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MINIMAL NONCOMMUTATIVE REVERSIBLE AND REFLEXIVE RINGS

Byung-Ok Kim and Yang Lee

Abstract. The reflexiveness and reversibility were introduced by Mason and Cohn respectively. The structures of minimal reversible rings and minimal reflexive rings are completely determined. The term minimal means having smallest cardinality.

Throughout this note all rings are associative with identity unless otherwise stated. Let R be a ring. The n by n full (resp. upper triangular) matrix ring over R is denoted by $\mathrm{Mat}_n(R)$ (resp. $U_n(R)$). Let $J(R)$ denote the Jacobson radical of R. $\mathbb{Z}_n$ denotes the ring of integers modulo n.

According to Cohn [2], a ring R is called *reversible* if $ab = 0$ implies $ba = 0$ for $a, b \in R$. Anderson-Camillo [1] used the term ZC_2 for the reversibility. A ring is called *reduced* if it has no nonzero nilpotent elements. Reduced rings are reversible via a simple computation. Commutative rings are clearly reversible, but the converse need not hold since there exist many kinds of noncommutative reduced rings. Note also that there are many kinds of non-reduced commutative rings (e.g., $\mathbb{Z}_{k^n}$ for any $k, n \geq 2$). A ring is called *Abelian* if every idempotent is central. Reversible rings are Abelian through a simple computation.

Due to Mason [12], a ring R is called *reflexive* if $aRb = 0$ implies $bRa = 0$ for $a, b \in R$. Semiprime rings are reflexive by a simple computation. Reversible rings are clearly reflexive, but the converse need not hold since there exist many kinds of non-Abelian semiprime rings (e.g., $\mathrm{Mat}_n(R)$ for a semiprime ring R and $n \geq 2$). Note that reflexive rings need not be Abelian as can be seen by $\mathrm{Mat}_n(R)$ ($n \geq 2$) over a semiprime ring R.

A ring R is called *semilocal* if $R/J(R)$ is semisimple Artinian, and R is called *semiperfect* if R is semilocal and idempotents can be lifted modulo $J(R)$. R is called *local* if $R/J(R)$ is a division ring. Local rings are clearly semilocal, and one important case of semiperfect rings is when the Jacobson radical is nil

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by [10, Proposition 3.6.1]. Local rings are Abelian through a simple computation. Note that $J(R)$ is nilpotent by [9, Theorem 2.4.12] when R is one-sided Artinian.

Lemma 1. (1) *The class of reversible rings is closed under subrings* (*possibly without identity*).

(2) *Let R_i $(i \in I)$ be rings. Then the following conditions are equivalent*:

(a) *R_i is reversible for all $i \in I$*;

(b) *The direct product of R_i $(i \in I)$ is reversible*;

(c) *The direct sum* (*possibly without identity*) *of R_i $(i \in I)$ is reversible.*

(3) *A ring R is semiperfect and reversible if and only if R is a finite direct sum of local reversible rings.*

Proof. (1) It is obvious that the reversibility is hereditary to subrings.

(2) Let R be the direct product of reversible rings R_i $(i \in I)$. Suppose that $(a_i)(b_i) = 0$ for $(a_i), (b_i) \in R$. Then $a_i b_i = 0$ for each $i \in I$. Since R_i is reversible, $b_i a_i = 0$ for each $i \in I$, entailing $(b_i)(a_i) = 0$. The proofs of remainder come from the result (1).

(3) We apply the proof of [5, Lemma 2.2(3)]. Suppose that R is a semiperfect reversible ring. Since R is semiperfect, R has a finite orthogonal set of local idempotents whose sum is 1 by [10, Proposition 3.7.2], say $R = \oplus \sum_{i=1}^{n} e_i R$ such that each $e_i R e_i$ is a local ring. Since reversible rings are Abelian, every $e_i R$ is an ideal of R with $e_i R = e_i R e_i$. Each $e_i R$ is a reversible ring by the result (2). Conversely assume that R is a finite direct sum of local reversible rings. Then R is semiperfect since local rings are semiperfect, and moreover R is reversible by the result (2). $\square$

We use $GF(p^n)$ to denote the Galois field of order p^n. Due to Feller [4], a ring is called *right* (*left*) *duo* if every right (left) ideal is two-sided. Right or left duo rings are clearly Abelian via a simple computation. Xue [13] proved that finite rings are right duo if and only if they are left duo. We first study minimal noncommutative reversible rings, analyzing the following example.

Example 2. Due to Xue [14, Example 2], let $R = \left\{ \left(\begin{smallmatrix} a & b \\ 0 & a^2 \end{smallmatrix}\right) \mid a, b \in GF(2^2) \right\}$ be a subring of $U_2(GF(2^2))$. Then

$$J(R) = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \in GF(2^2) \right\}, R/J(R) \cong GF(2^2),$$

and R is reversible by the computation in [5, Example 2.5]. In fact, letting $xy = 0$ for $x, y \in R \backslash 0$, both x and y must be contained in $J(R)$; hence we also get $yx = 0$. Thus R is a reversible ring of order 16. It is not difficult to check that R is duo.

Theorem 3. *If R is a minimal noncommutative reversible ring, then R is of order* 16 *and is isomorphic to the ring of all matrices of the form $\left(\begin{smallmatrix} a & b \\ 0 & a^2 \end{smallmatrix}\right)$ over $GF(2^2)$.*

Proof. We apply the proof of [5, Theorem 2.6]. Eldridge proved, in [3, Proposition], that if a ring A is of order p^3, p a prime, then A is isomorphic to $U_2(GF(p))$, and that if a finite ring has a cube free factorization, then it is commutative, in [3, Theorem]. Thus a minimal noncommutative reversible ring is of order 16, considering Example 2, since reversible rings are Abelian and $U_2(GF(p))$ is non-Abelian. Note that finite rings are semiperfect.

Let R be a minimal noncommutative reversible ring. Then since R is minimal, R is a local ring with the help of Lemma 1(3). We have three cases of $|J(R)| = 2$, $|J(R)| = 4$ or $|J(R)| = 8$, where $|J(R)|$ means the cardinality of $J(R)$. Note that $R/J(R)$ is a field and $J(R)$ is a vector space over $R/J(R)$. If $|J(R)| = 2$ then $R/J(R) \cong GF(2^3)$ and so $|J(R)| \geq 8$, a contradiction. Suppose $|J(R)| = 8$. Then $R/J(R) \cong GF(2)$ and by [7, Theorem 2.3.6], $J(R)$ has a basis $\{a, b, c\}$ over $GF(2)$ with $a^2 = ab = c$ and $ba = 0$; hence R is not reversible, a contradiction.

Consequently we have the only case of $|J(R)| = 4$, for R to be a minimal noncommutative reversible ring. In this case R is right duo (hence duo) by the argument in the proof of [5, Theorem 2.6]. This implies that R is a noncommutative duo ring of order 16 with $|J(R)| = 4$. Then R is isomorphic to the ring in Example 2 by [14, Theorem 3]. □

Local rings need not be reversible by the first and second rings in [14, Example 2], and reversible rings also need not be local since there exist many semiprimitive reduced rings but not division rings (e.g., the ring of integers, the first Weyl algebra over a field of characteristic zero).

Lemma 4. (1) *The class of reflexive rings is not closed under subrings.*

(2) *Let R_i $(i \in I)$ be rings. Then the following conditions are equivalent*:

(a) *R_i is reflexive for each $i \in I$*;

(b) *The direct product of R_i $(i \in I)$ is reflexive*;

(c) *The direct sum (possibly without identity) of R_i $(i \in I)$ is reflexive.*

(3) *Let R be an Abelian ring. Then R is a semiperfect reflexive ring if and only if R is a finite direct sum of local reflexive rings.*

Proof. (1) By [8, Example 2.8].

(2) Let R be the direct product of reflexive rings R_i $(i \in I)$. Suppose that $(a_i)R(b_i) = 0$ for $(a_i), (b_i) \in R$. Then $a_iR_ib_i = 0$ for each $i \in I$. Since R_i is reflexive, $b_iR_ia_i = 0$ for each $i \in I$, entailing $(b_i)R(a_i) = 0$. Let $e = (e_i) \in R$ such that $e_i = 1$ and $e_j = 0$ for all $j \neq i$. Then $e^2 = e$ and $eRe \cong R_i$. If R is reflexive, then R_i is also reflexive by [8, Theorem 2.7(1)]. The proof for the direct sum case is similar.

(3) The necessity is obtained by the same method as Lemma 1(3). Conversely assume that R is a finite direct sum of local reflexive rings. Then R is semiperfect since local rings are semiperfect, and moreover R is reflexive by the result (2). □

Note that $\left\{\left(\begin{smallmatrix} a & b \\ 0 & a^2 \end{smallmatrix}\right) \mid a, b \in GF(2^2)\right\}$ is reversible (hence reflexive) by Example 2, and that $\mathrm{Mat}_2(\mathbb{Z}_2)$ is reflexive since it is semiprime. $|\ |$ denotes the cardinality.

Theorem 5. *If R is a minimal noncommutative reflexive ring, then R is a ring of order 16 such that R is isomorphic to the ring of all matrices of the form $\left(\begin{smallmatrix} a & b \\ 0 & a^2 \end{smallmatrix}\right)$ over $GF(2^2)$ when R is Abelian, and $\mathrm{Mat}_2(\mathbb{Z}_2)$ when R is non-Abelian.*

Proof. For the proof of the case of Abelian, we apply one of [5, Theorem 2.6]. Eldridge proved that if a finite ring has a cube free factorization, then it is commutative, in [3, Theorem]; and that if a ring A is of order p^3, p a prime, then A is isomorphic to $U_2(GF(p))$, in [3, Proposition]. Thus every minimal noncommutative ring is isomorphic to $U_2(\mathbb{Z}_2)$. But $U_2(\mathbb{Z}_2)$ is not reflexive by [8, Example 2.8], and so a minimal noncommutative reflexive ring is of order 16, considering the reflexive rings $\left\{\left(\begin{smallmatrix} a & b \\ 0 & a^2 \end{smallmatrix}\right) \mid a, b \in GF(2^2)\right\}$ and $\mathrm{Mat}_2(\mathbb{Z}_2)$.

First let R be a minimal Abelian noncommutative reflexive ring. Then R is a local ring by Lemma 4(3) since R is minimal. Here we have three cases of $|J(R)| = 2$, $|J(R)| = 4$ or $|J(R)| = 8$, where $|J(R)|$ means the cardinality of $J(R)$. Note that $R/J(R)$ is a field and $J(R)$ is a vector space over $R/J(R)$. The case of $|J(R)| = 2$ is impossible by the proof of [5, Theorem 2.6]. Suppose $|J(R)| = 8$. Then $R/J(R) \cong GF(2)$ and by [7, Theorem 2.3.6], $J(R)$ has a basis $\{a, b, c\}$ over $GF(2)$ with $a^2 = ab = c$ and $ba = 0$. Then since every sum-factor of each element in bRa contains ba in its product expression, bRa must be zero. But $0 \neq ab \in aRb$ and so R is not reflexive, a contradiction. Consequently we have the only case of $|J(R)| = 4$, for R to be a minimal Abelian noncommutative reflexive ring. In this case R is isomorphic to $\left\{\left(\begin{smallmatrix} a & b \\ 0 & a^2 \end{smallmatrix}\right) \mid a, b \in GF(2^2)\right\}$ by the proof of [5, Theorem 2.6].

Secondly let R be a minimal non-Abelian (hence noncommutative) reflexive ring. Then R is not local since local rings are Abelian.

By the Wedderburn-Artin theorem, $R/J(R)$ is isomorphic to $\sum_{i=1}^{n} \mathrm{Mat}_{k_i}(D_i)$ for some k_i's and finite fields D_i's. In fact, $D_i = \mathbb{Z}_2$ for all i since R is minimal. Assume that $k_i = 1$ for all i. Here we have there cases of $|J(R)| = 2$, $|J(R)| = 4$, and $|J(R)| = 8$.

If $|J(R)| = 8$, then $R/J(R) \cong \mathbb{Z}_2$ and so R is local, a contradiction.

Let $|J(R)| = 2$. Then $R/J(R) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$, and since $J(R)$ is nilpotent and $|J(R)| = 2$ we have $J(R)^2 = 0$. We can obtain orthogonal nonzero idempotents e_1, e_2, e_3 with $e_1 + e_2 + e_3 = 1$ by [10, Proposition 3.7.2], and we have

$$R = \{x + y \mid x \in I, y \in J(R)\}$$

where $I = \{0, 1, e_1, e_2, e_3, 1-e_1, 1-e_2, 1-e_3\}$. Note that idempotents commute each other. Since R is non-Abelian, there exists $e \in \{e_1, e_2, e_3, 1-e_1, 1-e_2, 1-e_3\}$ and $r \in R$ such that $er - re \neq 0$ (otherwise, R is commutative). Consider the subset $e(er - re)Re$. Note that $er - re \in J(R)$, $e(er - re)e = 0$, and every element in $e(er - re)Re$ is of the form $e(er - re)(f + j)e$, where $f \in I$ and

$j \in J(R)$. But since $E = \{e_1, e_2, e_3\}$ is orthogonal and $J(R)^2 = 0$, we get

$$e(er - re)(f + j)e = e(er - re)je = 0 \text{ when } f \in E \text{ with } f \neq e$$

and

$$e(er - re)(f + j)e = e(er - re)ee + e(er - re)je = e(er - re)e = 0 \text{ when } f = e.$$

These entail $e(er - re)Re = 0$. Now since R is reflexive, $eRe(er - re) = 0$ and this implies $er - ere = 0$. Next consider the subset $eR(er - re)e$. Every element in $eR(er - re)e$ is of the form $e(f + j)(er - re)e$, where $f \in I$ and $j \in J(R)$. Similarly $e(f + j)(er - re)e = 0$ and this yields $eR(er - re)e = 0$. Also since R is reflexive, $(er - re)eRe = 0$ and this implies $re - ere = 0$. Consequently we get $er = ere = re$, a contradiction.

Let $|J(R)| = 4$. Then $R/J(R) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Since $|J(R)| = 4$ and $J(R)$ is nilpotent, we have one of the following three cases with the help of [7, Theorems 2.3.2 and 2.3.3] and their proofs:

(1) There exists a basis $\{a, b\}$ for an additive abelian group $J(R)$ such that $\operatorname{char} a = \operatorname{char} b = 2$ and $a^2 = b^2 = ab = ba = 0$ (equivalently, $J(R)^2 = 0$);

(2) There exists a basis $\{a, b\}$ for an additive abelian group $J(R)$ such that $a^2 = b, a^3 = 0$ and $J(R) = \langle a \rangle$, where $\langle a \rangle$ is the subring of $J(R)$ generated by a;

(3) There exists an element $a \in J(R)$ such that $J(R) = \{0, a, 2a, 3a\}$ and $\operatorname{char} a = 4$.

In this case of $|J(R)| = 4$, we also obtain nonzero idempotents e_1 and e_2, such that $e_1 + J(R) = (1, 0) + J(R)$ and $e_2 + J(R) = (0, 1) + J(R)$, with the help of [10, Proposition 3.6.2]. Note $e_1e_2, e_2e_1 \in J(R)$. Further, we have

$$R = \{x + y \mid x \in J, y \in J(R)\}$$

where $J = \{0, 1, e_1, e_2\}$. Since R is non-Abelian, there exists $e \in \{e_1, e_2\}$ and $r \in R$ such that $er - re \neq 0$ (if e_1, e_2 are both central, then we can find another such set $\{e_1, e_2\}$ in which e_1 or e_2 is non-central). Note that $er - re \in J(R)$, $e(er - re)e = 0$, and every element in $e(er - re)Re$ is of the form $e(er - re)(f + j)e$, where $f \in J$ and $j \in J(R)$.

Consider the subset $eJ(R)e$. Then we have three cases of $|eJ(R)e| = 0$, $|eJ(R)e| = 2$ or $|eJ(R)e| = 4$ (i.e., $eJ(R)e = J(R)$).

If $|eJ(R)e| = 0$, then $e(er - re)Re = 0$ from $er - re \in J(R)$.

If $|eJ(R)e| = 4$ (i.e., $eJ(R)e = J(R)$), then $er - re = e(er - re)e = 0$ from $er - re \in J(R)$ and so $e(er - re)Re = 0$.

Let $|eJ(R)e| = 2$. Then $(eJ(R)e)^2 = 0$ clearly. Consider the cases of (2) and (3). Then the elements in $J(R)$ commute and so for every $e(er - re)(f + j)e \in e(er - re)Re$, we get

$$\begin{aligned} e(er - re)(f + j)e &= e(er - re)fe + e(er - re)je \\ &= e(er - re)fe + eje(er - re)e \\ &= e(er - re)fe = efe(er - re)e = 0 \end{aligned}$$

since $(er - re) \in J(R)$ (for, $e(er - re)fe = 0$ when $f \in \{0, 1, e\}$, and $fe \in J(R)$ when $f \in \{e_1, e_2\}$ and $f \neq e$.) This yields $e(er - re)Re = 0$. Consider the case of (1). Then we have $J(R)^2 = 0$ and so $e(er - re)(f + j)e = e(er - re)fe = 0$ by the same reason as before, entailing $e(er - re)Re = 0$.

Summarizing, we get $e(er - re)Re = 0$ in any case of (1), (2) and (3). Since R is reflexive, $eRe(er - re) = 0$ and so we have $er = ere$. These yield $re = ere = er$, a contradiction to $er - re \neq 0$.

Therefore $k_i \geq 2$ for some i, say $i = 1$. But R is of order 16 and so we must have $R/J(R) \cong \mathrm{Mat}_2(\mathbb{Z}_2)$, entailing $J(R) = 0$. This concludes $R \cong \mathrm{Mat}_2(\mathbb{Z}_2)$. □

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Byung-Ok Kim
Department of Mathematics
Korea Science Academy
Pusan 614-103, Korea
E-mail address: bokim26@hanmail.net

Yang Lee
Department of Mathematics
Pusan National University
Pusan 609-735, Korea
E-mail address: ylee@pusan.ac.kr