Bull. Korean Math. Soc. **47** (2010), No. 5, pp. 1089–1103
DOI 10.4134/BKMS.2010.47.5.1089

LIGHTLIKE SUBMANIFOLDS OF A SEMI-RIEMANNIAN MANIFOLD WITH A SEMI-SYMMETRIC NON-METRIC CONNECTION

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Abstract. In this paper, we study lightlike submanifolds of a semi-Riemannian manifold admitting a semi-symmetric non-metric connection. We obtain a necessary and a sufficient condition for integrability of the screen distribution. Then we give the conditions under which the Ricci tensor of a lightlike submanifold with a semi-symmetric non-metric connection is symmetric. Finally, we show that the Ricci tensor of a lightlike submanifold of semi-Riemannian space form is not parallel with respect to the semi-symmetric non-metric connection.

1. Introduction

The idea of a semi-symmetric non-metric connection on a Riemannian manifold was introduced by Ageshe and Chafle [1]. They defined a linear connection on a Riemannian manifold admitting a semi-symmetric non-metric connection, and studied some properties of the curvature tensor of a Riemannian manifold with respect to the semi-symmetric non-metric connection. De and Kamilya [4] gave basic properties of a hypersurface of a Riemannian manifold with a semi-symmetric non-metric connection. Ageshe and Chafle [2] obtained the equations of Gauss, Codazzi and Ricci associated with a semi-symmetric non-metric connection and studied some properties of the submanifold of a space of constant curvature admitting a semi-symmetric non-metric connection. In an earlier paper [14], we studied lightlike hypersurfaces of a semi-Riemannian manifold admitting a semi-symmetric non-metric connection.

In this paper, we study lightlike submanifolds of a semi-Riemannian manifold with respect to the semi-symmetric non-metric connection because of

Received April 14, 2009.

2000 *Mathematics Subject Classification.* 53B15, 53B30, 53C05, 53C50.

Key words and phrases. lightlike submanifolds, semi-symmetric non-metric connection, Levi-Civita connection, Ricci tensor.

the following motivation: It is well known that while the geometry of semi-Riemannian manifold is fully developed, its counter part of lightlike submanifolds (for which the local geometry is completely different from the non-degenerate case) is relatively new and in a developing stage. When the need in general relativistic theories is considered, to study the general theory of lightlike submanifold for differential geometry is a very important topic. Several papers have been written on lightlike submanifolds in recent years (see [3], [7], [9] for instance) but the use of semi-symmetric non-metric connections has not been handled widely.

In the present paper, we have proved that on lightlike submanifold the connection induced from semi-symmetric non-metric connection is semi-symmetric non-metric, and also on screen distribution the connection induced from that connection is semi-symmetric non-metric connection. We have defined the induced geometrical objects with respect to the semi-symmetric non-metric connection on the triplet $(S(TM), S(TM^{\perp}), \mathrm{tr}(TM))$. Then we have investigated the integrability condition of the screen distribution with respect to the semi-symmetric non-metric connection. Also, we have given the conditions under which the Ricci tensor of a lightlike submanifold with respect to the semi-symmetric non-metric connection is symmetric. Moreover, we have shown that the Ricci tensor of a lightlike submanifold of semi-Riemannian space form is not parallel with respect to the semi-symmetric non-metric connection.

2. Preliminaries

Let $(\widetilde{M}, \widetilde{g})$ be a real $(m+n)$-dimensional semi-Riemannian manifold of constant index such that $1 \leq \nu \leq m+n-1$ and (M, g) be an m-dimensional submanifold of $\widetilde{M}$. In case $\widetilde{g}$ is degenerate on the tangent bundle TM of M, M is called a lightlike submanifold of $\widetilde{M}$. Denote by g the induced tensor field of $\widetilde{g}$ on M and suppose g is degenerate. Then, for each tangent space T_xM we consider

$$T_xM^{\perp} = \left\{ Y_x \in T_x\widetilde{M} \mid \widetilde{g}_x\left(Y_x, X_x\right) = 0, \forall X_x \in T_xM \right\}$$

which is a degenerate n-dimensional subspace of $T_x\widetilde{M}$. Thus, both T_xM and $T_xM^{\perp}$ are degenerate orthogonal subspaces but no longer complementary subspaces. For this case, there exists a subspace $\mathrm{Rad}T_xM = T_xM \cap T_xM^{\perp}$ called *radical* (*null*) *subspace*. If the mapping

$$\mathrm{Rad}TM : x \in M \longrightarrow \mathrm{Rad}T_xM$$

defines a smooth distribution on M of rank $r > 0$, the submanifold M of $\widetilde{M}$ is called *r-lightlike* (*r-degenerate*) *submanifold* and $\mathrm{Rad}TM$ is called the *radical* (*lightlike*) *distribution* on M. In the following, there are four possible cases:

Case 1. M is called a r-lightlike submanifold if $1 \leq r < \min\{m, n\}$.

Case 2. M is called a coisotropic submanifold if $1 < r = n < m$.

Case 3. M is called an isotropic submanifold if $1 < r = m < n$.

Case 4. M is called a totally lightlike submanifold if $1 < r = m = n$ [5].

In this paper, we have considered Case 1 where there exists a non-degenerate screen distribution $S(TM)$ which is a complementary vector subbundle to RadTM in TM. Therefore,

$$TM = \text{Rad}TM \perp S\,(TM)\,, \tag{2.1}$$

in which $\perp$ denotes orthogonal direct sum. Although $S(TM)$ is not unique, it is canonically isomorphic to the factor vector bundle $TM/\text{Rad}TM$. Denote an r-lightlike submanifold by $(M, g, S(TM), S(TM^{\perp}))$, where $S(TM^{\perp})$ is a complementary vector bundle of RadTM in $TM^{\perp}$ and $S(TM^{\perp})$ is non-degenerate with respect to $\widetilde{g}$. Let us define that tr(TM) is a complementary (but never orthogonal) vectors bundle to TM in $T\widetilde{M}_{|_M}$ and

$$\text{tr}(TM) = \text{ltr}(TM) \perp S(TM^{\perp}), \tag{2.2}$$

where ltr(TM) is an arbitrary lightlike transversal vector bundle of M. Then we have

$$\begin{aligned} T\widetilde{M}_{|_M} &= TM \oplus \text{tr}(TM) \\ &= (\text{Rad}TM \oplus \text{ltr}(TM)) \perp S(TM) \perp S(TM^{\perp}), \end{aligned} \tag{2.3}$$

where $\oplus$ denotes direct sum, but it is not orthogonal [5].

Now we assume that $\mathcal{U}$ is a local coordinate neighborhood of M. We consider the following local quasi-orthonormal field of frames on $\widetilde{M}$ along M:

$$\{\xi_1, \ldots, \xi_r, X_{r+1}, \ldots, X_m, N_1, \ldots, N_r, W_{r+1}, \ldots, W_n\}, \tag{2.4}$$

where $\{\xi_1, \ldots, \xi_r\}$ and $\{N_1, \ldots, N_r\}$ are lightlike basis of $\Gamma(\text{Rad}(TM)_{|_\mathcal{U}})$ and $\Gamma(\text{ltr}(TM)_{|_\mathcal{U}})$, respectively and $\{X_{r+1}, \ldots, X_m\}$ and $\{W_{r+1}, \ldots, W_n\}$ are orthonormal basis of $\Gamma(S(TM)_{|_\mathcal{U}})$ and $\Gamma(S(TM^{\perp})_{|_\mathcal{U}})$, respectively, where the following conditions are satisfied ([5])

$$\widetilde{g}(N_i, \xi_j) = \delta_{ij},\ 1 \leq i, j \leq r, \quad \widetilde{g}(N_i, N_j) = \widetilde{g}(N_i, X_k) = 0,\ r+1 \leq k \leq m,$$
$$X_k \in \Gamma(S(TM)_{|_\mathcal{U}}),\ N_i \in \Gamma(\text{ltr}(TM)_{|_\mathcal{U}}).$$

Example 2.1 ([5])**.** Consider in $\mathbb{R}^4_2$ the 1-lightlike submanifold M given by the equations:

$$x^3 = \frac{1}{\sqrt{2}}(x^1 + x^2),\ x^4 = \frac{1}{2}\log(1 + (x^1 - x^2)^2).$$

Then we have $TM = \text{Span}\{U_1, U_2\}$ and $TM^{\perp} = \text{Span}\{H_1, H_2\}$ where we set

$$U_1 = \sqrt{2}(1 + (x^1 - x^2)^2)\frac{\partial}{\partial x^1} + (1 + (x^1 - x^2)^2)\frac{\partial}{\partial x^3} + \sqrt{2}(x^1 - x^2)\frac{\partial}{\partial x^4},$$

$$U_2 = \sqrt{2}(1 + (x^1 - x^2)^2)\frac{\partial}{\partial x^2} + (1 + (x^1 - x^2)^2)\frac{\partial}{\partial x^3} - \sqrt{2}(x^1 - x^2)\frac{\partial}{\partial x^4},$$

and

$$H_1 = \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} + \sqrt{2}\frac{\partial}{\partial x^3},$$

$$H_2 = 2(x^2 - x^1)\frac{\partial}{\partial x^2} + \sqrt{2}(x^2 - x^1)\frac{\partial}{\partial x^3} + (1 + (x^1 - x^2)^2)\frac{\partial}{\partial x^4}.$$

It follows that $\mathrm{Rad}(TM)$ is a distribution on M of rank 1 spanned by $\xi = H_1$. Choose $S(TM)$ and $S(TM^{\perp})$ spanned by U_2 and H_2 which are timelike and spacelike respectively. Finally, the lightlike transversal vector bundle

$$\mathrm{ltr}(TM) = \mathrm{Span}\left\{N = -\frac{1}{2}\frac{\partial}{\partial x^1} + \frac{1}{2}\frac{\partial}{\partial x^2} + \frac{1}{\sqrt{2}}\frac{\partial}{\partial x^3}\right\},$$

and the transversal vector bundle

$$\mathrm{tr}(TM) = \mathrm{Span}\{N, H_2\}.$$

are obtained.

3. Semi-symmetric non-metric connection

Let $\widetilde{M}$ be an $(m+n)$-dimensional semi-Riemannian manifold with a semi-Riemannian metric $\widetilde{g}$ of index $1 \le \nu \le m+n-1$. A linear connection $\widetilde{\nabla}$ on $\widetilde{M}$ is called a semi-symmetric non-metric connection if

$$(\widetilde{\nabla}_{\widetilde{X}}\widetilde{g})(\widetilde{Y}, \widetilde{Z}) = -\widetilde{\pi}(\widetilde{Y})\widetilde{g}(\widetilde{X}, \widetilde{Z}) - \widetilde{\pi}(\widetilde{Z})\widetilde{g}(\widetilde{X}, \widetilde{Y})$$

and the torsion tensor $\widetilde{T}$ of $\widetilde{\nabla}$ satisfies

$$\widetilde{T}(\widetilde{X}, \widetilde{Y}) = \widetilde{\pi}(\widetilde{Y})\widetilde{X} - \widetilde{\pi}(\widetilde{X})\widetilde{Y}$$

for any $\widetilde{X}, \widetilde{Y}, \widetilde{Z} \in \Gamma(T\widetilde{M})$, where $\widetilde{\pi}$ is a 1-form on $\widetilde{M}$ [1].

We can now suppose that the semi-Riemannian manifold $\widetilde{M}$ admits a semi-symmetric non-metric connection $\widetilde{\nabla}$ given by

$$\widetilde{\nabla}_{\widetilde{X}}\widetilde{Y} = \overset{\circ}{\widetilde{\nabla}}_{\widetilde{X}}\widetilde{Y} + \widetilde{\pi}(\widetilde{Y})\widetilde{X} \tag{3.1}$$

for any $\widetilde{X}, \widetilde{Y} \in \Gamma(T\widetilde{M})$, where $\overset{\circ}{\widetilde{\nabla}}$ is a Levi-civita connection with respect to $\widetilde{g}$ and $\widetilde{\pi}$ is a 1-form associated with the vector field $\widetilde{Q}$ on $\widetilde{M}$ given by

$$\widetilde{\pi}(\widetilde{X}) = \widetilde{g}(\widetilde{X}, \widetilde{Q})$$

(see [1]). By using the first form of the decomposition (2.3), we can write

$$\widetilde{Q} = Q + \sum_{i=1}^{r}\lambda_i N_i + \sum_{\alpha=r+1}^{n}\lambda_\alpha W_\alpha, \tag{3.2}$$

where Q is a vector field and λ_a, $1 \le a \le n$ are real valued functions on M.

If we denote by $\overset{\circ}{\nabla}$ the symmetric linear connection induced on M from $\overset{\circ}{\widetilde{\nabla}}$ on $\widetilde{M}$, then we have the Gauss formula with respect to $\overset{\circ}{\nabla}$ for $X, Y \in \Gamma(TM)$, $N_i \in \Gamma(\mathrm{ltr}(TM))$, $1 \le i \le r$ and $W_\alpha \in \Gamma(S(TM^{\perp}))$, $r+1 \le \alpha \le n$

$$\overset{\circ}{\widetilde{\nabla}}_X Y = \overset{\circ}{\nabla}_X Y + \sum_{i=1}^{r}\overset{\circ}{h}{}^{\ell}_i(X, Y)N_i + \sum_{\alpha=r+1}^{n}\overset{\circ}{h}{}^{s}_\alpha(X, Y)W_\alpha, \tag{3.3}$$

where $\{\overset{\circ}{h}{}_i^{\ell}\}$ and $\{\overset{\circ}{h}{}_\alpha^{s}\}$ are called the local lightlike second fundamental forms and the local screen second fundamental forms of M which are symmetric bilinear forms [5]. Let us define the connection ∇ on M that is induced from the semi-symmetric non-metric connection $\widetilde{\nabla}$ on $\widetilde{M}$ given by the equation below is called the Gauss formula with respect to $\widetilde{\nabla}$ for any $X, Y \in \Gamma(TM)$

$$(3.4)\qquad \widetilde{\nabla}_X Y = \nabla_X Y + \sum_{i=1}^{r} h_i^{\ell}(X,Y)N_i + \sum_{\alpha=r+1}^{n} h_\alpha^{s}(X,Y)W_\alpha,$$

where $\{h_i^{\ell}\}$ and $\{h_\alpha^{s}\}$ are called the local lightlike second fundamental forms and the local screen second fundamental forms of M which are tensors of type $(0,2)$ on M.

In view of (3.1), we get

$$\widetilde{\nabla}_X Y = \overset{\circ}{\widetilde{\nabla}}_X Y + \widetilde{\pi}(Y)X$$

and therefore, by using (3.3) and (3.4), we can also write

$$(3.5)\qquad \nabla_X Y + \sum_{i=1}^{r} h_i^{\ell}(X,Y)N_i + \sum_{\alpha=r+1}^{n} h_\alpha^{s}(X,Y)W_\alpha$$
$$= \overset{\circ}{\nabla}_X Y + \sum_{i=1}^{r} \overset{\circ}{h}{}_i^{\ell}(X,Y)N_i + \sum_{\alpha=r+1}^{n} \overset{\circ}{h}{}_\alpha^{s}(X,Y)W_\alpha + \widetilde{\pi}(Y)X$$

from which we have

$$(3.6)\qquad \nabla_X Y = \overset{\circ}{\nabla}_X Y + \pi(Y)X,$$

where $\pi(Y) = \widetilde{\pi}(Y)$ and we also have

$$(3.7)\qquad h_i^{\ell} = \overset{\circ}{h}{}_i^{\ell} \text{ and } h_\alpha^{s} = \overset{\circ}{h}{}_\alpha^{s},\ 1 \le i \le r,\ r+1 \le \alpha \le n$$

for any $X, Y \in \Gamma(TM)$.

Taking account of (3.6) and the connection induced on lightlike submanifold from Levi-Civita connection is not metric, we get

$$(3.8)\qquad \begin{aligned}(\nabla_X g)(Y,Z) &= \sum_{i=1}^{r}\{h_i^{\ell}(X,Y)\eta_i(Z) + h_i^{\ell}(X,Z)\eta_i(Y)\}\\ &\quad -\pi(Y)g(X,Z) - \pi(Z)g(X,Y),\end{aligned}$$

where

$$(3.9)\qquad \eta_i(Z) = g(N_i, Z),\ 1 \le i \le r$$

for any $X, Y, Z \in \Gamma(TM)$ and $N_i \in \Gamma(\mathrm{ltr}(TM))$. We also obtain from (3.6)

$$(3.10)\qquad T(X,Y) = \pi(Y)X - \pi(X)Y$$

in which T denotes the torsion tensor of the connection ∇. Then from (3.8) and (3.10) we have:

Proposition 3.1. *The induced connection on a lightlike submanifold of a semi-Riemannian manifold with a semi-symmetric non-metric connection is also semi-symmetric non-metric connection.*

The Weingarten formulae with respect to $\overset{\circ}{\widetilde{\nabla}}$ is given by

$$\overset{\circ}{\widetilde{\nabla}}_X N_i = -\overset{\circ}{A}_{N_i} X + \overset{\circ}{\nabla}{}^{\ell}_X N_i + \overset{\circ}{D}{}^{s}(X, N_i),\ 1 \leq i \leq r, \tag{3.11}$$

and

$$\overset{\circ}{\widetilde{\nabla}}_X W_\alpha = -\overset{\circ}{A}_{W_\alpha} X + \overset{\circ}{D}{}^{\ell}(X, W_\alpha) + \overset{\circ}{\nabla}{}^{s}_X W_\alpha,\ r+1 \leq \alpha \leq n \tag{3.12}$$

for any $X \in \Gamma(TM)$, $N_i \in \Gamma(\mathrm{ltr}(TM))$ and $W_\alpha \in \Gamma(S(TM^\perp))$, where

$$\overset{\circ}{\nabla}{}^{\ell}_X : \Gamma(\mathrm{ltr}(TM) \longrightarrow \Gamma(\mathrm{ltr}(TM);\ \overset{\circ}{\nabla}{}^{\ell}_X(LV) = \overset{\circ}{D}{}^{\ell}_X(LV),$$

$$\overset{\circ}{\nabla}{}^{s}_X : \Gamma(S(TM^\perp)) \longrightarrow \Gamma(S(TM^\perp));\ \overset{\circ}{\nabla}{}^{s}_X(SV) = \overset{\circ}{D}{}^{s}_X(SV),$$

$$\overset{\circ}{D}{}^{\ell} : \Gamma(TM) \times \Gamma(S(TM^\perp)) \longrightarrow \Gamma(\mathrm{ltr}(TM));\ \overset{\circ}{D}{}^{\ell}(X, SV) = \overset{\circ}{D}{}^{\ell}_X(SV),$$

$$\overset{\circ}{D}{}^{s} : \Gamma(TM) \times \Gamma(\mathrm{ltr}(TM)) \longrightarrow \Gamma(S(TM^\perp));\ \overset{\circ}{D}{}^{s}(X, LV) = \overset{\circ}{D}{}^{s}_X(LV)$$

for any $V \in \Gamma(\mathrm{tr}(TM))$ such that L and S are the projection morphisms of $\mathrm{tr}(TM)$ on $\mathrm{ltr}(TM)$ and $S(TM^\perp)$ respectively. Also $\overset{\circ}{\nabla}{}^{\ell}$ and $\overset{\circ}{\nabla}{}^{s}$ are linear connections on $\mathrm{ltr}(TM)$ and $S(TM^\perp)$, respectively, $\overset{\circ}{A}_{N_i}$ and $\overset{\circ}{A}_{W_\alpha}$ are called the shape operators of M with respect to N_i and W_α, respectively [5]. We define

$$\rho_{ij}(X) = \widetilde{g}(\overset{\circ}{\nabla}{}^{\ell}_X N_i, \xi_j),\ 1 \leq i, j \leq r,$$

$$\sigma_{i\alpha}(X) = \varepsilon_\alpha \widetilde{g}(\overset{\circ}{D}{}^{s}(X, N_i), W_\alpha),\ r+1 \leq \alpha \leq n,\ 1 \leq i \leq r,$$

$$\gamma_{\alpha j}(X) = \widetilde{g}(\overset{\circ}{D}{}^{\ell}(X, W_\alpha), \xi_j),\ r+1 \leq \alpha \leq n,\ 1 \leq j \leq r,$$

$$\mu_{\alpha\beta}(X) = \varepsilon_\beta \widetilde{g}(\overset{\circ}{\nabla}{}^{s}_X W_\alpha, W_\beta),\ r+1 \leq \alpha, \beta \leq n$$

for any $X \in \Gamma(TM)$, $N_i \in \Gamma(\mathrm{ltr}(TM)$ and $W_\alpha \in \Gamma(S(TM^\perp))$. Thus, (3.11) and (3.12) become

$$\overset{\circ}{\widetilde{\nabla}}_X N_i = -\overset{\circ}{A}_{N_i} X + \sum_{j=1}^{r} \rho_{ij}(X) N_j + \sum_{\alpha=r+1}^{n} \sigma_{i\alpha}(X) W_\alpha, 1 \leq i \leq r \tag{3.13}$$

and

$$\overset{\circ}{\widetilde{\nabla}}_X W_\alpha = -\overset{\circ}{A}_{W_\alpha} X + \sum_{j=1}^{r} \gamma_{\alpha j}(X) N_j + \sum_{\beta=r+1}^{n} \mu_{\alpha\beta}(X) W_\beta,\ r+1 \leq \alpha \leq n. \tag{3.14}$$

Let us denote by P the projection morphism of TM on $S(TM)$ with respect to the decomposition (2.1). Then we can write $X = PX + \sum_{i=1}^{r} \eta_i(X)\xi_i$ for any $X \in \Gamma(TM)$. Next, by using (3.1) and (3.2) we have

$$\widetilde{\nabla}_X N_i = \overset{\circ}{\widetilde{\nabla}}_X N_i + \eta_i(Q)X,\ 1 \leq i \leq r,$$

$$\widetilde{\nabla}_X W_\alpha = \overset{\circ}{\widetilde{\nabla}}_X W_\alpha + \varepsilon_\alpha \lambda_\alpha X,\ r+1 \leq \alpha \leq n.$$

Substituting (3.13) and (3.14) into the above equations, we obtain

$$\widetilde{\nabla}_X N_i = (-\overset{\circ}{A}_{N_i} + \eta_i(Q)I)X + \sum_{j=1}^{r} \rho_{ij}(X)N_j + \sum_{\alpha=r+1}^{n} \sigma_{i\alpha}(X)W_\alpha,\ 1 \leq i \leq r, \tag{3.15}$$

$$\begin{aligned} \widetilde{\nabla}_X W_\alpha = (-\overset{\circ}{A}_{W_\alpha} + \varepsilon_\alpha \lambda_\alpha I)X + \sum_{i=1}^{r} \gamma_{\alpha j}(X)N_j \\ + \sum_{\beta=r+1}^{n} \mu_{\alpha\beta}(X)W_\beta,\ r+1 \leq \alpha \leq n, \end{aligned} \tag{3.16}$$

where I is the identity tensor and $\widetilde{\pi}(N_i) = \widetilde{g}(N_i, \widetilde{Q}) = \eta_i(Q)$. Let us define that A_{N_i}, $1 \leq i \leq r$, and A_{W_α}, $r+1 \leq \alpha \leq n$ are

$$A_{N_i} = \overset{\circ}{A}_{N_i} - \eta_i(Q)I,$$

and

$$A_{W_\alpha} = \overset{\circ}{A}_{W_\alpha} - \varepsilon_\alpha \lambda_\alpha I,$$

respectively. Then (3.15) and (3.16) become

$$\widetilde{\nabla}_X N_i = -A_{N_i}X + \sum_{j=1}^{r} \rho_{ij}(X)N_j + \sum_{\alpha=r+1}^{n} \sigma_{i\alpha}(X)W_\alpha,\ 1 \leq i \leq r \tag{3.17}$$

and

$$\widetilde{\nabla}_X W_\alpha = -A_{W_\alpha}X + \sum_{j=1}^{r} \gamma_{\alpha j}(X)N_j + \sum_{\beta=r+1}^{n} \mu_{\alpha\beta}(X)W_\beta,\ \ r+1 \leq \alpha \leq n \tag{3.18}$$

for any $X \in \Gamma(TM)$, $N_i \in \Gamma(\mathrm{ltr}(TM))$ and $W_\alpha \in \Gamma(S(TM^\perp))$. We will call (3.17) and (3.18) the Weingarten formulae with respect to $\widetilde{\nabla}$.

By using (3.1), (3.4), (3.17), (3.18) and taking into account that $\overset{\circ}{\widetilde{\nabla}}$ is a metric connection, we have

$$\widetilde{g}(N_i, A_{W_\alpha}X) = \sigma_{i\alpha}(X)\varepsilon_\alpha - \varepsilon_\alpha \lambda_\alpha \eta_i(X),\ 1 \leq i \leq r,\ r+1 \leq \alpha \leq n, \tag{3.19}$$

$$\widetilde{g}(A_{W_\alpha}X, Y) + \varepsilon_\alpha \lambda_\alpha g(X,Y) = \varepsilon_\alpha h^s_\alpha(X,Y) + \sum_{i=1}^{r} \gamma_{\alpha i}(X)\eta_i(Y), \tag{3.20}$$

$$h_i^{\ell}(X,Y)+\widetilde{g}(\nabla_X\xi_i,Y)=\lambda_i g(X,Y) \tag{3.21}$$

for any $X,Y\in\Gamma(TM)$, $\xi_i\in\Gamma(\mathrm{Rad}TM)$, $W_\alpha\in\Gamma(S(TM^\perp))$, $N_i\in\Gamma(\mathrm{ltr}(TM))$.

Now, we give geometrical objects for screen distribution. With the analysis of (2.1), we get

$$\overset{\circ}{\nabla}_X PY=\overset{\circ}{\nabla}{}^*_X PY+\sum_{i=1}^{r}\overset{\circ}{h}{}^*_i(X,PY)\xi_i, \tag{3.22}$$

$$\overset{\circ}{\nabla}_X\xi_i=-\overset{\circ}{A}{}^*_{\xi_i}X+\overset{\circ}{\nabla}{}^{*t}_X\xi_i,\ 1\le i\le r \tag{3.23}$$

for any $X,Y\in\Gamma(TM)$ and $\xi_i\in\Gamma(\mathrm{Rad}TM)$, $1\le i\le r$, where $\{\overset{\circ}{\nabla}{}^*_X PY, \overset{\circ}{A}{}^*_{\xi_i}X\}$ and $\{\sum_{i=1}^{r}\overset{\circ}{h}{}^*_i(X,PY)\xi_i, \overset{\circ}{\nabla}{}^{*t}_X\xi_i\}$ belong to $\Gamma(S(TM))$ and $\Gamma(\mathrm{Rad}TM)$, respectively. It follows that $\overset{\circ}{\nabla}{}^*$ and $\overset{\circ}{\nabla}{}^{*t}$ are metric linear connections on complementary distributions $S(TM)$ and $\mathrm{Rad}(TM)$, respectively, $\overset{\circ}{A}{}^*_{\xi_i}$ are shape operator of $S(TM)$ with respect to ξ_i, $\overset{\circ}{h}{}^*_i$ are bilinear forms on $\Gamma(TM)\times\Gamma(S(TM))$. Also by using (3.3) and (3.23) we obtain

$$\overset{\circ}{h}{}^{\ell}_i(X,PY)=\widetilde{g}(\overset{\circ}{A}{}^*_{\xi_i}X,PY),\ 1\le i\le r \tag{3.24}$$

for any $X,Y\in\Gamma(TM)$ [5]. We define

$$u_{ij}(X)=g(\overset{\circ}{\nabla}{}^{*t}_X\xi_i,N_j),\ 1\le i,j\le r$$

for any $X\in\Gamma(TM)$ and $\xi_i\in\Gamma(\mathrm{Rad}TM)$, $1\le i\le r$. Hence, (3.23) can be written as

$$\overset{\circ}{\nabla}_X\xi_i=-\overset{\circ}{A}{}^*_{\xi_i}X+\sum_{j=1}^{r}u_{ij}(X)\xi_j,\ 1\le i\le r. \tag{3.25}$$

Analogous to the equation (3.22) we have

$$\nabla_X PY=\nabla^*_X PY+\sum_{i=1}^{r}h^*_i(X,PY)\xi_i \tag{3.26}$$

where h^*_i is the second fundamental form of distribution $S(TM)$.

From (3.6), we get

$$\nabla_X PY=\overset{\circ}{\nabla}_X PY+\pi(PY)X. \tag{3.27}$$

Thus, using (3.22) and (3.26), we have

$$\nabla^*_X PY+\sum_{i=1}^{r}h^*_i(X,PY)\xi_i=\overset{\circ}{\nabla}{}^*_X PY+\sum_{i=1}^{r}\overset{\circ}{h}{}^*_i(X,PY)\xi_i+\pi(PY)X$$

from which

$$\nabla^*_X PY = \overset{\circ}{\nabla}{}^*_X PY + \pi(PY)PX \tag{3.28}$$

and

$$h^*_i(X, PY) = \overset{\circ}{h}{}^*_i(X, PY) + \eta_i(X)\pi(PY),\ 1 \le i \le r \tag{3.29}$$

for any $X, Y \in \Gamma(TM)$. In addition, substituting (3.25) into (3.6) we have

$$\nabla_X \xi_i = -A^*_{\xi_i} X + \sum_{j=1}^{r} u_{ij}(X)\xi_j, \tag{3.30}$$

where $A^*_{\xi_i} = \overset{\circ}{A}{}^*_{\xi_i} - \lambda_i I$, $1 \le i \le r$. By using (3.28) we get

$$(\nabla^*_X g)(PY, PZ) = -\pi(PY)g(PX, PZ) - \pi(PZ)g(PX, PY). \tag{3.31}$$

We also have from (3.28)

$$T^*(PX, PY) = \pi(PY)PX - \pi(PX)PY. \tag{3.32}$$

Then, in view of (3.31) and (3.32), we have:

Proposition 3.2. *The induced connection on a screen distribution of lightlike submanifold with a semi-symmetric non-metric connection is a semi-symmetric non-metric connection.*

Proposition 3.3. *Let $(M, g, S(TM), S(TM^\perp))$ be a lightlike submanifold of semi-Riemannian manifold $(\widetilde{M}, \widetilde{g})$ admitting a semi-symmetric non-metric connection. The screen distribution $S(TM)$ is integrable if and only if η_i, $1 \le i \le r$, are closed forms on $S(TM)$.*

Proof. The torsion tensor T of ∇ does not vanish, by using (3.10), (3.26), (3.30) and equality given by

$$X = PX + \sum_{i=1}^{r} \eta_i(X)\xi_i$$

we can get

$$\begin{aligned}
[X, Y] = {} & \nabla^*_X PY - \nabla^*_Y PX + \sum_{i=1}^{r} \eta_i(X)A^*_{\xi_i}Y - \eta_i(Y)A^*_{\xi_i}X \\
& + \sum_{i=1}^{r}\{h^*_i(X, PY) - h^*_i(Y, PX) + X(\eta_i(Y)) - Y(\eta_i(X))\}\xi_i \\
& + \sum_{i,j=1}^{r}\{\eta_i(Y)u_{ij}(X) - \eta_i(X)u_{ij}(Y)\}\xi_j \\
& + \{\pi(PX) + \sum_{i=1}^{r}\eta_i(X)\lambda_i\}PY - \{\pi(PY) + \sum_{i=1}^{r}\eta_i(Y)\lambda_i\}PX
\end{aligned} \tag{3.33}$$

$$+\sum_{i=1}^{r}\{\pi(PX)\eta_i(Y)-\pi(PY)\eta_i(X)\}\xi_i.$$

After that, by taking the scalar product of the equation above with N_i, $1 \leq i \leq r$, we have

$$\begin{aligned}\widetilde{g}([X,Y],N_i) &= h_i^*(X,PY)-h_i^*(Y,PX)+X(\eta_i(Y))-Y(\eta_i(X))\\ &+\sum_{j=1}^{r}\{\eta_i(Y)u_{ij}(X)-\eta_i(X)u_{ij}(Y)\}\\ &+\pi(PX)\eta_i(Y)-\pi(PY)\eta_i(X).\end{aligned}\tag{3.34}$$

From (3.9) and (3.34), we obtain

$$\begin{aligned}2d\eta_i(X,Y) &= h_i^*(Y,PX)-h_i^*(X,PY)+\sum_{j=1}^{r}\{\eta_i(X)\{u_{ij}(Y)+\pi(PY)\}\\ &-\eta_i(Y)\{u_{ij}(X)+\pi(PX)\}\}\end{aligned}$$

or

$$2d\eta_i(PX,PY)=h_i^*(PY,PX)-h_i^*(PX,PY),\ 1\leq i\leq r$$

which prove the assertion of the proposition. □

4. The Gauss and Codazzi equations

We denote by

$$\widetilde{R}(\widetilde{X},\widetilde{Y})\widetilde{Z}=\widetilde{\nabla}_{\widetilde{X}}\widetilde{\nabla}_{\widetilde{Y}}\widetilde{Z}-\widetilde{\nabla}_{\widetilde{Y}}\widetilde{\nabla}_{\widetilde{X}}\widetilde{Z}-\widetilde{\nabla}_{[\widetilde{X},\widetilde{Y}]}\widetilde{Z}$$

the curvature tensor of $\widetilde{M}$ with respect to the semi-symmetric non-metric connection $\widetilde{\nabla}$ and by

$$R(X,Y)Z=\nabla_X\nabla_YZ-\nabla_Y\nabla_XZ-\nabla_{[X,Y]}Z$$

that of M with respect to induced connection ∇, where $\widetilde{X}$, $\widetilde{Y}$, $\widetilde{Z}\in\Gamma(T\widetilde{M})$ and X, Y, $Z\in\Gamma(TM)$. Then by using (3.4), (3.17), (3.18), we get

$$\begin{aligned}\widetilde{R}(X,Y)Z &= R(X,Y)Z+\sum_{i=1}^{r}\{(\nabla_Xh_i^{\ell})(Y,Z)-(\nabla_Yh_i^{\ell})(X,Z)\\ &+h_i^{\ell}(\pi(Y)X-\pi(X)Y,Z)\}N_i\\ &+\sum_{\alpha=r+1}^{n}\{(\nabla_Xh_\alpha^s)(Y,Z)-(\nabla_Yh_\alpha^s)(X,Z)\\ &+h_\alpha^s(\pi(Y)X-\pi(X)Y,Z)\}W_\alpha+\sum_{i=1}^{r}\{h_i^{\ell}(X,Z)A_{N_i}Y\\ &-h_i^{\ell}(Y,Z)A_{N_i}X+\sum_{\alpha=r+1}^{n}h_\alpha^s(X,Z)A_{W_\alpha}Y-h_\alpha^s(Y,Z)A_{W_\alpha}X\end{aligned}\tag{4.1}$$

$$
\begin{aligned}
&+ \sum_{i,j=1}^{r} \{h_i^\ell(Y,Z)\rho_{ij}(X) - h_i^\ell(X,Z)\rho_{ij}(Y) \\
&+ \sum_{\alpha=r+1}^{n} h_\alpha^s(Y,Z)\gamma_{\alpha j}(X) - h_\alpha^s(X,Z)\gamma_{\alpha j}(Y)\}N_j \\
&+ \sum_{\alpha,\beta=r+1}^{n} \sum_{i=1}^{r} \{h_i^\ell(Y,Z)\sigma_{i\beta}(X) - h_i^\ell(X,Z)\sigma_{i\beta}(Y) \\
&+ h_\alpha^s(Y,Z)\mu_{\alpha\beta}(X) - h_\alpha^s(X,Z)\mu_{\alpha\beta}(Y)\}W_\beta
\end{aligned}
$$

for any $X, Y, Z \in \Gamma(TM)$, $N_i \in \Gamma(\mathrm{ltr}(TM))$ and $W_\alpha \in \Gamma(S(TM^\perp))$. From (3.26), (3.30) and (4.1), we have the Gauss and Codazzi equations of the lightlike submanifold with a semi-symmetric non-metric connection:

(4.2)

$$
\begin{aligned}
&\widetilde{g}(\widetilde{R}(X,Y)PZ, PU) \\
&= g(R(X,Y)PZ, PU) + \sum_{i=1}^{r} \{h_i^\ell(X,PZ)g(A_{N_i}Y, PU) - h_i^\ell(Y,PZ)g(A_{N_i}X, PU) \\
&\quad + \sum_{\alpha=r+1}^{n} h_\alpha^s(X,PZ)g(A_{W_\alpha}Y, PU) - h_\alpha^s(Y,PZ)g(A_{W_\alpha}X, PU),
\end{aligned}
$$

(4.3)

$$
\begin{aligned}
&\widetilde{g}(\widetilde{R}(X,Y)\xi_i, N_i) \\
&= g(R(X,Y)\xi_i, N_i) + \sum_{j=1}^{r} \{h_j^\ell(X,\xi_i)g(A_{N_j}Y, N_i) - h_j^\ell(Y,\xi_i)g(A_{N_j}X, N_i)\} \\
&\quad + \sum_{\alpha=r+1}^{n} \{h_\alpha^s(X,\xi_i)g(A_{W_\alpha}Y, N_i) - h_\alpha^s(Y,\xi_i)g(A_{W_\alpha}X, N_i)\},
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{g}(R(X,Y)\xi_i, N_i) &= h_i^*(Y, A_{\xi_i}^* X) - h_i^*(X, A_{\xi_i}^* Y) + 2du_{ii}(X,Y) \\
&\quad + \sum_{j=1}^{r} \{u_{ij}(Y)u_{ji}(X) - u_{ij}(X)u_{ji}(Y)\}. \qquad (4.4)
\end{aligned}
$$

Thus, from (4.1) we have the following proposition.

Proposition 4.1. *Let $(M, g, S(TM), S(TM^\perp))$ be a lightlike submanifold of semi-Riemannian manifold $(\widetilde{M}, \widetilde{g})$ admitting a semi-symmetric non-metric connection. If M is totally geodesic in $\widetilde{M}$, then*

$$\widetilde{R}(X,Y)Z = R(X,Y)Z, \ \forall X, Y, Z \in \Gamma(TM).$$

5. The Ricci tensor of lightlike submanifold with a semi-symmetric non-metric connection

Let $(M, g, S(TM), S(TM^{\perp}))$ be an m-dimensional lightlike submanifold of an $(m+n)$-dimensional semi-Riemannian manifold $(\widetilde{M}, \widetilde{g})$ admitting a semi-symmetric non-metric connection. Similar to the definition of the Ricci tensor of M with respect to the symmetric connection, the Ricci tensor of M with respect to the semi-symmetric non-metric connection is defined by

$$Ric(X,Y) = \text{trace}\{Z \to R(X,Z)Y\} \tag{5.1}$$

for any $X, Y, Z \in \Gamma(TM)$. Then the Ricci tensor of an m-dimensional lightlike submanifold M with respect to the semi-symmetric non-metric connection is given by

$$Ric(X,Y) = \sum_{i=1}^{r} \widetilde{g}(R(X,\xi_i)Y, N_i) + \sum_{k=r+1}^{m} \varepsilon_k g(R(X,X_k)Y, X_k), \tag{5.2}$$

where $\{X_{r+1}, \ldots, X_m\}$ is an orthonormal basis of screen distribution $\Gamma(S(TM))$.

Thus, by using (3.6), (4.1) and (5.2) we obtain

$$Ric(X,Y) - Ric(Y,X) = \overset{\circ}{Ric}(X,Y) - \overset{\circ}{Ric}(Y,X) + (m-1)d\pi(X,Y) \tag{5.3}$$

for any $X, Y \in \Gamma(TM)$.

From (5.3) we have:

Proposition 5.1. *Let $(M, g, S(TM), S(TM^{\perp}))$ be a lightlike submanifold of semi-Riemannian manifold $(\widetilde{M}, \widetilde{g})$ admitting a semi-symmetric non-metric connection. Then Ricci tensor of a lightlike submanifold with respect to the semi-symmetric non-metric connection is symmetric if and only if the Ricci tensor of a lightlike submanifold with respect to the symmetric non-metric connection is symmetric and the 1-form π is closed.*

We assume that the 1-form π is closed. In this case we can define the sectional curvature for a section in $\widetilde{M}$ with respect to the semi-symmetric non-metric connection (see [1]).

Now, suppose that the semi-symmetric non-metric connection $\widetilde{\widetilde{\nabla}}$ is of constant sectional curvature, then $\widetilde{R}(\widetilde{X}, \widetilde{Y})\widetilde{Z}$ should be in the form of

$$\widetilde{R}(\widetilde{X}, \widetilde{Y})\widetilde{Z} = c\{\widetilde{g}(\widetilde{X}, \widetilde{Z})\widetilde{Y} - \widetilde{g}(\widetilde{Y}, \widetilde{Z})\widetilde{X}\} \tag{5.4}$$

for any $\widetilde{X}, \widetilde{Y}, \widetilde{Z} \in \Gamma(T\widetilde{M})$, where c is a certain scalar. Thus, $\widetilde{M}$ is called a semi-Riemannian space form with respect to the semi-symmetric non-metric connection and is denoted by $\widetilde{M}(c)$.

Proposition 5.2. *Let $(M, g, S(TM), S(TM^{\perp}))$ be a lightlike submanifold of an $(m+n)$-dimensional semi-Riemannian space form $\widetilde{M}(c)$ with a semi-symmetric non-metric connection. Then we have*

$Ric(X,Y) = (m-1)cg(X,Y)$

$$
\begin{aligned}
&+\sum_{i=1}^{r} h_i^\ell(A_{N_i}X,Y) - \sum_{k=r+1}^{m} \varepsilon_k h_i^\ell(X,Y) g(A_{N_i}X_k, X_k)\} \\
(5.5) \qquad &+ \sum_{\alpha=r+1}^{n} \{h_\alpha^s(A_{W_\alpha}X,Y) - \sum_{k=r+1}^{m} \varepsilon_k h_\alpha^s(X,Y) g(A_{W_\alpha}X_k, X_k)\} \\
&- \sum_{i,j=1}^{r} \{h_i^\ell(X,Y)\eta_j(A_{N_i}\xi_j) - \sum_{\alpha=r+1}^{n} h_\alpha^s(X,Y)\eta_j(A_{W_\alpha}\xi_j)\}
\end{aligned}
$$

for any $X, Y, Z \in \Gamma(TM)$.

Proof. Taking (4.1) in (5.2) and considering (5.4), we obtain (5.5). □

Corollary 5.3. *Let* $(M, g, S(TM), S(TM^\perp))$ *be a lightlike submanifold of an* $(m+n)$*-dimensional semi-Riemannian space form* $\widetilde{M}(c)$ *admitting a semi-symmetric non-metric connection. If* M *is totally geodesic, then* M *is an Einstein manifold.*

Corollary 5.4. *Let* $(M, g, S(TM), S(TM^\perp))$ *be a lightlike submanifold of an* $(m+n)$*-dimensional semi-Riemannian space form* $\widetilde{M}(c)$ *with a semi-symmetric non-metric connection. Then the Ricci tensor of* M *is symmetric if and only if the shape tensors with respect to the semi-symmetric non-metric connection of* M *is symmetric with respect to the local lightlike second fundamental forms and the local screen fundamental forms of* M.

Proposition 5.5. *Let* $(M, g, S(TM), S(TM^\perp))$ *be a lightlike submanifold of a semi-Riemannian space form* $\widetilde{M}(c)$ *with a semi-symmetric non-metric connection. Then the Ricci tensor of* M *is not parallel with respect to the semi-symmetric non-metric connection.*

Proof. First of all, we compute the derivative of Ricci tensor of M with respect to the semi-symmetric non-metric connection. We define

$$(\nabla_Z Ric)(X,Y) = \nabla_Z Ric(X,Y) - Ric(\nabla_Z X, Y) - Ric(X, \nabla_Z Y).$$

Then, by using equations (3.8) and (5.5), we obtain

$$
\begin{aligned}
(5.6) \qquad &(\nabla_Z Ric)(X,Y) \\
&= c(m-1)(\nabla_Z g)(X,Y) + \sum_{i=1}^{r}\{(\nabla_Z h_i^\ell)(A_{N_i}X,Y) + h_i^\ell(\nabla_Z A_{N_i}X,Y)\} \\
&\quad + \sum_{\alpha=r+1}^{n}\{(\nabla_Z h_\alpha^s)(A_{W_\alpha}X,Y) + h_\alpha^s(\nabla_Z A_{W_\alpha}X,Y)\} \\
&\quad - \sum_{k=r+1}^{m}\sum_{i=1}^{r}\varepsilon_k\{h_i^\ell(X,Y)\nabla_Z g(A_{N_i}X_k, X_k)
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{\alpha=r+1}^{n} h_\alpha^s(X,Y)\nabla_Z g(A_{W_\alpha}X_k, X_k)\} \\
&- \sum_{k=r+1}^{m}\sum_{\alpha=r+1}^{n} \varepsilon_k (\nabla_Z h_\alpha^s)(X,Y) g(A_{W_\alpha}X_k, X_k) \\
&- \sum_{j=1}^{r}\sum_{\alpha=r+1}^{n} \{(\nabla_Z h_\alpha^s)(X,Y)\eta_j(A_{W_\alpha}\xi_j) + h_\alpha^s(X,Y)\nabla_Z \eta_j(A_{W_\alpha}\xi_j)\} \\
&- \sum_{i=1}^{r}\sum_{\alpha=r+1}^{n} \{h_i^\ell(A_{N_i}\nabla_Z X, Y) + h_\alpha^s(A_{W_\alpha}\nabla_Z X, Y)\} \\
&- \sum_{k=r+1}^{m}\sum_{i=1}^{r} \varepsilon_k (\nabla_Z h_i^\ell)(X,Y) g(A_{N_i}X_k, X_k) \\
&- \sum_{i,j=1}^{r} \{(\nabla_Z h_i^\ell)(X,Y)\eta_j(A_{N_i}\xi_j) + h_i^\ell(X,Y)\nabla_Z \eta_j(A_{N_i}\xi_j)\}.
\end{aligned}
$$

Since the term $(\nabla_Z g)(X,Y)$ in the right hand side of the equation (5.6) is not vanishing for any $X, Y, Z \in \Gamma(TM)$, we have the assertion of the theorem. □

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