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ON THE CHARACTER RINGS OF TWIST KNOTS

Fumikazu Nagasato

Dedicated to Professor Akio Kawauchi for his 60th birthday

Abstract. The Kauffman bracket skein module $\mathcal{K}_t(M)$ of a 3-manifold M becomes an algebra for $t = -1$. We prove that this algebra has no non-trivial nilpotent elements for M being the exterior of the twist knot in 3-sphere and, therefore, it is isomorphic to the $\mathrm{SL}_2(\mathbb{C})$-character ring of the fundamental group of M. Our proof is based on some properties of Chebyshev polynomials.

1. Introduction

In this paper, we show a property of the Kauffman bracket skein module (KBSM for short) by using polynomials $S_n(z)$ $(n \in \mathbb{Z})$ in an indeterminate z defined by the following recursive relation:

$$S_{n+2}(z) = zS_{n+1}(z) - S_n(z),\ S_1(z) = z,\ S_0(z) = 1.$$

The polynomial $S_n(z)$ can be transformed into *the Chebyshev polynomial of the second type* $U_n(z)$ defined by

$$U_{n+2}(z) = 2zU_{n+1}(z) - U_n(z),\ U_1(z) = 2z,\ U_0(z) = 1.$$

Indeed, $U_n(z) = S_n(2z)$ holds for any $n \geq 0$.

In Theorem 2.1 of [3], Bullock and LoFaro found that the KBSM $\mathcal{K}_t(E_{K_m})$ of the exterior E_{K_m} of an *m-twist knot* K_m $(m \geq 0)$ in 3-sphere $\mathbb{S}^3$ is generated as $\mathbb{C}[t, t^{-1}]$-module by $x^p y^q$ $(p, q \in \mathbb{Z}_{\geq 0}, m \geq q)$:

$$\mathcal{K}_t(E_{K_m}) = \mathrm{Span}_{\mathbb{C}[t,t^{-1}]}\{x^p y^q \mid p, q \in \mathbb{Z}_{\geq 0},\ m \geq q\},$$

where $x^p y^q$ means the isotopy class of the disjoint union of p parallel copies of an annulus $\tilde{x}$ and q parallel copies of an annulus $\tilde{y}$ in E_{K_m} (see Figure 1). Namely, the set $\{x^p y^q\}_{p,q\in\mathbb{Z}_{\geq 0},m\geq q}$ forms a basis of $\mathcal{K}_t(E_{K_m})$ as $\mathbb{C}[t, t^{-1}]$-module. In [5],

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all *the factorization relations* for $\mathcal{K}_t(E_{K_m})$ are given. For example, the element $R_m(t)$ in $\mathcal{K}_t(E_{K_m})$ expressed by

$$\begin{aligned} R_m(t) := \; & S_{m+1}(y) + (t^{-6} - t^{-2}x^2)S_m(y) + ((2t^4 + t^{-8})x^2 - t^{-4})S_{m-1}(y) \\ & -t^{-10}S_{m-2}(y) + 2x^2(t^{-2m-2} + t^{-2m-6}) \sum_{i=0}^{m-2} t^{2i}S_i(y) - t^{-2m-6}x^2 \end{aligned}$$

gives a factorization relation for $S_{m+1}(y)$ or y^{m+1}:

Theorem 1 (Theorem 4 in [5])**.** *$R_m(t) = 0$ in $\mathcal{K}_t(E_{K_m})$.*

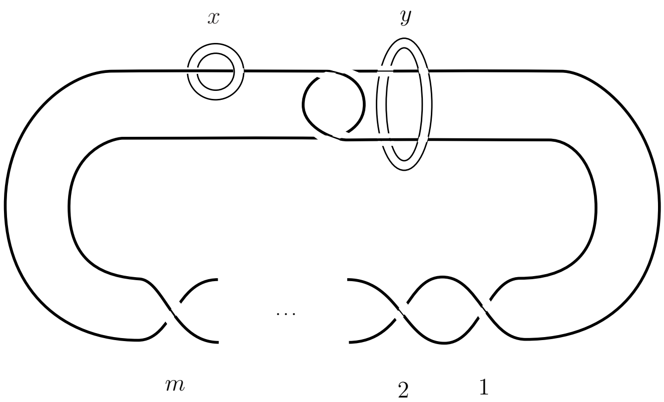
$K_m =$ 

FIGURE 1. an m-twist knot K_m in $\mathbb{S}^3$ and annuli $\tilde{x}$ and $\tilde{y}$ in E_{K_m}

By Lemma 7 of [5] the polynomial $R_m(-1)$ at $t = -1$, which has the following factorization

$$R_m(-1) = (y+2)\left(S_m(y) - S_{m-1}(y) + x^2 \sum_{i=0}^{m-1} S_i(y)\right),$$

has no repeated factors over the rational number field $\mathbb{Q}$. This paper generalizes this property to the case of the complex number field $\mathbb{C}$:

Theorem 2. *$R_m(-1)$ has no repeated factors over $\mathbb{C}$.*

In fact, Theorem 2 shows that the KBSM $\mathcal{K}_{-1}(E_{K_m})$ for $t = -1$, which is a $\mathbb{C}$-algebra generated by x and y, has no non-trivial nilpotent elements. Then it follows from Theorem 3 in Subsection 2.2 that $\mathcal{K}_{-1}(E_{K_m})$ is isomorphic to *the* $\mathrm{SL}_2(\mathbb{C})$*-character ring* $\chi(G_{K_m})$ of the twist knot K_m:

Corollary 1. *For any $m \in \mathbb{Z}_{\geq 0}$,*

$$\chi(G_{K_m}) \cong \mathcal{K}_{-1}(E_{K_m}) = \mathbb{C}[x,y]/\langle R_m(-1)\rangle,$$

where $\langle R_m(-1)\rangle$ means the ideal in $\mathbb{C}[x,y]$ generated by $R_m(-1)$.

In this paper, we explain these facts, especially Theorem 2 and Corollary 1 by using the polynomials $S_n(z)$.

2. Character rings

2.1. Character rings

Let G be a finitely generated and presented group, and $R(G)$ the set of all the representations $\rho : G \to \mathrm{SL}_2(\mathbb{C})$. For each element g in G, we can define a function t_g on $R(G)$ by $t_g(\rho) := \mathrm{tr}(\rho(g))$. Let T denote the ring generated by all the functions t_g, $g \in G$. By Proposition 1.4.1 in [4], the ring T is finitely generated. So we can fix a finite set $\{g_1, \ldots, g_n\}$ of G $(n > 0)$ such that $t_{g_1}, \ldots, t_{g_n}$ generate T. Then the image of $R(G)$ under the map

$$t : R(G) \to \mathbb{C}^n,\ t(\rho) := (t_{g_1}(\rho), \ldots, t_{g_n}(\rho)),\ \rho \in R(G)$$

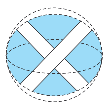
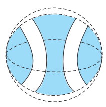
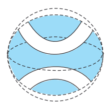
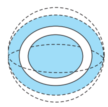
is a closed algebraic set (refer to Corollary 1.4.5 in [4]), denoted by $X(G)$. This algebraic set $X(G)$ is called the $\mathrm{SL}_2(\mathbb{C})$-character variety of G (For details, refer to [4]).

As $X(G)$ is an algebraic set, we can consider its coordinate ring as follows. Suppose $X(G)$ is an algebraic set in complex space $\mathbb{C}^m$ $(m > 0)$. Let $I(X(G))$ be the ideal of the polynomial ring $\mathbb{C}[x_1, \ldots, x_m]$ that consists entirely of polynomials vanishing on $X(G)$. Then the coordinate ring of $X(G)$, denoted by $\chi(G)$, is defined as the quotient polynomial ring $\mathbb{C}[x_1, \ldots, x_m]/I(X(G))$. This quotient ring $\chi(G)$ is called the $\mathrm{SL}_2(\mathbb{C})$-character ring of G. For a knot group G_K, which is the fundamental group of the exterior E_K, we call $\chi(G_K)$ the $\mathrm{SL}_2(\mathbb{C})$-character ring of the knot K.

2.2. Character rings from the KBSM point of view

The $\mathrm{SL}_2(\mathbb{C})$-character ring of a knot can be studied by using the KBSM. We quickly review a way to do it. For details, refer to [2], [6] and [7].

For a compact oriented 3-manifold M, let $\mathcal{L}_t(M)$ be the $\mathbb{C}[t, t^{-1}]$-module generated by all the isotopy classes of framed links in M (including the empty link $\emptyset$). Then the KBSM $\mathcal{K}_t(M)$ of M is defined as the quotient of $\mathcal{L}_t(M)$ by the Kauffman bracket skein relations as below:

 $-\, t$  $-\, t^{-1}$  , $L \sqcup$  $-\, (-t^2 - t^{-2})L,$

where L is any framed link in M and "$\sqcup$" means the disjoint union.

It is known that the KBSM $\mathcal{K}_{-1}(M)$ for $t = -1$ becomes a $\mathbb{C}$-algebra with a multiplication $\sqcup$ (the unit in $\mathcal{K}_{-1}(M)$ is the empty link $\emptyset$). For example, $\mathcal{K}_{-1}(E_{K_m})$ becomes $\mathbb{C}[x, y]/\langle R_m(-1)\rangle$. The $\mathbb{C}$-algebra $\mathcal{K}_{-1}(M)$ is naturally linked to the $\mathrm{SL}_2(\mathbb{C})$-character ring $\chi(\pi_1(M))$ of the fundamental group $\pi_1(M)$ as follows.

Theorem 3 ([2], [7]). *For any compact orientable 3-manifold M, there exists a surjective homomorphism Φ as $\mathbb{C}$-algebra*

$$\Phi : \mathcal{K}_{-1}(M) \to \chi(\pi_1(M)),$$

defined by $\Phi(K) := -t_{[K]}$, $\Phi(K_1 \sqcup \cdots \sqcup K_i) := \prod_{j=1}^{i} \Phi(K_i)$, where $[K]$ is an element of $\pi_1(M)$ represented by the knot K with an unspecified orientation and a base point. Moreover the kernel of Φ is the nilradical $\sqrt{0}$.

By Theorem 3, the $\mathrm{SL}_2(\mathbb{C})$-character ring $\chi(G_{K_m})$ of the m-twist knot K_m is isomorphic to $\mathcal{K}_{-1}(E_{K_m})/\sqrt{0}$. So if $\mathcal{K}_{-1}(E_{K_m})$ has no non-trivial nilpotent elements, then $\mathcal{K}_{-1}(E_{K_m})$ is isomorphic to $\chi(G_{K_m})$.

3. Proof of Theorem 2

To prove Theorem 2 we do not use any topological properties of the twist knots but some algebraic properties of the polynomials $S_n(z)$.

Proof of Theorem 2. Let $\widetilde{R}_m(x, y)$ be the factor

$$S_m(y) - S_{m-1}(y) + x^2 \sum_{i=0}^{m-1} S_i(y)$$

of $R_m(-1)$. First we show that $\widetilde{R}_m(x, y)$ has no repeated factors in the factorization over $\mathbb{C}$.

Assume that $\widetilde{R}_m(x, y)$ has a repeated factor in the factorization over $\mathbb{C}$. Then $\widetilde{R}_m(x, y)$ is reducible over $\mathbb{C}$. In this situation, we have the following two cases for the factorization of $\widetilde{R}_m(x, y)$ in terms of the variable x:

- $(ax + b)(cx + d)$, where $a, b, c, d \in \mathbb{C}[y]$,
- $(ax^2 + b)c$, where $a, b, c \in \mathbb{C}[y]$, $ax^2 + b$ is irreducible over $\mathbb{C}$.

We see that the first case never happens, because if it happens, then the equation

$$acx^2 + (ad + bc)x + bd = S_m(y) - S_{m-1}(y) + x^2 \sum_{i=0}^{m-1} S_i(y)$$

must hold. So we have

$$ac = \sum_{i=0}^{m-1} S_i(y),\ ad + bc = 0,\ bd = S_m(y) - S_{m-1}(y).$$

Let us focus on the degree of the above three equations in terms of y. Note that $\deg(S_m(y))$ is m by definition.

$$\deg(a) + \deg(c) = m - 1,\ \deg(a) + \deg(d) = \deg(b) + \deg(c),$$

$$\deg(b) + \deg(d) = m.$$

Combining these equations, we have

$$2\deg(a) = 2\deg(b) - 1,$$

i.e., an even number equals an odd number, a contradiction.

We can also check that the second case never happens by using algebraic properties of the polynomials $S_m(y)$ as follows. For the second case, the equation

$$acx^2 + bc = S_m(y) - S_{m-1}(y) + x^2 \sum_{i=0}^{m-1} S_i(y)$$

must hold. In particular, the following equation is required:

$$bc = S_m(y) - S_{m-1}(y).$$

Note that the Chebyshev polynomial $U_m(y) = S_m(2y)$ has the property

$$U_m(\cos\theta) = \sin(m+1)\theta / \sin\theta$$

(refer to [1] for example). By this property, for any integer $0 \leq i \leq m-1$, we obtain

$$\begin{aligned} & S_m\left(2\cos\frac{2i+1}{2m+1}\pi\right) - S_{m-1}\left(2\cos\frac{2i+1}{2m+1}\pi\right) \\ = \; & \frac{1}{\sin\frac{2i+1}{2m+1}\pi}\left(\sin\left(i+\frac{1}{2}+\frac{i+\frac{1}{2}}{2m+1}\right)\pi - \sin\left(i+\frac{1}{2}-\frac{i+\frac{1}{2}}{2m+1}\right)\pi\right) \\ = \; & 0. \end{aligned}$$

Note that the degree of $S_m(y) - S_{m-1}(y)$ in terms of y is m by definition. Thus we see that $S_m(y) - S_{m-1}(y)$ has the following nice factorization:

$$S_m(y) - S_{m-1}(y) = \prod_{i=0}^{m-1}\left(y - 2\cos\frac{2i+1}{2m+1}\pi\right).$$

So $S_m(y) - S_{m-1}(y)$ has no repeated factors. Hence bc, especially c has no repeated factors. Then $ax^2 + b$ must have a repeated factor, a contradiction. Therefore $\widetilde{R}_m(x,y)$ has no repeated factors over $\mathbb{C}$.

It is clear that $\widetilde{R}_m(x,y)$ does not have the factor $y+2$ (for example, use the fact that $S_m(-2) - S_{m-1}(-2) \neq 0$). These facts complete the proof. □

It follows from Theorem 2 and a property of radicals that $\sqrt{\langle R_m(-1)\rangle}$ coincides with $\langle R_m(-1)\rangle$. So we have

$$\mathcal{K}_{-1}(E_{K_m})/\sqrt{0} = \mathbb{C}[x,y]/\sqrt{\langle R_m(-1)\rangle} = \mathbb{C}[x,y]/\langle R_m(-1)\rangle = \mathcal{K}_{-1}(E_{K_m}),$$

which shows that $\mathcal{K}_{-1}(E_{K_m})$ has no non-trivial nilpotent elements. This fact and Theorem 3 prove Corollary 1.

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Department of Mathematics
Meijo University
Tempaku, Nagoya 468-8502, Japan
E-mail address: fukky@meijo-u.ac.jp