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INTEGRAL DOMAINS WITH A FREE SEMIGROUP OF $*$-INVERTIBLE INTEGRAL $*$-IDEALS

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Abstract. Let $*$ be a star-operation on an integral domain R, and let $\mathscr{I}_*^+(R)$ be the semigroup of $*$-invertible integral $*$-ideals of R. In this article, we introduce the concept of a $*$-coatom, and we then characterize when $\mathscr{I}_*^+(R)$ is a free semigroup with a set of free generators consisting of $*$-coatoms. In particular, we show that $\mathscr{I}_*^+(R)$ is a free semigroup if and only if R is a Krull domain and each v-invertible v-ideal is $*$-invertible. As a corollary, we obtain some characterizations of $*$-Dedekind domains.

1. Introduction

Let R be an integral domain, and let $*$ be a star-operation (defined later) on R. Let $\mathscr{F}_*(R)$ (resp., $\mathscr{I}_*(R)$, $\mathscr{P}(R)$) be the set of nonzero fractional $*$-ideals (resp., $*$-invertible $*$-ideals, principal ideals) of R. Then $\mathscr{F}_*(R)$ forms a commutative monoid under $*$-multiplication, that is, for any $A, B \in \mathscr{F}_*(R)$, $A * B := (AB)_*$. Moreover $\mathscr{I}_*(R)$ is a subgroup of $\mathscr{F}_*(R)$ and $\mathscr{P}(R)$ is a subgroup of $\mathscr{I}_*(R)$. Let $\mathscr{F}_*^+(R)$ (resp., $\mathscr{I}_*^+(R)$, $\mathscr{P}^+(R)$) be the positive cone of $\mathscr{F}_*(R)$ (resp., $\mathscr{I}_*(R)$, $\mathscr{P}(R)$) which consists of the nonzero integral $*$-ideals (resp., $*$-invertible $*$-ideals, principal ideals) of R. The structure of an integral domain R depends heavily on the properties of $\mathscr{F}_*(R)$, $\mathscr{F}_*^+(R)$, $\mathscr{P}(R)$, or $\mathscr{P}^+(R)$. For instance, it is well-known that R is a Dedekind domain if and only if $\mathscr{F}_d^+(R)$ is a free semigroup with base $\text{Spec}(R) \setminus \{0\}$. In [6], it was determined when $\mathscr{F}_d(R)$ is finitely generated as a monoid. Recently in [4], it was further investigated and extended to a commutative ring with zero divisors. In [8], among other things, it was shown that R is a Krull domain if and only if $\mathscr{F}_t^+(R)$ is a free semigroup with a set of free generators consisting of t-nonfactorable ideals. Later the concept of $*$-nonfactorability of ideals was further studied in

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[10, 18, 19]. In particular, it was shown in [10, Theorem 3.2] that if $\mathscr{F}_*^+(R)$ is a free semigroup with a set of free generators consisting of $*$-nonfactorable ideals, then R is a Krull domain. Finally, in [20], it was characterized when $\mathscr{I}_d^+(R)$ is a free semigroup with a system of generators consisting of coatoms.

In this article, we define the notion of $*$-coatoms for any star-operation $*$ and we then characterize when $\mathscr{I}_*^+(R)$ is a free semigroup with a set of free generators consisting of $*$-coatoms. More precisely, we show that $\mathscr{I}_*(R)$ is a free semigroup if and only if R is a Krull domain and $\mathscr{I}_v(R) = \mathscr{I}_*(R)$ if and only if every nonzero principal ideal of R can be expressed as a finite $*$-product of height-one prime ideals. In particular, if $*$ is of finite character, then $\mathscr{I}_*(R)$ is a free semigroup if and only if each nonzero $*$-locally principal ideal of R is $*$-invertible and R_M is a factorial domain for all $*$-maximal ideals M of R. As a byproduct, we obtain some characterizations of $*$-Dedekind domains.

Let R be an integral domain with quotient field K. Let $\mathscr{F}(R)$ be the set of nonzero fractional ideals of R. A mapping $A \mapsto A_*$ of $\mathscr{F}(R)$ into $\mathscr{F}(R)$ is called a *star-operation* on R if the following conditions are satisfied for all $a \in K \setminus \{0\}$ and $A, B \in \mathscr{F}(R)$:

(i) $(aR)_* = aR, (aA)_* = aA_*$;
(ii) $A \subseteq A_*$, if $A \subseteq B$, then $A_* \subseteq B_*$;
(iii) $(A_*)_* = A_*$.

It is easy to show that for all $A, B \in \mathscr{F}(R)$, $(AB)_* = (AB_*)_* = (A_*B_*)_*$. An $A \in \mathscr{F}(R)$ is called a $*$-*ideal* if $A = A_*$. A $*$-ideal is called a $*$-maximal ideal if it is maximal among proper integral $*$-ideals. We denote by $*$-Max(R) (resp., $*$-Spec(R)) the set of all $*$-maximal ideals (resp., prime $*$-ideals) of R.

Given any star-operation $*$ on R, we can construct another star-operation $*_f$ defined by $A_{*_f} := \bigcup\{J_* \mid J$ is a nonzero finitely generated subideal of $A\}$ for $A \in \mathscr{F}(R)$. Clearly, if $A \in \mathscr{F}(R)$ is finitely generated, then $A_* = A_{*_f}$. We say that $*$ is *of finite character* if $* = *_f$ and that $*_f$ is the *finite character star-operation induced by* $*$. It is well-known that if $* = *_f$, then $*$-Max$(R) \neq \emptyset$ when R is not a field; a $*$-maximal ideal is a prime ideal; each proper integral $*$-ideal is contained in a $*$-maximal ideal; each prime ideal minimal over a $*$-ideal is a prime $*$-ideal (in particular, each height-one prime ideal is a prime $*$-ideal); and $R = \bigcap_{P \in *-\mathrm{Max}(R)} R_P$. Let $*$ be any star-operation on R. An $A \in \mathscr{F}(R)$ is said to be $*$-*invertible* if $(AA^{-1})_* = R$, where $A^{-1} = \{x \in K | xA \subseteq R\}$. We say that $A \in \mathscr{F}(R)$ is of $*$-*finite type* if $A_* = B_*$ for some finitely generated ideal B of R. Also, $A \in \mathscr{F}(R)$ is said to be $*$-*locally principal* if AR_P is principal for all $*$-maximal ideals P of R. It is well-known that A is $*_f$-invertible if and only if A is of $*_f$-finite type and A is $*_f$-locally principal.

The most important examples of star-operations are (1) the d-operation defined by $A_d := A$, (2) the v-operation defined by $A_v := (A^{-1})^{-1}$, (3) the t-operation defined by $t := v_f$, and (4) the w-operation defined by $A_w := \{x \in K \mid Jx \subseteq A$ for some finitely generated ideal J with $J^{-1} = R\}$ for $A \in \mathscr{F}(R)$. Note that all star-operations above except for v are of finite character. For any star-operation $*$ on R and for any $A \in \mathscr{F}(R)$, we have that $A \subseteq A_* \subseteq A_v$ (i.e., $d \leq * \leq v$) and $A \subseteq A_{*_f} \subseteq A_t$; so $(A_*)_v = A_v = (A_v)_*$ and $(A_{*_f})_t = A_t = (A_t)_{*_f}$. In particular, a v-ideal (resp., t-ideal) is a $*$-ideal (resp., $*_f$-ideal). General references for any undefined terminology or notation are [15, 16].

2. When $\mathscr{I}_*^+(R)$ is a free semigroup

Throughout this section, R denotes an integral domain with quotient field K, $*$ is a star-operation on R, $*_f$ is the finite character star-operation on R induced by $*$, and $\mathscr{I}_*^+(R)$ is the semigroup of $*$-invertible integral $*$-ideals of R. In this section, we study when $\mathscr{I}_*^+(R)$ is a free semigroup.

As mentioned in the introduction, in [8], the authors introduced the concepts of $*$-nonfactorable ideals and (unique) $*$-factorable domains, and they then characterized several integral domains including Krull domains using these concepts. We say that an ideal N of R is *$*$-nonfactorable* if it is a proper $*$-ideal and $N = (AB)_*$, where A and B are ideals of R, implies either $A_* = R$ or $B_* = R$. We also say that an integral domain R is a *$*$-factorable domain* (resp., *unique $*$-factorable domain*) if every proper $*$-ideal of R can be factored (resp., factored uniquely) into a $*$-product of $*$-nonfactorable ideals.

Definition. An ideal A of $\mathscr{I}_*^+(R)$ is called a *$*$-coatom of $\mathscr{I}_*^+(R)$* if A is not expressible as a nontrivial $*$-product of $*$-invertible ideals of R.

We remark that the $*$-coatoms of $\mathscr{I}_*^+(R)$ coincide with the maximal elements of $\mathscr{I}_*^+(R)$ (with respect to set inclusion). By definition a $*$-coatom is a $*$-invertible $*$-ideal. Hence it is precisely a $*$-nonfactorable $*$-invertible $*$-ideal. We next recall some results from [21, Theorem 1.1] on $*$-invertible ideals, which are very useful in the subsequent arguments.

Lemma 2.1. *If I is a nonzero fractional ideal of R, then*

(1) $I_{*_f} \subseteq I_t$;

(2) *If I is $*$-invertible, then I is v-invertible and $I_* = I_v$;*

(3) *If I is $*_f$-invertible, then I is t-invertible and $I_* = I_{*_f} = I_t = I_v$.*

Lemma 2.2. *If R satisfies ACC on $*$-invertible $*$-ideals, then every $*$-invertible $*$-ideal is expressible as a (finite) $*$-product of $*$-coatoms.*

Proof. This follows from the following easy observation: Let $I \subseteq J$ be $*$-ideals of R with J, $*$-invertible. Then there exists a $*$-ideal A such that $I = (JA)_*$. □

The proof of the following lemma is essentially the same as that of [8, Lemma 11]. However for the sake of completeness we include its proof.

Lemma 2.3. *Suppose that $\mathscr{I}_*^+(R)$ is a free semigroup with a set of free generators consisting of $*$-coatoms, and let N be a proper $*$-invertible $*$-ideal of R. Then N is a $*$-coatom if and only if N is prime (if and only if N is $*$-nonfactorable).*

Proof. Let N be a $*$-coatom. Let $ab \in N$ for nonzero nonunits $a, b \in R$ with $a \notin N$. Then $aR = (P_1 \cdots P_r)_*$, where each P_i is a $*$-coatom. Note that each $P_i \neq N$ since $a \notin N$. Also, $bR = (Q_1 \cdots Q_s)_*$, where each Q_j is a $*$-coatom. Note that abN^{-1} is a proper $*$-invertible $*$-ideal (if $abN^{-1} = R$, then $N = ((abN^{-1})N)_* = abR = (aR)(bR)$ is not a $*$-coatom); so we may write $abN^{-1} = (M_1 \cdots M_u)_*$, where each M_k is a $*$-coatom. Thus $abR = (NM_1 \cdots M_u)_* = (P_1 \cdots P_r Q_1 \cdots Q_s)_*$. Since $\mathscr{I}_*^+(R)$ is a free semigroup with a set of free generators consisting of $*$-coatoms, we have that $N = Q_j$ for some j. Thus $b \in bR = (Q_1 \cdots Q_s)_* \subseteq N$, and hence N is prime. Conversely, it was already observed in [8] that a $*$-invertible prime $*$-ideal is $*$-nonfactorable, and hence a $*$-coatom. □

Lemma 2.4. *If every nonzero proper principal ideal of R decomposes into a $*$-product of prime $*$-ideals, then the set of the height-one primes equals the set of the $*$-invertible prime $*$-ideals.*

Proof. Let P be a $*$-invertible prime $*$-ideal of R and assume that the height of P is greater than 1. Then there exists a nonzero prime ideal $Q \subsetneq P$. For $0 \neq q \in Q$, we have $Q \supseteq qR = (P_1 \cdots P_s)_*$, where the P_i are prime $*$-ideals. It then follows that $P \supsetneq Q \supseteq P_i$ for some i $(1 \leq i \leq s)$. Thus $P_i = (PZ)_*$ for some $*$-invertible $*$-ideal $Z \neq R$. This implies $Z \subseteq P_i$. So if we replace Z by P_i in the equality $P_i = (PZ)_*$ and apply the $*$-invertibility of P_i, we get a contradiction. For the reverse inclusion, let P be a height-one prime ideal of R. Then P is minimal over a nonzero principal ideal $aR = (P_1 \cdots P_n)_*$, where the P_i are prime $*$-ideals. Thus $P = P_i$ for some i, and so P is a $*$-invertible prime $*$-ideal. □

Let $X^1(R)$ be the set of height-one prime ideals of R. An integral domain R is a Krull domain if (i) R_P is a rank-one DVR for each $P \in X^1(R)$, (ii) $R = \bigcap_{P \in X^1(R)} R_P$, and (iii) each nonzero element of R is contained in finitely many height-one prime ideals of R. It is well known that R is a Krull domain if and only if each nonzero proper principal ideal of R is a t-product of (t-invertible) prime ideals [17, Theorem 3.9].

We say that R is a $*$-*Schreier domain* if $\mathscr{I}_*(R)$ is a Riesz group. More precisely, R is a $*$-Schreier domain if whenever A, B_1, B_2 are $*$-invertible $*$-ideals of R and $A \supseteq B_1B_2$, then $A = (A_1A_2)_*$ for some ($*$-invertible) $*$-ideals A_1, A_2 of R with $A_i \supseteq B_i$ for $i = 1, 2$. We remark that the concepts of t-Schreier domains and d-Schreier domains (as the name of quasi-Schreier domains) were already introduced as a generalization of Prüfer v-multiplication domains and proved very useful in [5, 11, 12]. An integral domain R is called a $*$-*GCD domain* if the intersection of two $*$-invertible $*$-ideal is $*$-invertible (cf., [16, Definition 17.6]). Then d-GCD domains are exactly generalized GCD-domains, which are introduced in [1] and further investigated in [3]. For $I, J \in \mathscr{I}_*(R)$, an ordered group under the partial order $A \leq B \Leftrightarrow B \subseteq A$, $\sup(I, J)$ exists if and only if $I \cap J$ is $*$-invertible, and hence $\sup(I, J) = I \cap J$. It follows that

R is a $*$-GCD domain if and only if $\mathscr{I}_*(R)$ is a lattice-ordered group (cf., [3, Theorem 1]). Thus every $*$-GCD domain is a $*$-Schreier domain.

Theorem 2.5. *The following conditions are equivalent for an integral domain R.*

(1) *$\mathscr{I}_*^+(R)$ is a free semigroup with a set of free generators consisting of $*$-coatoms.*
(2) *R satisfies ACC on $*$-invertible $*$-ideals and R is a $*$-GCD domain.*
(3) *R satisfies ACC on $*$-invertible $*$-ideals and R is a $*$-Schreier domain.*
(4) *R satisfies ACC on $*$-invertible $*$-ideals and the $*$-coatoms are prime $*$-ideals.*
(5) *R is a Krull domain and $\mathscr{F}_v(R) = \mathscr{I}_*(R)$.*
(6) *R is a Krull domain and $\mathscr{F}_v^+(R) = \mathscr{I}_*^+(R)$.*
(7) *R satisfies ACC on $*$-invertible $*$-ideals and every nonzero principal ideal of R can be uniquely decomposed into a $*$-product of $*$-coatoms.*
(8) *Every nonzero principal ideal of R uniquely decomposes into a $*$-product of prime $*$-ideals.*
(9) *Every nonzero principal ideal of R can be written as a finite $*$-product of height-one prime ideals.*

Proof. (1) $\Rightarrow$ (2). Note that the ordered group $\mathscr{I}_*(R)$ is isomorphic to a direct sum of copies of $\mathbb{Z}$, the additive group of integers. Thus by the remark before Theorem 2.5 R is a $*$-GCD domain. The rest is clear.

(2) $\Rightarrow$ (3). This follows from the remark before Theorem 2.5.

(3) $\Rightarrow$ (4). It is easily seen that in a $*$-Schreier domain every $*$-coatom is a prime $*$-ideal.

(4) $\Rightarrow$ (1). The existence of a decomposition into $*$-coatoms follows from Lemma 2.2. From the primeness of the $*$-coatoms, it follows that they are free generators; that is, the equality $(P_1 \cdots P_m)_* = (Q_1 \cdots Q_s)_*$ implies that $m = s$ and that there exists a permutation σ such that $P_i = Q_{\sigma(i)}$.

(1) $\Rightarrow$ (7). This is clear.

(7) $\Rightarrow$ (4). Let P be a $*$-coatom of $\mathscr{I}_*^+(R)$, and let a, b be nonzero nonunits of R with $ab \in P$. It suffices to show that $a \in P$ or $b \in P$. By (7), there are some $*$-coatoms $P_1, \dots, P_n, Q_1, \dots, Q_k$ such that $aR = (P_1 \cdots P_n)_*$ and $bR = (Q_1 \cdots Q_k)_*$; in particular, $abR = (P_1 \cdots P_n Q_1 \cdots Q_k)_*$. Since $abR \subseteq P$, we have $I := abP^{-1} \subseteq R$ and $abR = (PI)_*$. Clearly, I is $*$-invertible, and hence $I = (P_1' \cdots P_l')_*$ for some $*$-coatoms P_i'; so $(P_1 \cdots P_n Q_1 \cdots Q_k)_* = abR = (PP_1' \cdots P_l')_*$. Hence by the uniqueness, we have $P = P_i$ or $P = Q_j$, and therefore $a \in P$ or $b \in P$.

(5) $\Leftrightarrow$ (6) $\Rightarrow$ (4). These are clear because each v-ideal is a $*$-ideal.

(7) $\Rightarrow$ (5). We first note that each nonzero nonunit of R can be written as a finite $*$-product of height-one prime $*$-ideals of R by Lemma 2.4 and the (7) $\Rightarrow$ (4) above. Next, let P be a height-one prime ideal of R. Then by Lemma 2.4, P is a $*$-invertible prime $*$-ideal. Thus we have that $P \subsetneq PP^{-1}$; so R_P is a rank-one DVR. Also, $aR = (P_1 \cdots P_n)_*$ implies that a is contained in a

finite number of height-one prime ideals of R. Suppose that $aR : bR \subsetneq R$ for $a, b \in R$. Then the proof of [10, Theorem 3.2] also shows that $aR : bR \subseteq P$ for some $P \in X^1(R)$, and thus $R = \bigcap_{P \in X^1(R)} R_P$ [15, Ex. 22, p. 52]. Thus R is a Krull domain.

Next, let I be a v-ideal of R. Then $I = (P_1 \cdots P_n)_v$ for some height-one prime ideals $P_1, \dots, P_n$ of R because R is a Krull domain. Note that each P_i is $*$-invertible by the above proof; so $P_1 \cdots P_n$ is $*$-invertible, and hence by Lemma 2.1 $I = (P_1 \cdots P_n)_*$. Since $I^{-1} = (P^{-1} \cdots P_n^{-1})_*$, we have $(II^{-1})_* = ((P_1 \cdots P_n)_* P^{-1} \cdots P_n^{-1})_* = ((P_1 P_1^{-1})_* \cdots (P_n P_n^{-1})_*)_* = R$. Thus I is $*$-invertible.

(1) $\Rightarrow$ (8). This follows from Lemma 2.3.

(8) $\Rightarrow$ (9). This follows from Lemma 2.4.

(9) $\Rightarrow$ (5). Note that by (9) each nonzero $a \in K$ can be uniquely written as $aR = (\prod\{P^{v_P(a)} \mid P \in X^1(R)\})_*$, where $v_P(a) \in \mathbb{Z}$ and $v_P(a) = 0$ for almost all P. Hence for each height-one prime P the assignment $v_P(a)$ defines a discrete valuation on K. Since $a \in R$ if and only if $v_P(a) \geq 0$ for all P, denoting by V_P the valuation ring of v_P, we have that $R = \bigcap V_P$ with finite character. Thus R is a Krull domain such that each v-ideal is $*$-invertible. $\square$

We remark that for $* = d$, Theorem 2.5 is essentially in [11, 14, 20].

3. When $\mathscr{I}_*^+(R)$ is a free semigroup for $* = *_f$

As in Section 2, we denote by R an integral domain with quotient field K, $*$ is a star-operation on R, $*_f$ is the finite character star-operation on R induced by $*$, and $\mathscr{I}_*^+(R)$ is the semigroup of $*$-invertible integral $*$-ideals of R. In this section, we study when $\mathscr{I}_*^+(R)$ is a free semigroup for $* = *_f$.

We first give an example of a star-operation $*$ such that $*$ is not of finite type, yet Theorem 2.5 holds for it. Following the referee's remark, such a star-operation has the potential of being very useful for answering other questions in the literature.

Example 3.1. *Let D be a π-domain, $\{X_\alpha\}$ an infinite set of indeterminates over D, and $\{Q_\lambda\}$ the set of (nonzero) finitely generated prime ideals of $D[\{X_\alpha\}]$. It is known that $D[\{X_\alpha\}]$ is a π-domain (cf.* [1, p. 200]) *and $D[\{X_\alpha\}] = \bigcap_\lambda D[\{X_\alpha\}]_{Q_\lambda}$. For each nonzero fractional ideal A of $D[\{X_\alpha\}]$, if we define*

$$A_* = \bigcap_\lambda AD[\{X_\alpha\}]_{Q_\lambda},$$

then $*$ is a star-operation on $D[\{X_\alpha\}]$ [2, Theorem 1]. *Again by* [2, Theorem 1], *if we let $M = (\{X_\alpha\})$, then $M_* = D[\{X_\alpha\}]$, because M is not contained in any of Q_λ. But $M_{*_f} \neq D[\{X_\alpha\}]$ by the definition of $*$. Thus $* \neq *_f$, i.e., $*$ is not of finite type. Finally, since $D[\{X_\alpha\}]$ is a π-domain, each nonzero principal ideal of $D[\{X_\alpha\}]$ is a finite product (and thus $*$-product since $d < *$)*

of height-one prime ideals. Thus $*$ is not of finite type but Theorem 2.5 holds for this star-operation.

The following result comes from Lemma 2.1 and [21, Corollary 4.5].

Lemma 3.2. *If P is a $*_f$-invertible prime ideal of R such that $P_{*_f} \subsetneq R$, then*
(a) *If I is a $*_f$-invertible $*_f$-ideal containing P, then $I = P = P_{*_f}$;*
(b) *P is a t-invertible t-maximal ideal.*

To characterize $*$-locally factorial Krull domains in terms of "$*$-LPI domain" in Theorem 3.7, we need the notion of a *$*$-LPI domain*, that is, an integral domain in which every nonzero $*$-locally principal ideal is $*$-invertible. This notion first appeared in [22] for $* = t$, in the context of integral domains, essentially Prüfer v-multiplication domains and slightly more general domains. Then it appears in [12] in the context of t-Schreier domains and in [7] as a defining property for "Locally Principal ideals are Invertible"-domains with $* = d$.

Lemma 3.3. *Let $*$ be of finite character on the integral domain R, and suppose that R is a $*$-LPI domain. If R_M satisfies ACC on principal ideals of R_M for every $*$-maximal ideal M of R, then R satisfies ACC on $*$-invertible $*$-ideals of R.*

Proof. Consider an increasing chain of $*$-invertible $*$-ideals: $I_1 \subseteq I_2 \subseteq \cdots$. Let $I := \varinjlim I_n$. Then I is a $*$-ideal of R, since $*$ is of finite character. It is sufficient to show that I is $*$-invertible.

For every $*$-maximal ideal M, we have $IR_M = I \otimes_R R_M = \varinjlim(I_n \otimes_R R_M) = \varinjlim(I_n R_M) = \varinjlim(a_n R_M) = aR_M$, where all a_n and a are nonzero elements of R_M. It then follows that I is $*$-invertible, since I is a $*$-locally principal ideal. □

Lemma 3.4 (cf. [13, Lemma 4.23])**.** *If P is a $*$-invertible prime $*$-ideal and Q is a $*$-invertible $*$-ideal such that $Q \nsubseteq P$, then $(P^k \cap Q)_* = (P^k Q)_*$ for all integers $k > 0$.*

Proof. We have $(P^k \cap Q)_* = (P^k A)_* = (QB)_*$ for some $*$-ideals A and B, and thus $(QB)_* \subseteq (P^k)_*$. Since $Q \nsubseteq P$, from [16, Theorem 13.2(iv)] we deduce that $B \subseteq (P^k)_*$. Thus we have $B = (P^k C)_*$ for some $*$-ideal C, and hence $(P^k \cap Q)_* = (QB)_* = (P^k QC)_* \subseteq (P^k Q)_*$. And the reverse inclusion is obvious. □

Proposition 3.5 (cf. [13, Corollary 4.24])**.** *If P_i $(1 \leq i \leq k)$ are $*$-invertible prime $*$-ideals of height-one such that $P_i \neq P_j$ for $i \neq j$, then $(\bigcap_1^k P_i^{n_i})_* = (\prod_1^k P_i^{n_i})_*$.*

Proof. We proceed by induction on the number k of ideals. For $k = 1$ the assertion is obvious. Assume that it is true for $k = s$, and deduce it for $k = s + 1$. We need to verify that $((\bigcap_1^s P_i^{n_i}) \cap P_{s+1}^{n_{s+1}})_* = ((\prod_1^s P_i^{n_i}) \cdot P_{s+1}^{n_{s+1}})_*$.

By induction hypothesis, $(\bigcap_1^s P_i^{n_i})_* = (\prod_1^s P_i^{n_i})_*$. Now the assertion follows from Lemma 3.4. $\square$

Lemma 3.6. *Let $*$ be of finite character on the integral domain R, and suppose that R is a $*$-LPI domain. If R_M is factorial for each $*$-maximal ideal M, then every nonzero principal ideal of R decomposes into a $*$-product of prime $*$-ideals.*

Proof. It follows from Lemma 3.3 that R satisfies ACC on $*$-invertible $*$-ideals of R, and hence every nonzero proper principal ideal aR of R decomposes into a $*$-product of $*$-coatoms, say $aR = (\prod_{i=1}^s P_i^{n_i})_*$ (all the P_i are distinct). We will prove that $\{P_i\}_{1\leq i\leq s}$ is the set of height-one prime ideals containing a, and $a \in (P_i^{n_i})_*$ and $a \notin (P_i^{n_i+1})_*$.

Consider the set $X_a^{(1)}$ of height-one prime ideals of R containing a.

(1) $X_a^{(1)}$ is not empty. Indeed, from the fact that $\frac{a}{1} \in Q$ for some height-one prime ideal Q of R_M (where M is a $*$-maximal ideal of R) it follows that $a \in Q \cap R \in X_a^{(1)}$.

(2) Every element P of $X_a^{(1)}$ is $*$-invertible. Indeed, let M be an arbitrary $*$-maximal ideal of R. Then PR_M is a height-one prime ideal of R_M if $P \subseteq M$, and so a t-ideal. Since R_M is factorial, PR_M is principal by [17, p. 284], and hence P is $*$-locally principal. Thus P is $*$-invertible.

(3) $X_a^{(1)}$ is finite. Indeed, if $P \in X_a^{(1)}$, then $(P_1^{n_1} \cdots P_s^{n_s})_* = aR \subseteq P$. Thus we have $P_i \subseteq P$. Since P is a $*$-invertible $*$-ideal and P_i is a $*$-coatom, $P = P_i$. Thus $X_a^{(1)}$ is finite.

(4) Let $X_a^{(1)} = \{Q_j\}_{j\in J}$, and also $a \in (Q_j^{m_j})_*$ and $a \notin (Q_j^{m_j+1})_*$. By Proposition 3.5, we have $(\prod_{j\in J} Q_j^{m_j})_* = (\bigcap_{j\in J} Q_j^{m_j})_* \supseteq (\prod_i^s P_i^{n_i})_*$. It remains to show that for every P_i there exists $Q_{j(i)} \in X_a^{(1)}$ such that $Q_{j(i)} = P_i$. Let M be a $*$-maximal ideal of R such that $aR \subseteq P_i \subseteq M$. In R_M there exists a minimal prime ideal $N \supseteq P_iR_M$. We have that $a \in P_i \subseteq N \cap R$ and $N \cap R$ is a height-one prime ideal in R, and thus it is $*$-invertible. From this we deduce that $N \cap R = Q_{j(i)} \in X_a^{(1)}$. The assertion is thus proved, since $Q_{j(i)} = P_i$. $\square$

Let V be a non-discrete valuation domain of (Krull) dimension 1. Then V does not have a v-maximal ideal, and hence V_M is a factorial domain for each v-maximal ideal M of V. However, each nonzero principal ideal of V cannot be expressed as a finite v-product of v-maximal ideals. Thus, in Lemma 3.6, we need the assumption that $*$ is of finite character.

In the following, we characterize $*$-locally factorial Krull domains.

Theorem 3.7. *If $*$ is of finite character, then the statements in Theorem 2.5 are equivalent to each of the following*:

(10) *R is a $*$-LPI domain and R_M is factorial for every $*$-maximal ideal M of R.*

(11) *R is a Krull domain and R_M is factorial for every $*$-maximal ideal M of R.*

Proof. (7) $\Rightarrow$ (10). Let I be a nonzero $*$-locally principal ideal of R. Then I is a t-ideal, and since R is a Krull domain, I is t-invertible. Therefore I is $*$-invertible by (7). Next, let M be a $*$-maximal ideal of R. Then R_M is a Krull domain; so to show that R_M is a factorial domain, it suffices to show that each height-one prime ideal of R_M is principal. Let Q be a height-one prime ideal of R_M. Then $Q = PR_M$ for some height-one prime ideal P of R. By (7), P is $*$-invertible, and hence $Q = PR_M$ is invertible. Thus Q is principal because R_M is quasi-local.

(10) $\Rightarrow$ (8). This follows from Lemma 3.3 and Lemma 3.6.

(9) $\Leftrightarrow$ (11). [16, Exercise 22.7]. □

We remark that if we take $* = d$ in Theorem 3.7, then it is well known that the statements in Theorem 3.7 are equivalent to R being a π-domain ([1, 17]). In the case of $* = t$ (resp., w), it follows from [17, Theorem 3.9] (resp., [18, Theorem 3.6]) that the equivalent conditions in Theorem 3.7 are equivalent to R being a Krull domain.

Recall that R is a *$*$-Dedekind domain* if every nonzero ideal of R is $*$-invertible, while R is a *$*$-Prüfer domain* if every nonzero finitely generated of R is $*$-invertible. Hence a Dedekind domain is a d-Dedekind domain; a Prüfer domain is a d-Prüfer domain; and a v-domain is a v-Prüfer domain. It is well known that each nonzero ideal of R is v-invertible if and only if R is completely integrally closed [15, Theorem 34.3]. Hence R is a v-Dedekind domain if and only if R is completely integrally closed.

Corollary 3.8. *Let $*$ be a finite character star-operation on an integral domain R. Then the following conditions are equivalent.*

(1) *R is a $*$-Dedekind domain.*
(2) *Every $*$-maximal ideal of R is $*$-invertible, and R satisfies ACC on $*$-invertible $*$-ideals of R.*
(3) *$\mathscr{I}_*^+(R)$ is a free semigroup with a system of generators consisting of $*$-coatoms, and $(I+J)_*$ is $*$-invertible for any $I, J \in \mathscr{I}_*^+(R)$.*
(4) *R is a $*$-Prüfer domain, and $\mathscr{I}_*^+(R)$ is a free semigroup with a system of generators consisting of $*$-coatoms.*
(5) *R is a Krull domain and $* = t$.*
(6) *R is a $*$-Prüfer domain and R satisfies ACC on $*$-invertible integral $*$-ideals of R.*
(7) *R is a $*$-Prüfer and $*$-factorable domain.*
(8) *R is a unique $*$-factorable domain and every prime $*$-ideal of R is $*$-maximal.*

Proof. (1) $\Rightarrow$ (4). Clearly, R is a $*$-Prüfer domain. Also, $\mathscr{I}_*^+(R)$ is free with base $*$-Spec$(R) = X^1(R)$ [16, Corollary 23.3 i)]. Now the second assertion follows from the fact that a $*$-invertible prime $*$-ideal is $*$-nonfactorable, and hence a $*$-coatom.

(4) $\Rightarrow$ (3). This follows because every $*$-finite type ideal of a $*$-Prüfer domain is $*$-invertible.

(3) $\Rightarrow$ (2). It suffices to show that every $*$-maximal ideal of R is $*$-invertible. Let M be a $*$-maximal ideal of R and let $0 \neq a \in M$. Then by (3), $aR = (P_1^{k_1} \cdots P_n^{k_n})_*$, where $P_1, \ldots, P_n$ are $*$-coatoms with $P_s \neq P_r$ for $s \neq r$. Hence $P_i \subseteq M$ for some i. We will show that $M = P_i$. Suppose for a contradiction that there exists $b \in M$ but $b \notin P_i$. Then $bR = (Q_1^{m_1} \cdots Q_l^{m_l})_* \subseteq M$, where $\{Q_i\}$ is a set of distinct $*$-coatoms of R. Hence $Q_k \subseteq M$ for some k. In addition, $Q_k \neq P_i$. Since $(Q_k + P_i)_*$ is a $*$-invertible $*$-ideal by (3), $(Q_k + P_i)_*$ is a t-invertible t-ideal of R. But then $M \supseteq (Q_k + P_i)_* = (Q_k + P_i)_t = R$, since Q_k and P_i are t-maximal (by Lemma 3.2). This is a contradiction.

(2) $\Rightarrow$ (1). By [16, Theorem 23.3], it suffices to show that every prime $*$-ideal of R is $*$-invertible. If R satisfies ACC on $*$-invertible $*$-ideals, then by Lemma 2.2, every $*$-invertible $*$-ideal is a $*$-product of $*$-coatoms. Thus by Lemma 2.3, every $*$-invertible $*$-ideal is a $*$-product of prime $*$-ideals. Hence it follows from hypothesis and Proposition 2.4 that every $*$-maximal ideal of R has height-one, that is, every prime $*$-ideal is $*$-maximal. Therefore again by hypothesis, every prime $*$-ideal of R is $*$-invertible.

(1) $\Rightarrow$ (5). Let I be a nonzero ideal of R. Then $(II^{-1})_* = R$ by (1), and hence I is t-invertible and $I_* = I_t$ by Lemma 2.2. Thus R is a Krull domain and $* = t$.

(5) $\Rightarrow$ (1). This is clear.

(4) $\Rightarrow$ (6). This follows from Theorem 3.7.

(6) $\Rightarrow$ (2). Let M be a $*$-maximal ideal of R. Choose a nonzero $a_1 \in M$. If $a_1 R = M$, then M is $*$-invertible; so assume $a_1 R \subsetneq M$. Choose another $a_2 \in M \setminus a_1 R$. If $(a_1, a_2)R = M$, then M is $*$-invertible because R is $*$-Prüfer. If $(a_1, a_2)R \subsetneq M$, choose an $a_3 \in M \setminus (a_1, a_2)R$. Repeating this process, we have $a_1 R \subseteq (a_1, a_2)R \subseteq \cdots \subseteq M$. Since R is $*$-Prüfer, each nonzero finitely generated ideal is $*$-invertible, and hence there is an integer n such that $M = (a_1, \ldots, a_n)R$. Thus M is $*$-invertible.

(5) $\Rightarrow$ (7). This follows from the fact that an integral domain R is a Krull domain if and only if R is a t-Prüfer and t-factorable domain ([8, Theorem 9]).

(7) $\Rightarrow$ (1). *Mutatis mutandis*, the proof is analogous to that of [8, Theorem 9].

(5) $\Rightarrow$ (8). This follows from the fact that an integral domain R is a Krull domain if and only if R is a unique t-factorable domain ([8, Theorem 12]).

(8) $\Rightarrow$ (5). This follows from [10, Theorem 3.2 and Corollary 3.3]. □

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