z})$ and it satisfies $[T_{i,1}]\cdot[T_{i,2}] = 1$, $[T_{i,1}]^2 = 0$ for all $i = 1, \ldots, b_1(Y)$. Then we perform ± 1-surgery on $T_{i,1}$ along the loop γ_i on $T_{i,1}$, so that we get a new smooth 4-manifold

$$Z := Y_{(T_{1,1},\gamma_1,\pm 1),(T_{2,1},\gamma_2,\pm 1),\ldots,(T_{b_1(Y),1},\gamma_{b_1(Y)},\pm 1)}.$$

Note that Z has the same homologies as X has because each ± 1-surgery does not change e and σ but it reduces the first Betti number b_1 up to 1. In general the resulting manifold Z is not simply connected, but if it is lucky in some cases, then one can prove that Z is simply connected and one gets a candidate of exotic smooth structures on X due to Freedman's classification theorem. Moreover, if one starts with a symplectic 4-manifold (Y, ω) as a model 4-manifold and if one can select each $T_{i,1}$ as Lagrangian tori, then one gets a new symplectic 4-manifold Z by performing a sequence of ± 1-Luttinger surgeries on $T_{i,1}$.

Now we explain how to construct infinitely many nondiffeomorphic 4-manifolds in homeomorphism type X. In order to do this, we first need Seiberg-Witten invariants.

The Seiberg-Witten invariant, denoted by SW_X, of an oriented closed smooth 4-manifold X is defined as an integer valued function on the set of $Spin^c$-structures over X. Since, for each $\mathfrak{s} \in Spin^c(X)$, there is a corresponding positive spinor bundle $W_{\mathfrak{s}}^+$ over X and $c(\mathfrak{s}) = PD(c_1(W_{\mathfrak{s}}^+))$ is a characteristic element of $H_2(X; \mathbb{Z})$, if $H_1(X; \mathbb{Z})$ has no 2-torsion, the Seiberg-Witten invariant of X can be considered as a function

$$SW_X : \{k \in H_2(X; \mathbb{Z}) \mid PD(k) \equiv w_2(X) \pmod 2\} \to \mathbb{Z}.$$

Note that, if $b_2^+(X) > 1$, it is a diffeomorphism invariant of X up to sign and only finitely many $Spin^c$-structures on X have a non-zero Seiberg-Witten invariant (In case $b_2^+(X) = 1$, the Seiberg-Witten invariant can be defined similarly, but it depends on the choice of chambers). The characteristic element $k \in H_2(X;\mathbb{Z})$, equivalently a corresponding *Spin*^{c}-structure $\mathfrak{s}$, is called a *SW-basic class* of X if $SW_X(k) \neq 0$.

Let X be a closed oriented smooth 4-manifold which contains a nullhomologous torus $\Lambda \subset X$ and $\gamma \subset \Lambda$ be a simple loop such that $[S^1_\gamma] = 0$ in $H_1(X \setminus N_\Lambda;\mathbb{Z})$. Suppose $X_{\Lambda,\gamma}(0)$ has a nontrivial Seiberg-Witten invariant and k_0 is a SW-basic class of $X_{\Lambda,\gamma}(0)$. Let T be the core torus of the surgery in $X_{\Lambda,\gamma}(0)$. Then T is an essential torus and $k_0 \cdot [T] = 0$ by adjunction inequality. Moreover the core torus Λ_n of the surgery in $X_{\Lambda,\gamma}(1/n)$ is also nullhomologous because its meridian μ_{Λ_n} represents the class $n[\lambda] + [\mu_\Lambda]$, which is homologous to a nontrivial class $[\mu_\Lambda]$ in $X_{\Lambda,\gamma}(1/n) \setminus N_{\Lambda_n} = X_{\Lambda,\gamma}(0) \setminus N_\Lambda$. Note that there are corresponding characteristic elements k_n in $H_2(X_{\Lambda,\gamma}(1/n);\mathbb{Z})$ and k in $H_2(X;\mathbb{Z})$, respectively, where k_n and k agree with the restriction of k_0 in $H_2(X \setminus N_\Lambda, \partial;\mathbb{Z})$. Then, by applying the product formula of J. Morgan, T. Mrowka, and Z. Szabo [12]

$$SW_{X_{\Lambda,\gamma}(1/n)}(k_n) = SW_X(k) + n\sum_{i\in\mathbb{Z}} SW_{X_{\Lambda,\gamma}(0)}(k_0 + 2i[T]).$$

R. Fintushel, D. Park, and R. Stern got the following theorem.

Theorem 2.3 ([9]). *Let X be a closed oriented smooth 4-manifold which contains a nullhomologous torus Λ and let $\gamma \subset \Lambda$ be a simple loop so that S^1_γ is nullhomologous in $X \setminus N_\Lambda$. If $X_{\Lambda,\gamma}(0)$ has nontrivial Seiberg-Witten invariant, then the set $\{X_{\Lambda,\gamma}(1/n) \mid n \in \mathbb{Z}^+\}$ contains infinitely many pairwise nondiffeomorphic 4-manifolds. Moreover, if $X_{\Lambda,\gamma}(0)$ has just one SW-basic class up to sign, then $\{X_{\Lambda,\gamma}(1/n) \mid n \in \mathbb{Z}^+\}$ are pairwise nondiffeomorphic.*

2.3. Building blocks

We review two basic building blocks, $\Sigma_2 \times \Sigma_2$ and $\mathrm{Sym}^2(\Sigma_3)$, to construct our exotic 4-manifolds. These two building blocks are originally studied by R. Fintushel, D. Park, and R. Stern for constructing small 4-manifolds [9].

Luttinger surgeries on $\Sigma_2 \times \Sigma_2$: Let us consider a symplectic 4-manifold $\Sigma_2 \times \Sigma_2$ with product symplectic structure ω. Note that $\Sigma_2 \times \Sigma_2$ has $e = 4$, $\sigma = 0$ and $b_1 = 8$. Then one can find 8 disjoint pairs of Lagrangian tori in $\Sigma_2 \times \Sigma_2$. Now perform seven ±1-Luttinger surgeries on disjoint Lagrangian tori in $\Sigma_2 \times \Sigma_2$ and one k-surgery as follows:

$$\begin{array}{llll} (a_1' \times c_1', a_1', -1), & (b_1' \times c_1'', b_1', -1), & (a_2' \times c_2', a_2', -1), & (b_2' \times c_2'', b_2', -1), \\ (a_2' \times c_1', c_1', +1), & (a_2'' \times d_1', d_1', +1), & (a_1' \times c_2', c_2', +1), & (a_1'' \times d_2', d_2', +k), \end{array}$$

where $\{a_1, b_1, a_2, b_2\}$, $\{c_1, d_1, c_2, d_2\}$ are the standard homotopy generators of Σ_2 based at x and y respectively and a_i', a_i'' are parallel copies of a_i in Σ_2 such that, when considering them as based loops, a_i' is homotopic to a_i and a_i'' is homotopic to $b_i a_i b_i^{-1}$ relative to the base point $\{x\} \times \{y\}$ (Similarly, one can choose parallel copies b_i', b_i'', b_i', b_i'', d_i', d_i'' for $i = 1, 2$). Then the following

18 relations are obtained in the fundamental group and it gives a family of homology $S^2 \times S^2$.

$$
\begin{array}{llll}
[b_1^{-1}, d_1^{-1}] = a_1, & [a_1^{-1}, d_1] = b_1, & [b_2^{-1}, d_2^{-1}] = a_2, & [a_2^{-1}, d_2] = b_2, \\
[d_1^{-1}, b_2^{-1}] = c_1, & [c_1^{-1}, b_2] = d_1, & [d_2^{-1}, b_1^{-1}] = c_2, & [c_2^{-1}, b_1]^k = d_2, \\
[a_1, c_1] = 1, & [a_1, c_2] = 1, & [a_1, d_2] = 1, & [b_1, c_1] = 1, \\
[a_2, c_1] = 1, & [a_2, c_2] = 1, & [a_2, d_1] = 1, & [b_2, c_2] = 1, \\
[a_1, b_1][a_2, b_2] = 1, & [c_1, d_1][c_2, d_2] = 1. & &
\end{array}
$$

Luttinger surgeries on $\mathrm{Sym}^2(\Sigma_3)$**:** Let $\mathrm{Sym}^2(\Sigma_3)$ be the 2-fold symmetric product of a genus 3 Riemann surface Σ_3, i.e., the quotient space of $\Sigma_3 \times \Sigma_3$ by using the involution $\tau : \Sigma_3 \times \Sigma_3 \to \Sigma_3 \times \Sigma_3$ defined by $\tau(v, w) = (w, v)$. Let $x \in \Sigma_3$ be a fixed base point of Σ_3 and let $\{f_i, g_i \,|\, 1 \leq 1 \leq 3\}$ be standard generators of $\pi_1(\Sigma_3, x)$. Let $z \in \mathrm{Sym}^2(\Sigma_3)$ be the image of (x, x), which will be considered as a fixed base point of $\mathrm{Sym}^2(\Sigma_3)$. Then $\pi_1(\mathrm{Sym}^2(\Sigma_3), z) = \mathbb{Z}^6$ and $\{f_i = f_i \times \{x\},\, g_i = g_i \times \{x\}|\ i = 1, 2, 3\}$ are generators of $\pi_1(\mathrm{Sym}^2(\Sigma_3), z)$. Note that $\mathrm{Sym}^2(\Sigma_3)$ also has $\sigma = -2$ and $e = 6$. In $\mathrm{Sym}^2(\Sigma_3)$ one can find 6 disjoint Lagrangian tori, so that one performs six ± 1-Luttinger surgeries as follows:

$$
\begin{array}{lll}
(f_1' \times f_2', f_2', -1), & (f_1'' \times g_2', g_2', -1), & (f_1' \times f_3'.f_1', -1), \\
(g_1' \times f_3'', g_1', -1), & (f_2' \times f_3', f_3', -1), & (f_2'' \times g_3', g_3', -1).
\end{array}
$$

In this way, R. Fintushel, D. Park, and R. Stern got a simply connected 4-manifold, which is homeomorphic but not diffeomorphic to $\mathbb{CP}^2 \sharp 3\overline{\mathbb{CP}}^2$.

Similarly, if one starts with $\mathrm{Sym}^2(\Sigma_g)$ $(g \geq 3)$ as a model 4-manifold and one performs (± 1)-Luttinger surgeries on $2g$ disjoint Lagrangian tori, then one can get a family of exotic $(g^2 - 3g + 1)\mathbb{CP}^2 \sharp (g^2 - 2g)\overline{\mathbb{CP}}^2$'s.

3. Proof of Theorem 1.1

In this section we prove the main result of this paper in two steps. We first construct an infinite family of homology $(2n+2l-1)\mathbb{CP}^2 \sharp (2n+4l-1)\overline{\mathbb{CP}}^2$'s by using a symplectic fiber sum and a Luttinger surgery. And then we will show that they are in fact all simply connected.

3.1. Construction

Let us consider n copies of $\Sigma_2 \times \Sigma_2$ with fixed base points (x_i, y_i) such that $x_1 = x_2 = \cdots = x_n \in \Sigma_2$ and $y_1 = y_2 = \cdots = y_n \in \Sigma_2$, and let $\{a_{i,j}, a'_{i,j}, a''_{i,j}, \ldots, d_{i,j}, d'_{i,j}, d''_{i,j}\}$ be the simple closed curves in the i-th copy of $\Sigma_2 \times \Sigma_2$ such that $a'_{i,j}$, $a''_{i,j}$ are parallel copies of $a_{i,j}$ and $a'_{i,j}$ is homotopic to $a_{i,j}$ and $a''_{i,j}$ is homotopic to $b_{i,j} a_{i,j} b_{i,j}^{-1}$ relative to the base point (x_i, y_i). Similarly, we choose parallel copies $b'_{i,j}$, $b''_{i,j}$, $b'_{i,j}$, $b''_{i,j}$, $d'_{i,j}$, $d''_{i,j}$ for $j = 1, 2$.

We define a symplectic 4-manifold Z_n inductively as follows: Let us denote $Z_1 = \Sigma_2 \times \Sigma_2$. Note that, when performing a symplectic fiber sum, we locally perturb the symplectic form so that $a''_{i,1} \times d'_{i,2}$ becomes a symplectic torus of self-intersection 0 while all other disjoint Lagrangian tori are still Lagrangian and $a''_{i,1} \times d''_{i,2}$ is a symplectic torus of self-intersection 0 in the i-th copy of $\Sigma_2 \times \Sigma_2$. Let $Z_i = Z_{i-1}\sharp_{a''_{i-1,1}\times d''_{i-1,2}=a''_{i,1}\times d'_{i,2}}(\Sigma_2 \times \Sigma_2)$ be a symplectic 4-manifold obtained by taking a symplectic fiber sum along symplectic torus $a''_{i-1,1} \times d''_{i-1,2}$ in Z_{i-1} and $a''_{i,1} \times d'_{i,2}$ in the i-th copy of $\Sigma_2 \times \Sigma_2$.

We also define $S_1 = \mathrm{Sym}^2(\Sigma_3)$ and $S_s = S_{s-1}\sharp_{f''_{s-1,1}\times g''_{s-1,2}=f''_{s,1}\times g'_{s,2}}\mathrm{Sym}^2(\Sigma_3)$ for $s \geq 2$ by performing a symplectic fiber sum along torus $f''_{s-1,1} \times g''_{s-1,2}$ and $f''_{s,1} \times g'_{s,2}$, where $\{f_{i,1}, g_{i,1}, f_{i,2}, g_{i,2}, f_{i,3}, g_{i,3}\}$ are generators of the s-th copy of $\pi_1(\mathrm{Sym}^2(\Sigma_3))$ corresponding to base point z_s and we consider $\{f'_{s,j}, f''_{s,j}, g'_{s,j}, g''_{s,j}\}$ as based loops in $\mathrm{Sym}^2(\Sigma_3)$ such that $f'_{s,j}$ is homotopic to $f_{s,j}$ and $f''_{s,j}$ is homotopic to $g_{s,j}f_{s,j}g_{s,j}^{-1}$ ([9] for details).

Next, for each integer $n \geq 0$, $l \geq 1$, we define a symplectic 4-manifold $Y_{l,n}$ by

$$Y_{l,0} = S_l, \quad Y_{l,n} = S_l\sharp_{f''_{s,1}\times g''_{s,2}=a''_{1,1}\times d'_{1,2}}Z_n.$$

Then we are able to construct a desired family of exotic 4-manifolds, denoted by $\widetilde{Y}_{l,n,k}$, from $Y_{l,n}$ by performing $7n+5l$ times ± 1-Luttinger surgeries and one more surgery as follows: 7 times ± 1-Luttinger surgeries on each copy of $\Sigma_2 \times \Sigma_2$ -

$$\begin{aligned}&(a'_{i,1} \times c'_{i,1}, a'_{i,1}, -1), (b'_{i,1} \times c''_{i,1}, b'_{i,1}, -1), (a'_{i,2} \times c'_{i,2}, a'_{i,2}, -1),\\&(b'_{i,2} \times c''_{i,2}, b'_{i,2}, -1), (a'_{i,2} \times c'_{i,1}, c'_{i,1}, +1), (a''_{i,2} \times d'_{i,1}, d'_{i,1}, +1),\\&(a'_{i,1} \times c'_{i,2}, c'_{i,2}, +1),\end{aligned}$$

5 times ± 1-Luttinger surgeries on each copy of $\mathrm{Sym}^2(\Sigma_3)$ -

$$\begin{aligned}&(f'_{i,1} \times f'_{i,2}, f'_{i,2}, -1), \ (f'_{i,1} \times f'_{i,3}.f'_{i,1}, -1), \ (g'_{i,1} \times f''_{i,3}, g'_{i,1}, -1),\\&(f'_{i,2} \times f'_{i,3}, f'_{i,3}, -1), \ (f''_{i,2} \times g'_{i,3}, g'_{i,3}, -1),\end{aligned}$$

and one more k-surgery on $f''_{1,1} \times g'_{1,2}$ along $g'_{1,2}$.

3.2. Computation of $\pi_1(\widetilde{Y}_{l,n,k})$

We take a base point (x_i, y_i) in each i-th copy of $\Sigma_2 \times \Sigma_2$ and z_s for the s-th copy of $\mathrm{Sym}^2(\Sigma_3)$. Note that $\pi_1(\Sigma_2 \times \Sigma_2, (x_i, y_i))$ is generated by $\{a_{i,1}, a_{i,2}, b_{i,1}, b_{i,2}, c_{i,1}, c_{i,2}, d_{i,1}, d_{i,2}\}$ and $\pi_1(\mathrm{Sym}^2(\Sigma_3), z_s)$ is generated by $\{f_{s,j}, g_{s,j} \,|\, j = 1, 2, 3\}$.

Now we first find a group presentation of $\pi_1(\widetilde{Y}_{l,n,k}, z_1)$. To do this we have to find a path from z_1 to a base point of $\Sigma_2 \times \Sigma_2$ or $\mathrm{Sym}^2(\Sigma_3)$ located in $Y_{l,n}$. We may assume that $z_s \in \partial\nu(f''_{s,1} \times g'_{s,2})$, the fixed base point of s-th copy of $\mathrm{Sym}^2(\Sigma_3)$. Since the complement of six disjoint Lagrangian tori in $\mathrm{Sym}^2(\Sigma_3)$, on which we perform a surgery, is path connected, there are paths η_s, $1 \leq s \leq l$, from z_s to a fixed point $Q_s \in \partial\nu(f''_{s,1} \times g''_{s,2})$. We may assume that η_s is located on $g_{s,1} \times f_{s,2}$ and a symplectic fiber sum identifies Q_s and z_{s+1}, $1 \leq s < l$. Similarly, since the complement of eight disjoint Lagrangian tori in $\Sigma_2 \times \Sigma_2$, on which we perform a surgery, is path connected, there are paths η_{l+i}, $1 \le i \le n$, from (x_i, y_i) to a fixed point $Q_{l+i} \in \partial\nu(a''_{i,1} \times d''_{i,2})$ such that η_{l+i} is located on $b_{i,1} \times c_{i,2}$. We can also assume that $(x_i, y_i) \in \partial\nu(a''_{i,1} \times d'_{i,2})$ and we identify Q_{l+i} and (x_{i+1}, y_{i+1}), $i = 0, 1, \ldots, n-1$, when performing a symplectic fiber sum. Let γ_i be a path from z_1 to z_s or (x_i, y_i) which is defined as follows: γ_1 is

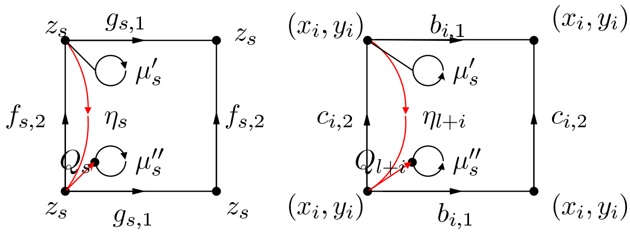


FIGURE 1

a constant path based at z_1 and $\gamma_s = \eta_1 \cdot \eta_2 \cdots \eta_{s-1}$ for $2 \le s \le l$. In the same way, $\gamma_{l+i} = \eta_1 \cdot \eta_2 \cdots \eta_l \cdot \eta_{l+1} \cdots \eta_{l+i-1}$ be a path connecting z_1 and (x_i, y_i).

In this article, we use the notation $\alpha(\beta) = \alpha \cdot \beta \cdot \alpha^{-1}$ for two paths α, β. Note that each symplectic fiber sum between $\mathrm{Sym}^2(\Sigma_3)$ gives the following 3 relations:

(1)
$$\begin{aligned}
&\gamma_{s-1}(g_{s-1,1}\ f_{s-1,1}\ g_{s-1,1}^{-1}) = \gamma_s(g_{s,1}\ f_{s,1}\ g_{s,1}^{-1}),\\
&\gamma_{s-1}(f_{s-1,2}\ g_{s-1,2}\ f_{s-1,2}^{-1}) = \gamma_s(g_{s,2}),\\
&\gamma_{s-1}(\mu''_{s-1}) = \gamma_s(\mu'^{-1}_s),
\end{aligned}$$

as loops based at z_1, where μ'_{s-1} is a meridian of $f''_{s-1,1} \times g'_{s-1,2}$ and μ''_{s-1} is a meridian of $f''_{s-1,1} \times g''_{s-1,2}$ which are considered as loops based at z_{s-1}. When we perform a symplectic fiber sum between S_l and $\Sigma_2 \times \Sigma_2$, we also have to add the following relations:

(2)
$$\begin{aligned}
&\gamma_{l+1}(b_{1,1}\ a_{1,1}\ b_{1,1}^{-1}) = \gamma_l(g_{l,1}\ f_{l,1}\ g_{l,1}^{-1}),\\
&\gamma_{l+1}(d_{1,2}) = \gamma_l(f_{l,2}\ g_{l,2}\ f_{l,2}^{-1}), \quad \gamma_l(\mu''_l) = \gamma_{l+1}(\mu'^{-1}_{l+1}).
\end{aligned}$$

Similarly, performing a symplectic fiber sum between $Y_{l,i-1}$ $(i \ge 2)$ and $\Sigma_2 \times \Sigma_2$, the following 3 relations hold:

(3)
$$\begin{aligned}
&\gamma_{l+i-1}(b_{i-1,1}\ a_{i-1,1}\ b_{i-1,1}^{-1}) = \gamma_{l+i}(b_{i,1}\ a_{i,1}\ b_{i,1}^{-1}),\\
&\gamma_{l+i-1}(c_{i-1,2}\ d_{i-1,2}\ c_{i-1,2}^{-1}) = \gamma_{l+i}(d_{i,2}),\\
&\gamma_{l+i-1}(\mu''_{l+i-1}) = \gamma_{l+i}(\mu'^{-1}_{i+i}),
\end{aligned}$$

where μ'_{l+i-1} is a meridian of $a''_{i-1,1} \times d'_{i-1,2}$ and μ''_{l+i-1} is a meridian of $a''_{i-1,1} \times d''_{i-1,2}$ which are considered as a loop based at (x_{i-1}, y_{i-1}).

Note that, after performing 7 times ±1-Luttinger surgeries in each copy of $\Sigma_2 \times \Sigma_2$ lying in $Y_{l,n}$, we have the following 17 relations:

$$
\begin{array}{llll}
(4) & [b_{i,1}^{-1}, d_{i,1}^{-1}] = a_{i,1}, & [a_{i,1}^{-1}, d_{i,1}] = b_{i,1}, & [b_{i,2}^{-1}, d_{i,2}^{-1}] = a_{i,2}, \\
& [a_{i,2}^{-1}, d_{i,2}] = b_{i,2}, & [d_{i,1}^{-1}, b_{i,2}^{-1}] = c_{i,1}, & [c_{i,1}^{-1}, b_{i,2}] = d_{i,1}, \\
& [d_{i,2}^{-1}, b_{i,1}^{-1}] = c_{i,2}, & [a_{i,1}, c_{i,1}] = 1, & [a_{i,1}, c_{i,2}] = 1, \\
& [a_{i,1}, d_{i,2}] = 1, & [b_{i,1}, c_{i,1}] = 1, & [a_{i,2}, c_{i,1}] = 1, \\
& [a_{i,2}, c_{i,2}] = 1, & [a_{i,2}, d_{i,1}] = 1, & [b_{i,2}, c_{i,2}] = 1, \\
& [a_{i,1}, b_{i,1}][a_{i,2}, b_{i,2}] = 1, & [c_{i,1}, d_{i,1}][c_{i,2}, d_{i,2}] = 1, &
\end{array}
$$

as loops based at (x_i, y_i). Furthermore, after performing 5 times ±1-Luttinger surgeries in each copy of $\mathrm{Sym}^2(\Sigma_3)$ lying in $Y_{l,n}$, we also have the following 14 relations:

(5)

$$
\begin{array}{llll}
[g_{s,1}^{-1}, g_{s,2}^{-1}] = f_{s,2}, & [g_{s,1}^{-1}, g_{s,3}^{-1}] = f_{s,1}, & [f_{s,1}^{-1}, g_{s,3}] = g_{s,1}, & [g_{s,2}^{-1}, g_{s,3}^{-1}] = f_{s,3}, \\
[g_{s,2}, f_{s,3}^{-1}] = g_{s,3}, & [f_{s,1}, g_{s,1}] = 1, & [f_{s,1}, f_{s,2}] = 1, & [f_{s,1}, g_{s,2}] = 1, \\
[f_{s,1}, f_{s,3}] = 1, & [g_{s,1}, f_{s,3}] = 1, & [f_{s,2}, g_{s,2}] = 1, & [f_{s,2}, f_{s,3}] = 1, \\
[f_{s,2}, g_{s,3}] = 1, & [f_{s,3}, g_{s,3}] = 1, & &
\end{array}
$$

as loops based at z_s and it gives

$$
\begin{aligned}
(6) \qquad & \gamma_s(f_{s,1}) = \gamma_s(f_{s,2}) = \gamma_s(g_{s,1}) = 1, \\
& \gamma_s([g_{s,2}^{-1}, g_{s,3}^{-1}]) = \gamma_s(f_{s,3}), \quad \gamma_s([g_{s,2}, f_{s,3}^{-1}]) = \gamma_s(g_{s,3})
\end{aligned}
$$

as loops based at z_1 by the same method as in [9], i.e.,

$$g_{s,1} = [f_{s,1}^{-1}, g_{s,3}] = [f_{s,1}^{-1}, [g_{s,2}, f_{s,3}^{-1}]] = 1$$

because $[f_{s,1}, f_{s,3}] = 1$ and $[f_{s,1}, g_{s,2}] = 1$.

Note that, since $g_{1,1} = 1$ due to the relation (6), the last surgery $(f''_{1,1} \times g'_{1,2}, g'_{1,2}, +k)$ gives a relation

$$(7) \qquad g_{1,2} = ([g_{1,1}, f_{1,2}^{-1}])^k = 1,$$

so that the relations (1) and (7) give $\gamma_s(g_{s,2}) = 1$ for each $s = 2, \dots, l$ and if we apply this to the relation (6), we also get

$$(8) \qquad \gamma_s(f_{s,m}) = \gamma_s(g_{s,m}) = 1 \text{ for all } s = 1, \dots, l, \quad m = 1, 2, 3.$$

Hence the relations (8) and (2) imply

$$\gamma_{l+1}(a_{1,1}) = 1 = \gamma_{l+1}(d_{1,2})$$

as loops based at z_1 and if we apply this to the relation (4) again, then we also get

$$(9) \qquad \gamma_{l+1}(a_{1,j}) = \gamma_{l+1}(b_{1,j}) = \gamma_{l+1}(c_{1,j}) = \gamma_{l+1}(d_{1,j}) = 1 \text{ for } j = 1, 2.$$

Now, by an induction argument using relations (9), (3) and (4), we get

(10)

$$\gamma_{l+i}(a_{i,j}) = \gamma_{l+i}(b_{i,j}) = \gamma_{l+i}(c_{i,j}) = \gamma_{l+i}(d_{i,j}) = 1 \text{ for all } 1 \le i \le n,\ j = 1, 2.$$

Therefore we have $\pi_1(\widetilde{Y}_{l,n,k}, z_1) = 1$ from the relations (8) and (10).

3.3. Exotic smooth structures

Since $e(\mathrm{Sym}^2(\Sigma_3)) = 6$ and $\sigma(\mathrm{Sym}^2(\Sigma_3)) = -2$, we can get easily $e(Y_{l,n}) = 4n + 6l$, $\sigma(Y_{l,n}) = -2l$ and $e(\widetilde{Y}_{l,n,k}) = 4n + 6l$, $\sigma(\widetilde{Y}_{l,n,k}) = -2l$, $b_1(\widetilde{Y}_{l,n,k}) = 0$, so that we have $b_2^+(\widetilde{Y}_{l,n,k})) = 2n + 2l - 1$, and $b_2^-(\widetilde{Y}_{l,n,k})) = 2n + 4l - 1$. Furthermore, since there are at least $3l$ disjoint tori of square -1 in $\widetilde{Y}_{l,n,k}$, descended from each copy of $\mathrm{Sym}^2(\Sigma_3)$, the manifold $\widetilde{Y}_{l,n,k}$ is nonspin, so that it is homeomorphic to the connected sum $(2n + 2l - 1)\mathbb{CP}^2\sharp(2n + 4l - 1)\overline{\mathbb{CP}}^2$.

Finally, by applying Theorem 2.3 to $\widetilde{Y}_{l,n,1}$, we conclude that a family of 4-manifolds $\{\widetilde{Y}_{l,n,k} \mid k \ge 1\}$ contain infinitely many pairwise non-diffeomorphic exotic $(2n + 2l - 1)\mathbb{CP}^2\sharp(2n + 4l - 1)\overline{\mathbb{CP}}^2$'s.

Remark. Note that the fundamental group $\pi_1(\widetilde{Y}_{l,n,k})$ is trivial due to the presence of $\mathrm{Sym}^2(\Sigma_3)$. But, in the case $l = 0$, we do not know whether it is trivial or not as mentioned in [9].

4. More examples

In the section we explore more examples which can be obtained by using the same method as above but with different building blocks.

Let us denote by

$$X^g_{n,l} = \underbrace{\mathrm{Sym}^2(\Sigma_g)\sharp\cdots\sharp\mathrm{Sym}^2(\Sigma_g)}_{l\ge 1}\sharp\underbrace{(\Sigma_2\times\Sigma_2)\sharp\cdots\sharp(\Sigma_2\times\Sigma_2)}_{n}$$

a symplectic 4-manifold obtained by doing symplectic fiber sums along

- $a_1'' \times d_2''$ in $\Sigma_2 \times \Sigma_2$ and $a_1'' \times d_2'$ in $\Sigma_2 \times \Sigma_2$
- $f_1'' \times g_2''$ in $\mathrm{Sym}^2(\Sigma_g)$ and $f_1'' \times g_2'$ in $\mathrm{Sym}^2(\Sigma_g)$
- $f_1'' \times g_2''$ in $\mathrm{Sym}^2(\Sigma_g)$ and $a_1'' \times d_2'$ in $\Sigma_2 \times \Sigma_2$.

Then, by an easy computation of Euler number and signature, we know that $\sigma(X^g_{n,l}) = l(1-g)$, $e(X^g_{n,l}) = 4n + l(2g^2 - 5g + 3)$ and $b_1(X^g_{n,l}) = nb_1(\Sigma_2 \times \Sigma_2) + lb_1(\mathrm{Sym}^2(\Sigma_g)) - (n + l - 1) = 7n + (2g-1)l + 1$. Moreover we can find $7n + (2g-1)l + 1$ disjoint tori in $X^g_{n,l}$ which consist of $7n + (2g-1)l$ Lagrangian tori and one symplectic torus. So $X^g_{n,l}$ can be a Symplectic 4-manifold as a model for exotic $(2n + l(g-1)(g-2) - 1)\mathbb{CP}^2\sharp(2n + l(g-1)^2 - 1)\overline{\mathbb{CP}}^2$.

Now, by performing $7n + (2g - 1)l$ times ± 1-Luttinger surgeries on each Lagrangian tori and one more k-surgery on a symplectic torus in $X^g_{n,l}$, we construct a candidate $Z^g_{n,l}$ for exotic $(2n + l(g-1)(g-2) - 1)\mathbb{CP}^2\sharp(2n + l(g-1)^2 - 1)\overline{\mathbb{CP}}^2$'s. And then, by the same way as in the proof of Theorem 1.1

above, we can prove that $\pi_1(Z^g_{n,l}) = 1$. Therefore we get an infinite family of nondiffeomorphic exotic $(2n+l(g-1)(g-2)-1)\mathbb{CP}^2\sharp(2n+l(g-1)^2-1)\overline{\mathbb{CP}}^2$'s. These examples are lying on $c_1^2 = 8\chi_h - l(g-1)$ in the (χ_h, c_1^2)-plane with $\chi_h = \frac{e+\sigma}{4}$ and $c_1^2 = 3\sigma + 2e$.

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