ON m-CONVEX SETS IN PRECONVEXITY SPACES

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Abstract. In this paper, we introduce the concepts of m-convex set, mc-convex function and mc*-convex function. We study basic properties for m-convex sets and characterization for such functions.

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1. Introduction

In [1], Guay introduced the concept of preconvexity spaces defined by a binary relation on the power set $P(X)$ of a set $X$ and investigated some properties. He showed that a preconvexity on a set yields a convexity space in the same manner as a proximity [3] yields a topological space. In this paper, we introduce and study the m-convex sets induced by convex sets on a preconvexity space. In fact, every m-convex set is a convex set but the collection of all m-convex sets on a preconvexity space has some special properties as studied in section 2. We also introduce the concepts of mc-convex functions and mc*-continuous functions defined by m-convex sets and convex sets. In particular, the mc-convex function is a generalization of convex function on preconvexity spaces. In section 3, we investigate characterizations for such functions and relationships among c-convex function, mc-convex function and mc*-continuous function.

Definition 1 ([1]). Let $X$ be a nonempty set. A binary relation $\sigma$ on $P(X)$ is called a preconvexity on $X$ if the relation satisfies the following properties; we write $x \sigma A$ for $\{x\} \sigma A$:

1. If $A \subseteq B$, then $A \sigma B$.
2. If $A \sigma B$ and $B = \emptyset$, then $A = \emptyset$.
3. If $A \sigma B$ and $b \sigma C$ for all $b \in B$, then $A \sigma C$.
4. If $A \sigma B$ and $x \in A$, then $x \sigma B$.

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The pair \((X, \sigma)\) is called a preconvexity space. A convexity is a reflexive and transitive relation. In a preconvexity space \((X, \sigma)\), \(G(A) = \{x \in X : x \sigma A\}\) is called the convexity hull of a subset \(A\). \(A\) is said to be convex \([1]\) if \(G(A) = A\).

**Theorem 1** \([1]\). For a preconvexity space \((X, \sigma)\),

1. \(G(\emptyset) = \emptyset\).
2. \(A \subseteq G(A)\) for all \(A \subseteq X\).
3. If \(A \subseteq B\), then \(G(A) \subseteq G(B)\).
4. \(G(G(A)) = G(A)\) for \(A \subseteq X\).

**Theorem 2** \([1]\). If \(\sigma\) is a preconvexity on \(X\) and \(A \subseteq X\), then \(G(A) = \cap \{C : G(C) = C \text{ and } A \subseteq C\}\).

**Theorem 3** \([1]\). Let \(\sigma\) be a preconvexity on \(X\) and \(A, B \subseteq X\). Then

1. \(A \sigma B \text{ iff } A \subseteq G(B)\).
2. \(A \sigma B \text{ iff } G(A) \sigma G(B)\).

**Definition 2** \([1]\). Let \(\sigma_1, \sigma_2\) be two preconvexities on the convexity spaces \((X, \sigma_1)\) and \((Y, \sigma_2)\), respectively. A function \(f : X \rightarrow Y\) is said to be \(c\)-convex if \(A \sigma_1 B\) implies \(f(A) \sigma_2 f(B)\).

**Lemma 1** \([2]\). Let \((X, \sigma)\) be a preconvexity space. Then for all \(A \subseteq X\), \(G(A) \sigma A\).

### 2. \(m\)-convex sets on preconvexity spaces

**Definition 3.** Let \((X, \sigma)\) be a preconvexity space, \(x \in X\) and \(A \subseteq X\). Then \(A\) is called an \(m\)-convex set if \(x \in A \cup F\), whenever \(x \sigma (A \cup F)\) for every convex set \(F\).

**Theorem 4.** Let \((X, \sigma)\) be a preconvexity space. For an \(m\)-convex set \(A\) and a convex set \(F\), \(A \cup F\) is a convex set.

**Proof.** If \(x \in G(A \cup F)\), then \(x \sigma (A \cup F)\). By definition of \(m\)-convex set, \(x \in (A \cup F)\) and \(G(A \cup F) = A \cup F\). Therefore, \(A \cup F\) is convex. \(\square\)

**Theorem 5.** Let \((X, \sigma)\) be a preconvexity space. Then

1. Both \(\emptyset\) and \(X\) are \(m\)-convex.
2. If \(A\) and \(B\) are \(m\)-convex, then \(A \cup B\) is \(m\)-convex.
3. For \(\alpha \in J\), if \(A_\alpha\) is \(m\)-convex, then \(\cap_{\alpha \in J} A_\alpha\) is \(m\)-convex.

**Proof.** (1) For each convex set \(F\), if \(x \sigma (F \cup \emptyset)\), then \(x \in G(F)\). Since \(G(F) = F\), it implies \(x \in F \cup \emptyset\) and hence \(\emptyset\) is \(m\)-convex. From \(X = G(X)\), obviously it follows that \(X\) is \(m\)-convex.

(2) Let \(A\) and \(B\) be \(m\)-convex subsets. For each convex set \(F\), let \(x \sigma ((A \cup B) \cup F)\). By Theorem 4, we know that \(B \cup F\) is a convex set. Since \(A\) is an \(m\)-convex set and \(x \sigma (A \cup (B \cup F))\), \(x \in (A \cup (B \cup F))\). It implies that \(A \cup B\) is \(m\)-convex.
Theorem 6. Let \( x \in m \)-convex set \( F \) be convex. 

**Proof.** For each \( m \)-convex set \( F \), since \( \emptyset \) is a convex set, by Theorem 4, \( F \cup \emptyset = F \) is convex. \( \square \)

Example 1. Let \( X = \{a, b, c\} \) and a topology \( \tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\} \). Consider a family \( S = \{A \subseteq X : A \subseteq \text{cl}(\text{int}(A))\} \), where \( \text{cl} \) and \( \text{int} \) denote closure and interior operators, respectively, in the topological space \( (X, \tau) \). Set \( \text{scl}(A) = \cap\{F : A \subseteq F \text{ and } X \setminus F \in S\} \). Define \( A \sigma B \) iff \( \text{scl}(A) \subseteq \text{scl}(B) \). Then \( \sigma \) is a preconvexity on \( (X, \sigma) \). Note that:

1. \( \{\emptyset, \{a\}, \{b\}, \{b, c\}, \{a, c\}, X\} \) is the collection of all convex subsets on \( (X, \sigma) \);
2. \( \{\emptyset, \{c\}, \{b, c\}, \{a, c\}, X\} \) is the collection of all \( m \)-convex subsets on \( (X, \sigma) \).

This fact shows that in Theorem 6 the converse may not be true.

Definition 4. Let \( (X, \sigma) \) be a preconvexity space and \( A \subseteq X \).

The set \( \text{mh}(A) = \cap\{F \subseteq X : A \subseteq F, \; F \text{ is an } m \text{-convex set}\} \) is called the \( m \)-closure of \( A \).

Lemma 2. Let \( (X, \sigma) \) be a preconvexity space and \( A \subseteq X \). Then

\[ G(A) \subseteq \text{mh}(A). \]

**Proof.** For some \( x \in G(A) \), suppose on the contrary that \( x \notin \text{mh}(A) \). Then there exists an \( m \)-convex set \( F \) such that \( A \subseteq F \) and \( x \notin F \). Since every \( m \)-convex set is convex, \( G(A) \subseteq G(F) = F \) and so \( x \notin G(A) \). It contradicts that \( x \in G(A) \). This completes that \( G(A) \subseteq \text{mh}(A) \). \( \square \)

From Theorem 5, we get the next theorem:

Theorem 7. Let \( (X, \sigma) \) be a preconvexity space. Then

1. \( \text{mh}(\emptyset) = \emptyset \).
2. \( A \subseteq \text{mh}(A) \) for \( A \in X \).
3. \( \text{mh}(\text{mh}(A)) = \text{mh}(A) \) for \( A \in X \).
4. \( \text{mh}(A \cup B) = \text{mh}(A) \cup \text{mh}(B) \) for \( A, B \in X \).

3. \( mc^* \)-convex functions and \( mc \)-convex functions

In this section, we introduce the concepts of \( mc \)-convex functions and \( mc^* \)-continuous functions defined by \( m \)-convex sets and convex sets. We investigate characterizations for such functions and relationships among \( c \)-convex function, \( mc \)-convex function and \( mc^* \)-continuous function.
Definition 5. Let \((X, \sigma)\) and \((Y, \mu)\) be two preconvexity spaces. A function \(f : X \rightarrow Y\) is said to be

1. \(mc^*\)-convex if \(f^{-1}(U)\) is \(m\)-convex for each \(m\)-convex set \(U\) in \(Y\);
2. \(mc\)-convex if \(f^{-1}(U)\) is convex for each \(m\)-convex set \(U\) in \(Y\).

Theorem 8. Let \(f : X \rightarrow Y\) be a function on two preconvexity spaces \((X, \sigma)\) and \((Y, \mu)\). Then the following are equivalent:

1. \(f\) is \(mc^*\)-continuous.
2. \(f(mh(A)) \subseteq mh(f(A))\) for \(A \subseteq X\).
3. \(mh(f^{-1}(B)) \subseteq f^{-1}(mh(B))\) for \(B \subseteq Y\).

Proof. (1) ⇒ (2) Let \(F\) be any \(m\)-convex set in \(Y\) containing \(f(A)\). Then \(f^{-1}(F)\) is an \(m\)-convex set containing \(A\). Since \(mh(A)\) is the smallest \(m\)-convex set containing \(A\), \(A \subseteq mh(A) \subseteq f^{-1}(F)\), and \(f(A) \subseteq f(mh(A)) \subseteq F\). This implies that \(f(mh(A)) \subseteq mh(f(A))\).

(2) ⇒ (3) Obvious.

(3) ⇒ (1) Obvious. □

Theorem 9. Let \(f : X \rightarrow Y\) be a function on two preconvexity spaces \((X, \sigma)\) and \((Y, \mu)\). Then the following are equivalent:

1. \(f\) is \(mc\)-convex.
2. \(f(G(A)) \subseteq mh(f(A))\) for \(A \subseteq X\).
3. \(G(f^{-1}(B)) \subseteq f^{-1}(mh(B))\) for \(B \subseteq Y\).

Proof. (1) ⇒ (2) Let \(F\) be any \(m\)-convex set in \(Y\) containing \(f(A)\); then \(f^{-1}(F)\) is a convex set containing \(A\). So \(A \subseteq G(A) \subseteq f^{-1}(F)\) and \(f(A) \subseteq f(G(A)) \subseteq F\). This implies that \(f(G(A)) \subseteq mh(f(A))\).

(2) ⇒ (3) For \(B \subseteq Y\), it is \(f(G(f^{-1}(B))) \subseteq mh(B)\) by (2). Thus we get the result.

(3) ⇒ (1) It is obvious. □

Theorem 10. Let \(f : X \rightarrow Y\) be a function on two preconvexity spaces \((X, \sigma)\) and \((Y, \mu)\). Then if \(f\) is \(c\)-convex, then it is \(mc\)-convex.

Proof. Let \(F\) be any \(m\)-convex set in \(Y\). By Theorem 6 and Lemma 1, \(F\) is convex and \(G(F) \sigma F\). From \(f\) is \(c\)-convex and Lemma 2,

\[f(G(F)) \sigma f(F) \subseteq G(f(F)) \subseteq mh(f(F))\]

Hence by Theorem 9, \(f\) is \(mc\)-convex. □

Remark 1. Finally, we have the following implications but the converses are not true in general.

\[mc^*\)-continuous ⇒ \(mc\)-convex ⇔ \(c\)-convex \]
Example 2. Let $X = \{a, b, c\}$ and a topology $\tau_2 = \emptyset, \{a\}, X$. Consider a family $S = \{A \subseteq X : A \subseteq cl(int(A))\}$, where $cl$ and $int$ denote closure and interior operators, respectively, in the topological space $(X, \tau_2)$. Set $scl(A) = \cap\{F : A \subseteq F \text{ and } X - F \in S\}$. Define $A \mu B$ iff $scl(A) \subseteq scl(B)$. Then $\mu$ is a preconvexity on $(X, \mu)$ and

1. $C = \{\emptyset, \{b\}, \{c\}, \{b, c\}, X\}$ is the collection of all convex subsets in $(X, \mu)$;
2. $C$ is also the collection of all $m$-convex subsets of $(X, \mu)$.

Consider the preconvexity $\sigma$ defined in Example 1. Then we obtain the following things:

(i) The identity function $f : (X, \sigma) \to (X, \mu)$ is $mc$-convex. For an $m$-convex set $A = \{b\}$ in $(X, \mu)$, $f^{-1}(A)$ is not $m$-convex in $(X, \sigma)$. Therefore, $f$ is not $mc^*$-convex.

(ii) Let us define a function $f : (X, \mu) \to (X, \sigma)$ as the following: $f(a) = f(b) = b, f(c) = c$. Then $f$ is $m$-convex. For a convex set $F = \{b\}$ in $(X, \sigma)$, $f^{-1}(F) = \{a, b\}$ is not convex in $(X, \mu)$. Consequently, $f$ is not $c$-convex.

References


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