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STOCHASTIC COMPARISON OF THE FIRST PASSAGE TIME DISTRIBUTIONS IN THE M/G/1 RETRIAL QUEUE

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Abstract. This paper gives the stochastic comparison results for the first passage time distributions in the M/G/1 retial queue.

1. Introduction

Retrial queues are queueing systems in which arriving customers who find all servers occupied may retry for service again after a random amount of time. Retrial queues have been widely used to model many problems in telephone systems, call centers, telecommunication networks, computer networks and computer systems, and in daily life. Detailed overviews for retrial queues can be found in the bibliographies [1, 2, 3], the surveys [5, 7], and the books [4, 6].

Retrial queues are characterized by the following feature: If the server is idle when a customer arrives from outside the system, this customer begins to be served immediately and leaves the system after the service is completed. On the other hand, any customer who finds the server busy upon arrival joins a retrial group, called an orbit, and then attempts service after a random amount of time. If the server is idle when a customer from the orbit attempts service, this customer receives service immediately and leaves the system after the service completion. Otherwise the customer comes back to the orbit immediately and repeats the retrial process.

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We consider the M/G/1 retrial queue where customers arrive from outside the system according to a Poisson process with rate λ and service times are independent and identically distributed. Let B denote a generic random variable representing the service time, and F_B be a distribution function of the service time. The retrial time, i.e., the time interval between two consecutive attempts made by a customer in the orbit, is exponentially distributed with mean ν^{-1}. The arrival process, the service times, and the retrial times are assumed to be mutually independent. The traffic load ρ is defined as $\rho = \lambda \mathbb{E}B$. We assume that $\rho < 1$ for stability of the system.

Let $N(t)$ be the number of customers in the orbit at time t, and $C(t)$ is 1 if the server is busy at time t and 0 otherwise. Since $\{(N(t), C(t)) : t \geq 0\}$ is not a Markov process, we introduce a supplementary random variable: if $C(t) = 1$, we define $X(t)$ as the elapsed service time of the customer who is in service at time t and if $C(t) = 0$, we define $X(t) = 0$. Then $\{(N(t), C(t), X(t)) : t \geq 0\}$ is a Markov process. We assume that every sample path of the Markov process is right continuous. Define the first passage times

$$\begin{aligned}\tau_n &= \inf\{t > 0 : N(t) = n, C(t) = 1, X(t) = 0\}, n = 0, 1, 2, \ldots, \\ \sigma_n &= \inf\{t > 0 : N(t) = n, C(t) = 0\}, n = 0, 1, 2, \ldots,\end{aligned}$$

that is, τ_n is the first time a service starts with n customers in the orbit, σ_n is the first time a service ends with n customers in the orbit. Let

$$\begin{aligned}G_n(x) &= \mathbb{P}(\sigma_n \leq x \mid N(0) = n, C(0) = 1, X(0) = 0), n = 0, 1, 2, \ldots, \\ H_n(x) &= \mathbb{P}(\tau_{n-1} \leq x \mid N(0) = n, C(0) = 1, X(0) = 0), n = 1, 2, 3, \ldots,\end{aligned}$$

that is, $G_n(x)$ is the distribution function of the time interval that starts when a service starts with n customers in the orbit and ends when a service completes with n customers in the orbit, and $H_n(x)$ is the distribution function of the time interval that starts when a service starts with n customers in the orbit and ends when a service starts with $n-1$ customers in the orbit.

Clearly

$$\overline{G}_n(x) \leq \overline{H}_n(x), \qquad n = 1, 2, 3, \ldots,$$

where $\overline{F}$ is the complementary distribution function of F, i.e., $\overline{F}(x) = 1 - F(x)$, $x \in \mathbb{R}$. We also have the stochastic comparison on the first passage times, as stated in the following theorem. The proofs are given in Section 3.

THEOREM 1.1. *We have*

$$\overline{G}_n(x) \geq \overline{G}_m(x),\ x \in \mathbb{R}, \quad \text{if } 0 \leq n \leq m, \tag{1.1}$$

$$\overline{H}_n(x) \geq \overline{H}_m(x),\ x \in \mathbb{R}, \quad \text{if } 0 \leq n \leq m. \tag{1.2}$$

2. Stochastic comparison of Markov renewal sequences

In this section we provide a stochastic comparison result for Markov renewal sequences. We begin with recalling the definition of Markov renewal sequence. Let $\{(\mathcal{N}_n, \mathcal{S}_n) : n = 0, 1, 2, \ldots\}$ be a sequence of random vectors taking values in $\{0, 1, 2, \ldots\} \times [0, \infty)$ such that $0 = \mathcal{S}_0 \leq \mathcal{S}_1 \leq \mathcal{S}_2 \leq \cdots$. The process $\{(\mathcal{N}_n, \mathcal{S}_n) : n = 0, 1, 2, \ldots\}$ is said to be a Markov renewal sequence (MRS) if

$$\begin{aligned}&\mathbb{P}(\mathcal{N}_{n+1} = j, \mathcal{S}_{n+1} - \mathcal{S}_n \leq x \mid (\mathcal{N}_0, \mathcal{S}_0), (\mathcal{N}_1, \mathcal{S}_1), \ldots, (\mathcal{N}_n, \mathcal{S}_n)) \\ &= \mathbb{P}(\mathcal{N}_{n+1} = j, \mathcal{S}_{n+1} - \mathcal{S}_n \leq x \mid \mathcal{N}_n).\end{aligned}$$

We denote $\mathcal{K}_{ij}(x) = \mathbb{P}(\mathcal{N}_{n+1} = j, \mathcal{S}_{n+1} - \mathcal{S}_n \leq x \mid \mathcal{N}_n = i)$, $i, j = 0, 1, 2, \ldots$, $x \geq 0$. The matrix $\mathcal{K}(x) = (\mathcal{K}_{ij}(x))$ is called the kernel of the MRS $\{(\mathcal{N}_n, \mathcal{S}_n) : n = 0, 1, 2, \ldots\}$.

The following lemma, which will be used to prove Theorem 1.1, may be found in the open literature, but for completeness we prove this.

LEMMA 2.1. *For $k = 1, 2$, we consider an MRS $\{(\mathcal{N}_n^{(k)}, \mathcal{S}_n^{(k)}) : n = 0, 1, 2, \ldots\}$ with kernel $\mathcal{K}^{(k)}(x) = (\mathcal{K}_{ij}^{(k)}(x))$, $i, j = 0, 1, 2, \ldots$, $x \geq 0$. Suppose that*

(i) *For $k = 1, 2$,*

$$\mathcal{K}_{0j}^{(k)}(x) = \begin{cases} U(x) & \text{if } j = 0, \\ 0 & \text{if } j \geq 1, \end{cases}$$

where $U(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$

(ii) *For $k = 1, 2$, $\mathcal{K}_{ij}^{(k)}(x) = 0$ if $j < i - 1$.*

(iii) *For each $i = 1, 2, \ldots$, there is a random vector $(\mathcal{X}^{(1)}, \mathcal{Y}^{(1)}, \mathcal{X}^{(2)}, \mathcal{Y}^{(2)})$ such that*

$$\mathbb{P}(\mathcal{X}^{(1)} \leq \mathcal{X}^{(2)}, \mathcal{Y}^{(1)} \leq \mathcal{Y}^{(2)}) = 1, \tag{2.1}$$

$$\mathbb{P}(\mathcal{X}^{(k)} = j, \mathcal{Y}^{(k)} \leq x) = \mathcal{K}_{ij}^{(k)}(x),\ x \geq 0, j = 0, 1, 2, \ldots, k = 1, 2. \tag{2.2}$$

(iv) *For each $i = 1, 2, \ldots$, $\lim_{n\to\infty} \mathbb{P}(\mathcal{N}_n^{(2)} = 0 \mid \mathcal{N}_0^{(2)} = i) = 1$.*

Then, for each $i_0 \geq 1$, and $x \geq 0$,

$$\mathbb{P}(\lim_{n\to\infty} \mathcal{S}_n^{(1)} > x \mid \mathcal{N}_0^{(1)} = i_0) \leq \mathbb{P}(\lim_{n\to\infty} \mathcal{S}_n^{(2)} > x \mid \mathcal{N}_0^{(2)} = i_0).$$

Proof. We note that $(\mathcal{X}^{(1)}, \mathcal{Y}^{(1)}, \mathcal{X}^{(2)}, \mathcal{Y}^{(2)}) = (0,0,0,0)$ satisfies (2.1) and (2.2) if $i = 0$. Hence, for each $i = 0, 1, 2, \ldots$, there is $(\mathcal{X}^{(1)}, \mathcal{Y}^{(1)}, \mathcal{X}^{(2)}, \mathcal{Y}^{(2)})$ such that (2.1) and (2.2) hold.

Now we construct a sequence of random vectors $(\mathcal{X}_n^{(1)}, \mathcal{Y}_n^{(1)}, \mathcal{X}_n^{(2)}, \mathcal{Y}_n^{(2)})$, $n = 0, 1, 2, \ldots$, by induction on n as follows: Let $(\mathcal{X}_0^{(1)}, \mathcal{Y}_0^{(1)}, \mathcal{X}_0^{(2)}, \mathcal{Y}_0^{(2)}) = (i_0, 0, i_0, 0)$. Suppose that $(\mathcal{X}_k^{(1)}, \mathcal{Y}_k^{(1)}, \mathcal{X}_k^{(2)}, \mathcal{Y}_k^{(2)})$, $k = 0, 1, \ldots, n$, are defined. Choose $(\mathcal{X}_{n+1}^{(1)}, \mathcal{X}_{n+1}^{(2)}, \mathcal{Y}_{n+1}^{(1)}, \mathcal{Y}_{n+1}^{(2)})$ such that

$$\mathbb{P}(\mathcal{X}_{n+1}^{(1)} \leq \mathcal{X}_{n+1}^{(2)}, \mathcal{Y}_{n+1}^{(1)} \leq \mathcal{Y}_{n+1}^{(2)}) = 1,$$
$$\mathbb{P}(\mathcal{X}_{n+1}^{(k)} = j, \mathcal{Y}_{n+1}^{(k)} \leq x \mid \mathcal{X}_n^{(2)}) = \mathcal{K}_{\mathcal{X}_n^{(2)} j}^{(k)}(x),\ k = 1, 2.$$

Let $(\mathcal{N}_n^{(2)}, \mathcal{S}_n^{(2)}) = (\mathcal{X}_n^{(2)}, \sum_{k=1}^n \mathcal{Y}_k^{(2)})$, $n = 0, 1, 2, \ldots$. Then $\{(\mathcal{N}_n^{(2)}, \mathcal{S}_n^{(2)}) : n = 0, 1, 2, \ldots\}$ is the MRS with kernel $\mathcal{K}^{(2)}(x) = (\mathcal{K}_{ij}^{(2)}(x))$. Now we construct an MRS with kernel $\mathcal{K}^{(1)}(x) = (\mathcal{K}_{ij}^{(1)}(x))$. Define random times $0 = \iota_0 < \iota_1 < \iota_2 < \cdots$ by

$$\iota_{n+1} = \inf\{k > \iota_n : \mathcal{X}_{\iota_n+1}^{(1)} = \mathcal{X}_k^{(2)}\},\ n = 0, 1, 2, \ldots.$$

Let $(\mathcal{N}_n^{(1)}, \mathcal{S}_n^{(1)}) = (\mathcal{X}_{\iota_n}^{(2)}, \sum_{k=1}^n \mathcal{Y}_{\iota_{k-1}+1}^{(1)})$, $n = 0, 1, 2, \ldots$. Then it can be observed that $\{(\mathcal{N}_n^{(1)}, \mathcal{S}_n^{(1)}) : n = 0, 1, 2, \ldots\}$ is the MRS with kernel $\mathcal{K}^{(1)}(x)$. Since

$$\mathcal{S}_n^{(1)} = \sum_{k=1}^n \mathcal{Y}_{\iota_{k-1}+1}^{(1)} \leq \sum_{k=1}^n \mathcal{Y}_{\iota_{k-1}+1}^{(2)} \leq \sum_{k=1}^{\iota_n} \mathcal{Y}_k^{(2)} = \mathcal{S}_{\iota_n}^{(2)},\ n = 0, 1, 2, \ldots,$$

we have

$$\lim_{n\to\infty} \mathcal{S}_n^{(1)} \leq \lim_{n\to\infty} \mathcal{S}_{\iota_n}^{(2)} = \lim_{n\to\infty} \mathcal{S}_n^{(2)}. \tag{2.3}$$

Since $\mathcal{N}_0^{(1)} = \mathcal{N}_0^{(2)} = i_0$, (2.3) completes the proof of the lemma. □

3. Proof of Theorem 1.1

3.1. Proof of (1.2)

Suppose that a service begins at time 0. Denote by $0 = T_0 < T_1 < T_2 < \cdots$ the service initiation epochs. Then $\{(N(T_n), T_n) : n =$

$0, 1, 2, \ldots\}$ is an MRS with kernel

$$\mathcal{K}(x) = \begin{bmatrix} \mathcal{K}_{00}(x) & \mathcal{K}_{01}(x) & \mathcal{K}_{02}(x) & \cdots \\ \mathcal{K}_{10}(x) & \mathcal{K}_{11}(x) & \mathcal{K}_{12}(x) & \cdots \\ 0 & \mathcal{K}_{21}(x) & \mathcal{K}_{22}(x) & \cdots \\ 0 & 0 & \mathcal{K}_{32}(x) & \cdots \\ & & \ddots & \ddots \end{bmatrix},$$

where

$$\mathcal{K}_{ij}(x) = \begin{cases} \frac{\lambda}{j\nu+\lambda} a_{j-i} * E_{j\nu+\lambda}(x) + \frac{(j+1)\nu}{(j+1)\nu+\lambda} a_{j-i+1} * E_{(j+1)\nu+\lambda}(x) & \text{if } j \geq i, \\ \frac{i\nu}{i\nu+\lambda} a_0 * E_{i\nu+\lambda}(x) & \text{if } j = i-1, \\ 0 & \text{if } j < i-1, \end{cases}$$

with $a_l(y) = \frac{1}{\mathbb{E}B} \int_y^\infty e^{-\lambda t} \frac{(\lambda t)^l}{l!} dF_B(t)$, and E_α denoting the exponential distribution with rate α. We note that $H_n(x)$ is given by

$$H_n(x) = \mathbb{P}(T_{k_{n-1}} \leq x \mid N(T_0) = n), \tag{3.1}$$

where

$$k_{n-1} = \inf\{k \geq 1 : N(T_k) = n-1\}.$$

Consider an MRS $\{(\mathcal{N}_k^{(n)}, \mathcal{S}_k^{(n)}) : k = 0, 1, 2, \ldots\}$ with kernel

$$\mathcal{K}^{(n)}(x) = \begin{bmatrix} U(x) & 0 & 0 & \cdots \\ \mathcal{K}_{n,n-1}(x) & \mathcal{K}_{nn}(x) & \mathcal{K}_{n,n+1}(x) & \cdots \\ 0 & \mathcal{K}_{n+1,n}(x) & \mathcal{K}_{n+1,n+1}(x) & \cdots \\ 0 & 0 & \mathcal{K}_{n+2,n+1}(x) & \cdots \\ & & \ddots & \ddots \end{bmatrix}.$$

Then equation (3.1) can be written as

$$H_n(x) = \mathbb{P}(\lim_{k\to\infty} \mathcal{S}_k^{(n)} \leq x \mid \mathcal{N}_0^{(n)} = 1).$$

We prove (1.2) for $n \leq m$. By Lemma 2.1, it suffices to show that for every $i = 1, 2, 3, \ldots$, there is a random vector $(\mathcal{X}^{(1)}, \mathcal{Y}^{(1)}, \mathcal{X}^{(2)}, \mathcal{Y}^{(2)})$ such that

$$\mathbb{P}(\mathcal{X}^{(1)} \leq \mathcal{X}^{(2)}, \mathcal{Y}^{(1)} \leq \mathcal{Y}^{(2)}) = 1,$$
$$\mathbb{P}(\mathcal{X}^{(1)} = j, \mathcal{Y}^{(1)} \leq x) = \mathcal{K}_{m-1+i,m-1+j}(x),\ x \geq 0, j = 0, 1, 2, \ldots,$$
$$\mathbb{P}(\mathcal{X}^{(2)} = j, \mathcal{Y}^{(2)} \leq x) = \mathcal{K}_{n-1+i,n-1+j}(x),\ x \geq 0, j = 0, 1, 2, \ldots.$$

Now suppose that a service starts at time 0, and that there are $m-1+i$ customers in the orbit at time 0, excluding the one starting service. We divide the $m-1+i$ customers in the orbit into two groups, group 1 and group 2; group 1 consists of $n-1+i$ customers, and group 2 consists of the other $m-n$ customers. We call group 3, which consists of the customers arriving exogenously during the first service time.

We define the following epochs. Let δ_1 be the time of the first exogenous arrival after the first service completion, δ_2 the first retrial epoch of customers in groups 1 and 3 after the first service completion, and δ_3 the first retrial epoch of customer in group 2 after the first service completion. Assume that $\mathcal{L}$ customers arrive exogenously during the first service time, i.e., $\mathcal{L}$ is the number of customers in group 3. Let $\mathcal{Y}^{(1)} \equiv \min\{\delta_1, \delta_2, \delta_3\}$, i.e., $\mathcal{Y}^{(1)}$ is the second service initiation epoch for the retrial queue where the first service starts at time 0 with $m-1+i$ customers in the orbit. Then the number of customers in the orbit immediately after the beginning of the second service is given by

$$\begin{cases} m-1+i+\mathcal{L} & \text{if } \delta_1 = \min\{\delta_1, \delta_2, \delta_3\}, \\ m+i+\mathcal{L}-2 & \text{if } \delta_1 > \min\{\delta_1, \delta_2, \delta_3\}. \end{cases}$$

Let

$$\mathcal{X}^{(1)} = \begin{cases} i+\mathcal{L} & \text{if } \delta_1 = \min\{\delta_1, \delta_2, \delta_3\}, \\ i+\mathcal{L}-1 & \text{if } \delta_1 > \min\{\delta_1, \delta_2, \delta_3\}. \end{cases} \tag{3.2}$$

Then

$$\begin{aligned} \mathbb{P}(\mathcal{X}^{(1)} = j, \mathcal{Y}^{(1)} \le x) &= \mathcal{K}_{m-1+i,m-1+j}(x) \\ &= \mathcal{K}_{ij}^{(m)}(x). \end{aligned}$$

On the other hand, let $\mathcal{Y}^{(2)} \equiv \min\{\delta_1, \delta_2\}$, i.e., $\mathcal{Y}^{(2)}$ is the second service initiation epoch for the retrial queue where the first service starts at time 0 with $n-1+i$ customers in the orbit. Then the number of customers in the orbit immediately after the beginning of the second service is given by

$$\begin{cases} n-1+i+\mathcal{L} & \text{if } \delta_1 = \min\{\delta_1, \delta_2\}, \\ n+i+\mathcal{L}-2 & \text{if } \delta_1 > \min\{\delta_1, \delta_2\}. \end{cases}$$

Let

$$\mathcal{X}^{(2)} = \begin{cases} i+\mathcal{L} & \text{if } \delta_1 = \min\{\delta_1, \delta_2\}, \\ i+\mathcal{L}-1 & \text{if } \delta_1 > \min\{\delta_1, \delta_2\}. \end{cases} \tag{3.3}$$

Then

$$\begin{aligned} \mathbb{P}(\mathcal{X}^{(2)} = j, \mathcal{Y}^{(2)} \le x) &= \mathcal{K}_{n-1+i,n-1+j}(x) \\ &= \mathcal{K}_{ij}^{(n)}(x). \end{aligned}$$

By (3.2) and (3.3), we have $\mathcal{X}^{(1)} \le \mathcal{X}^{(2)}$. Clearly, $\mathcal{Y}^{(1)} \le \mathcal{Y}^{(2)}$. Therefore, the proof is complete.

3.2. Proof of (1.1)

Conditioning on the first service time, the number of exogenous arrivals during the first service time and the following exogenous arrival time or retrial time, we have

$$G_n(x) = a_0(x) + \sum_{k=1}^{\infty} a_k * E_{(n+k)\nu+\lambda} * \mathcal{H}_{n,k} * G_n(x), \tag{3.4}$$

where

$$\begin{aligned}\mathcal{H}_{n,k} \quad &= \tfrac{(n+k)\nu}{(n+k)\nu+\lambda} H_{n+k-1} * \cdots * H_{n+1} \\ &\quad + \tfrac{\lambda}{(n+k)\nu+\lambda} H_{n+k} * \cdots * H_{n+1}.\end{aligned}$$

Here $H_{n+k-1} * \cdots * H_{n+1}$ with $k = 1$ is interpreted as U. Equation (3.4) can be written as

$$G_n(x) = \sum_{l=0}^{\infty} \Big(\sum_{k=1}^{\infty} a_k * E_{(n+k)\nu+\lambda} * \mathcal{H}_{n,k} \Big)^{*l} * a_0(x). \tag{3.5}$$

Since the distribution functions $E_{(n+k)\nu+\lambda}$ and $\mathcal{H}_{n,k}$ decrease stochastically as n increases, (3.5) implies that G_n decreases stochastically as n increases. This proves (1.1).

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