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ON HYPONORMALITY OF TOEPLITZ OPERATORS WITH POLYNOMIAL AND SYMMETRIC TYPE SYMBOLS

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Abstract. In [6], it was shown that hyponormality for Toeplitz operators with polynomial symbols can be reduced to classical Schur's algorithm in function theory. In [6], Zhu has also given the explicit values of the Schur's functions Φ_0, Φ_1 and Φ_2. Here we explicitly evaluate the Schur's function Φ_3. Using this value we find necessary and sufficient conditions under which the Toeplitz operator T_φ is hyponormal, where φ is a trigonometric polynomial given by $\varphi(z) = \sum_{n=-N}^{N} a_n z_n \, (N \geq 4)$ and satisfies the condition $\bar{a}_N \begin{pmatrix} a_{-1} \\ a_{-2} \\ a_{-4} \\ \vdots \\ a_{-N} \end{pmatrix} = a_{-N} \begin{pmatrix} \bar{a}_1 \\ \bar{a}_2 \\ \bar{a}_4 \\ \vdots \\ \bar{a}_N \end{pmatrix}$. Finally we illustrate the easy applicability of the derived results with a few examples.

1. Introduction

A bounded linear operator T on a Hilbert space is said to be hyponormal if its self commutator $[T^*, T] := T^*T - TT^*$ is positive semi definite. Given $\varphi \in L^\infty(\mathbb{T})$, the Toeplitz operator with symbol φ is the operator T_φ on the Hardy space $H^2(\mathbb{T})$ of the unit circle $\mathbb{T} = \partial\mathbb{D}$ defined by $T_\varphi f := P(\varphi.f)$, where $f \in H^2(\mathbb{T})$ and P denotes the orthogonal projection that maps $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$.

We are interested in the following question: "if φ is a trigonometric polynomial, then when is the trigonometric Toeplitz operator T_φ hyponormal?" If φ is a trigonometric polynomial of the form $\varphi(z) = \sum_{n=-m}^{N} a_n z^n$ where a_{-m} and a_N are non zero, then it was shown in [4] and [2] that hyponormality of T_φ implies $m \leq N$ and $|a_{-m}| \leq |a_N|$. In [2] it was shown that for $\varphi(z) = \sum_{n=-m}^{N} a_n z^n$, if $|a_{-m}| = |a_N| \neq 0$, then T_φ is hyponormal if and only

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if the coefficients of φ satisfy the following 'symmetry' condition:

$$\bar{a}_N \begin{pmatrix} a_{-1} \\ a_{-2} \\ \vdots \\ a_{-m} \end{pmatrix} = a_{-m} \begin{pmatrix} \bar{a}_{N-m+1} \\ \bar{a}_{N-m+2} \\ \vdots \\ \bar{a}_N \end{pmatrix}.$$

But if $|a_{-m}| \neq |a_N|$, then the case of arbitrary polynomial φ, though solved in principle by Cowen's Theorem [1] or Zhu's Theorem [6], is in practice very complicated. In [5], the hyponormality of T_φ was studied when $\varphi(z) = \sum_{n=-N}^{N} a_n z^n$ satisfies a 'partial' symmetry condition:

$$(1) \qquad \bar{a}_N \begin{pmatrix} a_{-m} \\ a_{-(m+1)} \\ \vdots \\ a_{-N} \end{pmatrix} = a_{-N} \begin{pmatrix} \bar{a}_m \\ \bar{a}_{m+1} \\ \vdots \\ \bar{a}_N \end{pmatrix} \quad \text{for} \quad m-1 \leq \tfrac{N}{2}.$$

The authors gave in the paper a necessary and sufficient condition for the hyponormality of T_φ when φ satisfies (1) and has the property $|\bar{a}_N a_{-(m-1)} - a_{-N}\bar{a}_{m-1}| = |a_N|^2 - |a_{-N}|^2$. Further, in [3] we find a complete criterion for hyponormality of T_φ when φ satisfies (1).

In this paper, we consider the trigonometric polynomial $\varphi(z) = \sum_{n=-N}^{N} a_n z^n$ with $|a_{-N}| < |a_N|$ such that

$$\bar{a}_N \begin{pmatrix} a_{-1} \\ a_{-2} \\ a_{-4} \\ \vdots \\ a_{-N} \end{pmatrix} = a_{-N} \begin{pmatrix} \bar{a}_1 \\ \bar{a}_2 \\ \bar{a}_4 \\ \vdots \\ \bar{a}_N \end{pmatrix} \text{ and } \bar{a}_N a_{-3} \neq a_{-N}\bar{a}_3,$$

and we give necessary and sufficient conditions for hyponormality of T_φ.

2. Evaluation of Schur's function Φ_3

In [6], Kehe Zhu applied Schur's algorithm to the Schur's function Φ_N in order to determine hyponormality of Toeplitz operators with polynomial symbols. We begin the section with the brief description of Zhu's idea.

Suppose that $f(z) = \sum_{j=0}^{\infty} c_j z^j$ is in the closed unit ball of $H^\infty(\mathbb{T})$. If $f_0 = f$, define by induction a sequence $\{f_n\}$ of functions in the closed unit ball of $H^\infty(\mathbb{T})$ as follows:

$$f_{n+1}(z) = \frac{f_n(z) - f_n(0)}{z(1 - \overline{f_n(0)} f_n(z))}, \quad |z| < 1, \quad n = 0, 1, 2, \ldots.$$

We write $f_n(0) = \Phi_n(c_0, \ldots, c_n)$, $n = 0, 1, 2, \ldots$, where Φ_n is a function of $n+1$ complex variables. We call the Φ_n's *Schur's functions.* Zhu's theorem can now be written as follows:

Theorem 2.1 ([6]). *If $\varphi(z) = \sum_{n=-N}^{N} a_n z^n$, where $a_N \neq 0$ and if*

$$\begin{pmatrix} \bar{c_0} \\ \bar{c_1} \\ \vdots \\ \bar{c}_{N-1} \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & \cdots & a_{N-1} & a_N \\ a_2 & a_3 & \cdots & a_N & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_N & 0 & \cdots & 0 & 0 \end{pmatrix}^{-1} \begin{pmatrix} \bar{a}_{-1} \\ \bar{a}_{-2} \\ \vdots \\ \bar{a}_{-N} \end{pmatrix},$$

then T_φ is hyponormal if and only if $|\Phi_n(c_0, \dots, c_n)| \leq 1$ for each $n = 0, 1, \dots,$ $N-1$.

Note that each $\Phi_n(c_0, \dots, c_n)$ is a rational function of the form

$$\Phi_n(c_0, \dots, c_n) = \frac{F_n(c_0, \dots, c_n)}{G_n(c_0, \dots, c_n)},$$

where F_n and G_n are polynomials. Thus the inequalities $|\Phi_n(c_0, \dots, c_n)| \leq 1$ should be understood as $|F_n(c_0, \dots, c_n)| \leq |G_n(c_0, \dots, c_n)|$.

In [6] Zhu also listed the first three Schur's functions as follows:

$$\begin{aligned} \Phi_0(c_0) &= c_0, \\ \Phi_1(c_0, c_1) &= \frac{c_1}{1 - |c_0|^2}, \\ \Phi_2(c_0, c_1, c_2) &= \frac{c_2(1 - |c_0|^2) + \bar{c_0} {c_1}^2}{(1 - |c_0|^2)^2 - |c_1|^2}. \end{aligned}$$

Here in this section we evaluate $\Phi_3(c_0, c_1, c_2, c_3)$ using the Schur's algorithm, and we get,

$$\begin{aligned} &\Phi_3(c_0, c_1, c_2, c_3) \\ =&\frac{((1 - |c_0|^2)^2 - |c_1|^2)((1 - |c_0|^2)c_3 + \bar{c_0} c_1 c_2) + (c_2(1 - |c_0|^2) + \bar{c_0} {c_1}^2)(\bar{c_0}(1 - |c_0|^2)c_1 + \bar{c_1} c_2)}{((1 - |c_0|^2)^2 - |c_1|^2)^2 - |c_2(1 - |c_0|^2) + \bar{c_0} {c_1}^2|^2}. \end{aligned}$$

We briefly outline the steps leading to this result. For the function $f(z) = \sum_{j=0}^{\infty} c_j z^j$ in the closed unit ball of $H^\infty(\mathbb{T})$:

Step(1): $f_0(z) = f(z)$ and so $\Phi_0(c_0) = f_0(0) = f(0) = c_0$.

Step(2): $f_1(z) = \frac{f_0(z) - f_0(0)}{z(1 - \overline{f_0(0)} f_0(z))} = \frac{\sum_{k=1}^{\infty} c_k z^{k-1}}{1 - |c_0|^2 - \bar{c_0} \sum_{k=1}^{\infty} c_k z^k}$ and so $\Phi_1(c_0, c_1) = f_1(0) = \frac{c_1}{1 - |c_0|^2}$.

Step(3): $f_2(z) = \frac{f_1(z) - f_1(0)}{z(1 - \overline{f_1(0)} f_1(z))} = \frac{N}{zD}$, where

$$\begin{aligned} N &= \frac{(1 - |c_0|^2) \sum_{k=2}^{\infty} c_k z^{k-1} + \bar{c_0} c_1 \sum_{k=1}^{\infty} c_k z^k}{(1 - \bar{c_0} \sum_{k=0}^{\infty} c_k z^k)(1 - |c_0|^2)}, \\ D &= \frac{((1 - |c_0|^2)^2 - |c_1|^2) - \bar{c_0}(1 - |c_0|^2) \sum_{k=1}^{\infty} c_k z^k - \bar{c_1} \sum_{k=2}^{\infty} c_k z^{k-1}}{(1 - \bar{c_0} \sum_{k=0}^{\infty} c_k z^k)(1 - |c_0|^2)}. \end{aligned}$$

Therefore,

$$f_2(z) = \frac{(1 - |c_0|^2) \sum_{k=2}^{\infty} c_k z^{k-2} + \bar{c_0} c_1 \sum_{k=1}^{\infty} c_k z^{k-1}}{((1 - |c_0|^2)^2 - |c_1|^2) - \bar{c_0}(1 - |c_0|^2) \sum_{k=1}^{\infty} c_k z^k - \bar{c_1} \sum_{k=2}^{\infty} c_k z^{k-1}}$$

and

$$\Phi_2(c_0, c_1, c_2) = f_2(0) = \frac{c_2(1-|c_0|^2) + \bar{c_0}{c_1}^2}{(1-|c_0|^2)^2 - |c_1|^2}.$$

Step(4):

$$f_3(z) = \frac{f_2(z) - f_2(0)}{z(1 - \overline{f_2(0)} f_2(z))} = \frac{N}{D},$$

where

$$N = ((1-|c_0|^2)^2 - |c_1|^2)\left((1-|c_0|^2)\sum_{k=2}^{\infty} c_k z^{k-2} + \bar{c_0}c_1 \sum_{k=1}^{\infty} c_k z^{k-1}\right)$$
$$-(c_2(1-|c_0|^2) + \bar{c_0}{c_1}^2)\left(((1-|c_0|^2)^2 - |c_1|^2) - \bar{c_0}(1-|c_0|^2)\sum_{k=1}^{\infty} c_k z^k - \bar{c_1}\sum_{k=2}^{\infty} c_k z^{k-1}\right)$$

and

$$D = ((1-|c_0|^2)^2 - |c_1|^2)\left(((1-|c_0|^2)^2 - |c_1|^2) - \bar{c_0}(1-|c_0|^2)\sum_{k=1}^{\infty} c_k z^k - \bar{c_1}\sum_{k=2}^{\infty} c_k z^{k-1}\right)$$
$$-(\bar{c_2}(1-|c_0|^2) + c_0\bar{c}_1^2)\left((1-|c_0|^2)\sum_{k=2}^{\infty} c_k z^{k-2} + \bar{c_0}c_1 \sum_{k=1}^{\infty} c_k z^{k-1}\right).$$

Thus,

$$\Phi_3(c_0, c_1, c_2, c_3)$$
$$= f_3(0)$$
$$= \frac{((1-|c_0|^2)^2 - |c_1|^2)\,((1-|c_0|^2)c_3 + \bar{c_0}c_1c_2) + (c_2(1-|c_0|^2) + \bar{c_0}{c_1}^2)(\bar{c_0}(1-|c_0|^2)c_1 + \bar{c_1}c_2)}{((1-|c_0|^2)^2 - |c_1|^2)^2 - |c_2(1-|c_0|^2) + \bar{c_0}{c_1}^2|^2}.$$

3. Hyponormality condition

Theorem 3.1. *Suppose $\varphi(z) = \sum_{n=-N}^{N} a_n z^n$ is such that*

$$\bar{a}_N \begin{pmatrix} a_{-1} \\ a_{-2} \\ a_{-4} \\ \vdots \\ a_{-N} \end{pmatrix} = a_{-N} \begin{pmatrix} \bar{a}_1 \\ \bar{a}_2 \\ \bar{a}_4 \\ \vdots \\ \bar{a}_N \end{pmatrix} \text{ and } \alpha := \frac{\det\begin{pmatrix} a_{-3} & a_{-N} \\ \bar{a}_3 & \bar{a}_N \end{pmatrix}}{|a_N|^2 - |a_{-N}|^2}.$$

(1) *If $N = 4$, then T_φ is hyponormal if and only if*
   (a) $|\alpha| \le 1$,
   (b) $|\bar{\alpha}a_{-4} - a_3| \le |a_4|\left(\frac{1}{|\alpha|} - |\alpha|\right)$,
   (c) $|(1-|\alpha|^2)(\bar{a}_3^2 - \bar{a}_2\bar{a}_4 - \bar{a}_3\bar{a}_{-4}\alpha) + \alpha(\bar{a}_{-4}\alpha - \bar{a}_3)(\bar{a}_{-4} - \bar{a}_3\bar{\alpha})|$
   $\le |\alpha|\left(|a_4|^2\left(\frac{1}{|\alpha|} - |\alpha|\right)^2 - |a_{-4}\bar{\alpha} - a_3|^2\right)$.

(2) *If $N \ge 5$, then T_φ is hyponormal if and only if*
   (a) $|\alpha| \le 1$,
   (b) $\left|\frac{a_{N-1}}{a_N}\right| \le \frac{1}{|\alpha|} - |\alpha|$,

(c) $$\left|\tfrac{1}{\alpha}\left(\tfrac{a_{N-1}}{a_N}\right)^2-\left(\tfrac{1}{\alpha}-\bar{\alpha}\right)\tfrac{a_{N-2}}{a_N}\right| \ \leq\ \left(\tfrac{1}{|\alpha|}-|\alpha|\right)^2-\left|\tfrac{a_{N-1}}{a_N}\right|^2.$$

Remark 3.2. Here we assume that $\det\left(\begin{smallmatrix} a_{-3} & a_{-N} \\ \bar{a}_3 & \bar{a}_N \end{smallmatrix}\right)$ and $|a_N|^2-|a_{-N}|^2$ are non-zero, because otherwise the results follow from [2] and [5]. Thus, $a_{-3}\bar{a}_N \neq a_{-N}\bar{a}_3$ and $|a_{-N}| < |a_N|$.

Proof. (1) When $N=4$.

Using Proposition 1 [5], c_0, c_1, c_2, c_3 are determined as follows:

(2) $$c_0=\frac{a_{-4}}{\bar{a}_4},$$

(3) $$c_1=(\bar{a}_4)^{-1}(a_{-3}-c_0\bar{a}_3),$$

(4) $$c_2=(\bar{a}_4)^{-1}(a_{-2}-c_0\bar{a}_2-c_1\bar{a}_3),$$

(5) $$c_3=(\bar{a}_4)^{-1}(a_{-1}-c_0\bar{a}_1-c_1\bar{a}_2-c_2\bar{a}_3).$$

Calculating and simplifying we get

(6) $$c_1=(\bar{a}_4)^{-2}\left(|a_4|^2-|a_{-4}|^2\right)\alpha,$$

(7) $$c_2=(\bar{a}_4)^{-1}\left(a_{-2}-\frac{a_{-4}}{\bar{a}_4}\bar{a}_2-c_1\bar{a}_3\right)=-\frac{c_1\bar{a}_3}{\bar{a}_4},$$

(8) $$c_3=\left(\frac{\bar{a}_3^2-\bar{a}_2\bar{a}_4}{\bar{a}_4^2}\right)c_1,$$

(9) $$1-|c_0|^2=1-\frac{|a_{-4}|^2}{|a_4|^2}=|a_4|^{-2}\left(|a_4|^2-|a_{-4}|^2\right),$$

(10) $$\frac{|c_1|}{1-|c_0|^2}=|\alpha|.$$

We also have,

(11) $$\Phi_0=c_0,$$

(12) $$\Phi_1=\frac{c_1}{1-|c_0|^2},$$

(13) $$\Phi_2=\frac{c_2(1-\mid c_0\mid^2)+\bar{c_0}{c_1}^2}{(1-|c_0|^2)^2-|c_1|^2},$$

(14)
$$\Phi_3=\frac{\left(\left(1-|c_0|^2\right)^2-|c_1|^2\right)\left(\left(1-|c_0|^2\right)c_3+\bar{c_0}c_1c_2\right)+\left(c_2\left(1-|c_0|^2\right)+\bar{c_0}{c_1}^2\right)\left(\bar{c_0}\left(1-|c_0|^2\right)c_1+\bar{c_1}c_2\right)}{\left((1-|c_0|^2)^2-|c_1|^2\right)^2-|c_2(1-|c_0|^2)+\bar{c_0}{c_1}^2|^2}.$$

Using expressions (6) to (10) in (11), (12), (13) and (14) and simplifying, we get

$|\Phi_0|\leq 1$ if and only if $|a_{-4}|\leq|a_4|$,
$|\Phi_1|\leq 1$ if and only if $|\alpha|\leq 1$,
$|\Phi_2|\leq 1$ if and only if $|\bar{a}_{-4}\alpha-\bar{a}_3|\leq|a_4|\left(\frac{1}{|\alpha|}-|\alpha|\right)$,

$|\Phi_3| \leq 1$ if and only if

$$\begin{aligned}&\left|(1-|\alpha|^2)(\bar{a}_3^2-\bar{a}_2\bar{a}_4-\bar{a}_3\bar{a}_{-4}\alpha)+\alpha(\bar{a}_{-4}\alpha-\bar{a}_3)(\bar{a}_{-4}-\bar{a}_3\bar{\alpha})\right|\\ &\leq |\alpha|\left(|a_4|^2\left(\frac{1}{|\alpha|}-|\alpha|\right)^2-|\bar{a}_{-4}\alpha-\bar{a}_3|^2\right).\end{aligned}$$

The result follows from Theorem 2.1.

(2) When $N > 4$.

Let $c_0, c_1, \ldots, c_{N-1}$ be determined by Proposition 1 [5]. Then,

$$\begin{aligned}c_0 &= \frac{a_{-N}}{\bar{a}_N},\\ c_1 &= (\bar{a}_N)^{-1}(a_{-(N-1)}-c_0\bar{a}_{N-1}) = (\bar{a}_N)^{-2}(\bar{a}_N a_{-(N-1)}-a_{-N}\bar{a}_{N-1}) = 0,\\ &\vdots\\ c_{N-4} &= (\bar{a}_N)^{-1}(a_{-4}-c_0\bar{a}_4-c_1\bar{a}_5-\ldots-c_{N-5}\bar{a}_{N-1}) = 0,\\ c_{N-3} &= (\bar{a}_N)^{-1}(a_{-3}-c_0\bar{a}_3-c_1\bar{a}_4-\ldots-c_{N-4}\bar{a}_{N-1})\\ &= (\bar{a}_N)^{-2}(a_{-3}\bar{a}_N-a_{-N}\bar{a}_3),\\ c_{N-2} &= (\bar{a}_N)^{-1}(a_{-2}-c_0\bar{a}_2-c_1\bar{a}_3-c_2\bar{a}_4-\ldots-c_{N-3}\bar{a}_{N-1}),\\ &= -\left(\frac{\bar{a}_{N-1}}{\bar{a}_N}\right)c_{N-3},\\ c_{N-1} &= (\bar{a}_N)^{-1}(a_{-1}-c_0\bar{a}_1-c_1\bar{a}_2-\ldots-c_{N-3}\bar{a}_{N-2}-c_{N-2}\bar{a}_{N-1})\\ &= (\bar{a}_N)^{-1}(-c_{N-1}\bar{a}_{N-2}+\frac{\bar{a}_{N-1}}{\bar{a}_N}c_{N-3}\bar{a}_{N-1})\\ &= \left(\left(\frac{\bar{a}_{N-1}}{\bar{a}_N}\right)^2-\frac{\bar{a}_{N-2}}{\bar{a}_N}\right)c_{N-3}.\end{aligned}$$

Thus, $b_p(z) = c_0 + c_{N-3}z^{N-3} + c_{N-2}z^{N-2} + c_{N-1}z^{N-1}$ is the unique analytic polynomial of degree less than N satisfying $\varphi - b_p\bar{\varphi} \in H^\infty$. By our assumption we have $c_{N-3} \neq 0$. Thus, by Proposition 3 [5],

$$\begin{aligned}\Phi_0 &= c_0,\\ \Phi_1 &= \Phi_2 = \ldots = \Phi_{N-4} = 0,\\ \Phi_{N-3} &= \frac{c_{N-3}}{1-|c_0|^2},\\ \Phi_{N-2} &= \frac{c_{N-2}}{(1-|c_0|^2)(1-|\Phi_{N-3}|^2)},\\ \Phi_{N-1} &= \frac{(1-|\Phi_{N-3}|^2)c_{N-1}c_{N-3}+|\Phi_{N-3}|^2c_{N-2}^2}{c_{N-3}(1-|c_0|^2)(1-|\Phi_{N-3}|^2)^2(1-|\Phi_{N-2}|^2)}.\end{aligned}$$

Computing and simplifying from these relations, we get,

$|\Phi_0| \le 1$ if and only if $|a_{-N}| \le |a_N|$, which is always true since we are taking $|a_{-N}| < |a_N|$,

$|\Phi_{N-3}| \le 1$ if and only if $|\alpha| \le 1$,

$|\Phi_{N-2}| \le 1$ if and only if $|\frac{a_{N-1}}{a_N}| \le \frac{1}{|\alpha|} - |\alpha|$,

$|\Phi_{N-1}| \le 1$ if and only if $\left| \frac{1}{\alpha} \left(\frac{a_{N-1}}{a_N} \right)^2 - \left(\frac{1}{\alpha} - \bar{\alpha} \right) \frac{a_{N-2}}{a_N} \right| \le \left(\frac{1}{|\alpha|} - |\alpha| \right)^2 - \left| \frac{a_{N-1}}{a_N} \right|^2$.

The result follows from Theorem 2.1. □

4. Examples

Example 1. For $\varphi(z) = z^{-4} + \lambda z^{-3} + 2z^{-1} + 4z^3 + 2z^6$ we want to determine the values of λ for which the Toeplitz operator T_φ is hyponormal.

By Proposition 2 [5], T_φ is hyponormal if and only if T_ψ is hyponormal where $\psi(z) = z^{-4} + \lambda z^{-3} + 2z^{-1} + 4z + 2z^4$. Expressing $\psi(z)$ as $\sum_{n=-N}^{N} a_n z^n$ and comparing with the given expression, we have, $N = 4$ and also $\bar{a}_4 \begin{pmatrix} a_{-1} \\ a_{-2} \\ a_{-4} \end{pmatrix} = a_{-4} \begin{pmatrix} \bar{a}_1 \\ \bar{a}_2 \\ \bar{a}_4 \end{pmatrix}$. Hence, we can apply Case 1 of Theorem 3.1 to determine the values of λ for which T_ψ will be hyponormal. It may be noted that if $\lambda = 0$, then T_ψ is hyponormal by Theorem 1.4 [2].

As $\alpha = \begin{vmatrix} a_{-3} & a_{-4} \\ \bar{a}_3 & \bar{a}_4 \end{vmatrix} / (|a_4|^2 - |a_{-4}|^2) = \frac{2\lambda}{3}$, thus refering to the notations used in Theorem 3.1 we have,

(i) $|\alpha| \le 1$ if and only if $|\lambda| \le \frac{3}{2} = 1.5$,

(ii) $|\bar{\alpha} a_{-4} - a_3| \le |a_4| \left(\frac{1}{|\alpha|} - |\alpha| \right)$ if and only if $|\lambda| \le \sqrt{\frac{3}{2}} = 1.22$, correct upto two decimal places,

(iii)

$$\begin{aligned} &|(1 - |\alpha|^2)(\bar{a}_3^2 - \bar{a}_2 \bar{a}_4 - \bar{a}_3 \bar{a}_{-4} \alpha) + \alpha(\bar{a}_{-4}\alpha - \bar{a}_3)(\bar{a}_{-4} - \bar{a}_3 \bar{\alpha})| \\ \le\ & |\alpha| \left(|a_4|^2 \left(\frac{1}{|\alpha|} - |\alpha| \right)^2 - |a_{-4}\bar{\alpha} - a_3|^2 \right) \\ &\text{if and only if } 12|\lambda|^4 - 6|\lambda|^3 - 72|\lambda|^2 + 81 \ge 0 \\ &\text{if and only if } |\lambda| \le 1.13. \end{aligned}$$

This can be seen from the graph of $12|\lambda|^4 - 6|\lambda|^3 - 72|\lambda|^2 + 81 = 0$ in Figure 1:

Thus, T_φ is hyponormal if and only if $|\lambda| \le 1.13$ (correct upto two decimal places).

Example 2. If $\varphi(z) = z^{-4} + \lambda z^{-3} + z^{-2} + 2z^{-1} + 1 + 4z + 2z^2 + 2z^4$, then using Theorem 3.1 (Case 1) as in Example 1, we get that the Toeplitz operator T_φ is hyponormal if and only if $|\lambda| \le 0.95$ (correct upto two decimal places).

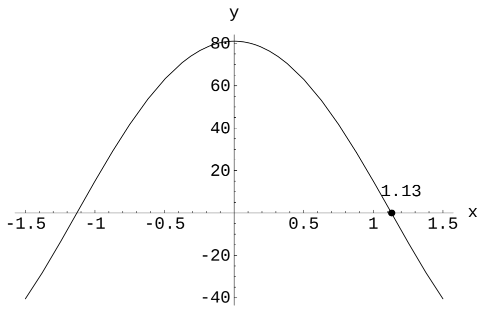

Figure 1

Example 3. If $\varphi(z) = 2z^{-5} + 2z^{-4} + \lambda z^{-3} + 4z^{-2} + 2z^{-1} + 3z + 6z^2 + 3z^3 + 3z^4 + 3z^5$, then using Theorem 3.1 (Case 2) we get that the Toeplitz operator T_φ is hyponormal if and only if $0.34 \leq \lambda \leq 0.58$ (correct upto two decimal places).

Example 4 . If $\varphi(z) = z^{-6} + 3z^{-5} + z^{-4} + 2z^{-2} + z^{-1} + 2z + 4z^2 + \lambda z^3 + 2z^4 + 6z^5 + 2z^6$, then using Theorem 3.1 (Case 2) we get that the Toeplitz operator T_φ is hyponormal if and only if $|\lambda| \leq 0.32$ (correct upto two decimal places).

5. Conclusion

If for $N \geq 4$ and $\varphi(z) = \sum_{n=-N}^{N} a_n z^n$ with $|a_N| > |a_{-N}|$, we have

$$\bar{a}_N \begin{pmatrix} a_{-1} \\ a_{-2} \\ a_{-4} \\ \vdots \\ a_{-N} \end{pmatrix} = a_{-N} \begin{pmatrix} \bar{a}_1 \\ \bar{a}_2 \\ \bar{a}_4 \\ \vdots \\ \bar{a}_N \end{pmatrix} \text{ and } \bar{a}_N a_{-3} \neq a_{-N} \bar{a}_3.$$

Then for $N \geq 6$ we can also apply Theorem 1 [3]. Particularly, for very large N, application of Theorem 1 [3] would significantly reduce the work load in determining the hyponormality of T_φ. However, this theorem do not give a ready set of conditions to finally determine hyponormality. In view of this, Theorem 3.1 offers an alternative method to determine hyponormality of T_φ under the given restrictions on φ.

For $N \geq 3$, if

$$\bar{a}_N \begin{pmatrix} a_{-1} \\ a_{-3} \\ \vdots \\ a_{-N} \end{pmatrix} = a_{-N} \begin{pmatrix} \bar{a}_1 \\ \bar{a}_3 \\ \vdots \\ \bar{a}_N \end{pmatrix} \text{ and } \bar{a}_N a_{-2} \neq a_{-N} \bar{a}_2,$$

then Theorem 8 [5] gives a similar alternative method to determine hyponormality of T_φ.

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