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ALGORITHMIC SOLUTION FOR $M/M/c$ RETRIAL QUEUE WITH PH_2-RETRIAL TIMES†

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ABSTRACT. We present an algorithmic solution for the stationary distribution of the $M/M/c$ retrial queue in which the retrial times of each customer in orbit are of phase type distribution of order 2. The system is modeled by the level dependent quasi-birth-and-death (LDQBD) process.

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1. Introduction

Consider the retrial queue which consists of an orbit with infinite capacity and a service facility that has c servers and no waiting space. When an arriving customer finds that all servers are busy, the customer joins a virtual pool of blocked customers called orbit and repeats its request after a random amount of time, called retrial time until the customer gets into the service facility. Retrial queues have been used to model problems in many application areas including telephone, computer and communication systems.

The phase type distributions (PH-distribution) can be used to approximate general distribution and to fit the observed data. The algorithmic solutions for retrial queues with PH service times are extensively exploited [7]. However, the literature about the explicit or algorithmic solution for the retrial queues with PH retrial time is very limited but there some approximations [5, 10, 13] and system properties such as stability condition [8] and stochastic order relation [10]. For comprehensive survey of the main results and literature, see [3, 6].

In this paper, we consider an $M/M/c$ retrial queue with PH retrial time. The main difficulty of analyzing the system with PH retrial times is due to the fact that the number of phases for describing the behavior of the customers in

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orbit becomes extremely large as the number of customers in orbit increases and computational feasibility is quickly trapped into the curse of dimension. For example, the models in [1, 2] use the phase type distribution of order m^n as a remaining retrial time in the system with PH retrial time of order m denoted by PH_m when there are n customers in orbit. The $M/M/c$ retrial queue with PH_2 retrial time can be modeled by the level dependent quasi-birth-and-death (LDQBD) process whose size of block matrix corresponding to the level n is of $(c+1)(n+1)$ by considering the number of customers in each phase is counted instead of considering the phases of all customers. So many approximations are compared with the exact solutions of the system PH_2 retrial time and Gauss-Seidel iterative method is suggested for computing the exact ones e.g. [5, 13]. Besides on the computational feasibility, PH_2 distribution can be used to match the first three moments of a distribution of nonnegative random variables [12].

The objectives of this paper is to present an effective algorithm for computing the stationary distribution of the number of customers in the system for $M/M/c$ retrial queue with PH_2 retrial time.

In section 2, the mathematical model is described in detail. The algorithmic solutions are proposed in section 3. Numerical results and conclusions are presented in section 4 and 5, respectively.

2. Model description

Consider an $M/M/c$ retrial queue which consists of infinite-capacity orbit and the service facility with c identical servers and no waiting positions. Service times of customers are independent of each other and have a common exponential distribution with parameter μ. Customers arrive according to a Poisson process with rate λ. The customer who finds that all servers are busy upon its arrival joins orbit and tries to its luck again after a random amount of time until the customer gets into the service facility. The access of each customer from orbit to the service facility is governed by the phase type distribution $PH(\boldsymbol{\alpha}, \boldsymbol{\Gamma})$ of order 2, where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$ $(\alpha_1 + \alpha_2 = 1)$ and $\boldsymbol{\Gamma}$ is a 2×2 matrix with

$$\boldsymbol{\Gamma} = \begin{pmatrix} -(\gamma_{12}+\gamma_1) & \gamma_{12} \\ \gamma_{21} & -(\gamma_{21}+\gamma_2) \end{pmatrix}, \quad \boldsymbol{\Gamma}^0 = -\boldsymbol{\Gamma}\mathbf{e} = \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}.$$

Let $N(t)$ be the number of customers in orbit, $C(t)$ the number of busy servers and $N_1(t)$ the number of customers in phase 1 at time t. Then the stochastic process $\boldsymbol{X} = \{(N(t), C(t), N_1(t)), t \geq 0\}$ is a Markov chain with state space $\mathcal{S} = \{(n, i, j), n \geq 0, 0 \leq i \leq c, 0 \leq j \leq n\}$. Denote the set $\boldsymbol{n} = \{(n, i, j), 0 \leq i \leq c, 0 \leq j \leq n\}$ by level $\boldsymbol{n}$ and let $\boldsymbol{k}_n = \{(n, k, j), 0 \leq j \leq n\}$. Writing the states lexicographic order, the generator of $\boldsymbol{X}$ is given by

$$Q = \begin{pmatrix} Q_1^{(0)} & Q_0^{(0)} & & & \\ Q_2^{(1)} & Q_1^{(1)} & Q_0^{(1)} & & \\ & Q_2^{(2)} & Q_1^{(2)} & Q_0^{(2)} & \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

The $Q_0^{(n)}$ is of $(n+1)(c+1) \times (n+2)(c+1)$ matrix of the form

$$Q_0^{(n)} = \begin{array}{c} \\ \mathbf{0}_n \\ \mathbf{1}_n \\ \vdots \\ \mathbf{c}_n \end{array} \begin{array}{c} \begin{array}{cccc} \mathbf{0}_{n+1} & \mathbf{1}_{n+1} & \cdots & \mathbf{c}_{n+1} \end{array} \\ \begin{pmatrix} & & & \\ & O & & \\ & & & \\ & & & \lambda A^{(n)} \end{pmatrix} \end{array}, \tag{1}$$

where $A^{(n)}$ is the $(n+1) \times (n+2)$ matrix with (i, j) component $(0 \le i \le n, 0 \le j \le n+1)$

$$[A^{(n)}]_{ij} = \begin{cases} \alpha_2, & j = i \\ \alpha_1, & j = i+1 \\ 0, & \text{otherwise} \end{cases}$$

The $Q_1^{(n)}$ is of $(n+1)(c+1) \times (n+1)(c+1)$ matrix of the form

$$Q_1^{(n)} = \begin{pmatrix} B_0^{(n)} & \lambda I_{n+1} & & & \\ \mu I_{n+1} & B_1^{(n)} & \lambda I_{n+1} & & \\ & \ddots & \ddots & \ddots & \\ & & (c-1)\mu I_{n+1} & B_{c-1}^{(n)} & \lambda I_{n+1} \\ & & & c\mu I_{n+1} & B_c^{(n)} \end{pmatrix},$$

where I_{n+1} is the identity matrix of order $n+1$ and $B_k^{(n)}$ is the tridiagonal matrix of size $(n+1) \times (n+1)$. The (i, j) component of $B_k^{(n)}$ is as follows: for $0 \le k \le c-1$, $0 \le i, j \le n$

$$[B_k^{(n)}]_{ij} = \begin{cases} -(\lambda + k\mu + i(\gamma_1 + \gamma_{12}) + (n-i)(\gamma_2 + \gamma_{21})), & j = i \\ i\gamma_{12}, & j = i-1 \\ (n-i)\gamma_{21}, & j = i+1 \\ 0, & \text{otherwise} \end{cases}$$

and

$$[B_c^{(n)}]_{ij} = \begin{cases} -(\lambda + c\mu + i(\gamma_{12} + \gamma_1\alpha_2) + (n-i)(\gamma_{21} + \gamma_2\alpha_1)), & j = i \\ i(\gamma_{12} + \gamma_1\alpha_2), & j = i-1 \\ (n-i)(\gamma_{21} + \gamma_2\alpha_1), & j = i+1 \\ 0, & \text{otherwise} \end{cases}$$

The $Q_2^{(n)}$ is of $(n+1)(c+1) \times n(c+1)$ matrix of the form

$$Q_2^{(n)} = \begin{array}{c} \\ \mathbf{0}_n \\ \mathbf{1}_n \\ \vdots \\ (\mathbf{c-1})_n \\ \mathbf{c}_n \end{array} \begin{array}{c} \begin{array}{ccccc} \mathbf{0}_{n-1} & \mathbf{1}_{n-1} & \mathbf{2}_{n-1} & \cdots & \mathbf{c}_{n-1} \end{array} \\ \begin{pmatrix} & C^{(n)} & & & \\ & & C^{(n)} & & \\ & & & \ddots & \\ & & & & C^{(n)} \\ & & & & \end{pmatrix} \end{array}, \tag{2}$$

where $C^{(n)}$ is the $(n+1)\times n$ matrix with (i,j) component $(0\le i\le n, 0\le j\le n-1)$

$$[C^{(n)}]_{ij}=\begin{cases}(n-i)\gamma_2, & j=i\\ i\gamma_1, & j=i-1\\ 0, & \text{otherwise}\end{cases}$$

3. Algorithmic Solution

We assume that $\rho=\frac{\lambda}{c\mu}<1$ which guarantees the existence of the stationary distribution of $\boldsymbol{X}$ [8] and let $x_{nij}=P(N=n, C=i, J=j)$, where N, C and J are the stationary versions of $N(t)$, $C(t)$ and $J(t)$, respectively. Write the stationary distribution of Q in the block partitioned form $\boldsymbol{x}=(\boldsymbol{x}_0,\boldsymbol{x}_1,\boldsymbol{x}_2,\cdots)$ with $\boldsymbol{x}_n=(x_{nij}, 0\le i\le c, 0\le j\le n)$, $n\ge 0$. In order to calculate the stationary distribution $\boldsymbol{x}$, it is necessary to have the family of the matrices $\{R_k, k\ge 0\}$ which are the minimal nonnegative solutions to the systems of equations

$$Q_0^{(k-1)}+R^{(k)}Q_1^{(k)}+R^{(k)}(R^{(k+1)}Q_2^{(k+1)})=0,\ \ k\ge 1. \tag{3}$$

It follows from the special structure of the matrix $Q_0^{(k-1)}$ that

$$R^{(k)}=Q_0^{(k-1)}(-Q_1^{(k)}-R^{(k+1)}Q_2^{(k+1)})^{-1} \tag{4}$$

has the following formula

$$R^{(k)}=\begin{pmatrix} & O & & \\ R_0^{(k)} & R_1^{(k)} & \cdots & R_c^{(k)}\end{pmatrix},\ \ k\ge 1, \tag{5}$$

where $R_j^{(k)}$ is the $k\times(k+1)$ matrix and O is the $kc\times(k+1)c$ matrix. Thus it follows from Bright and Taylor [4] that the stationary distribution $\boldsymbol{x}=(\boldsymbol{x}_i, i\ge 0)$ is given by

$$\begin{aligned}\boldsymbol{x}_n &= \boldsymbol{x}_0\left(\prod_{k=1}^{n}R^{(k)}\right)\\ &= x_{0c0}\left(\prod_{k=1}^{n-1}R_c^{(k)}\right)[R_0^{(n)}\ R_1^{(n)}\ \cdots\ R_c^{(n)}],\ \ n\ge 1,\end{aligned} \tag{6}$$

and $\boldsymbol{x}_0$ is the unique solution of the equation

$$\boldsymbol{x}_0(Q_1^{(0)}+R^{(1)}Q_2^{(1)})=0 \tag{7}$$

with the normalizing condition

$$\boldsymbol{x}_0\mathbf{e}_{c+1}+x_{0c0}\left(\sum_{n=1}^{\infty}\left(\prod_{k=1}^{n-1}R_c^{(k)}\right)[R_0^{(n)}\ R_1^{(n)}\ \cdots\ R_c^{(n)}]\mathbf{e}_{(n+1)(c+1)}\right)=1, \tag{8}$$

where $\mathbf{e}_n$ is the $n \times 1$ matrix whose components are all 1. Once $\boldsymbol{x}$ is obtained, the distribution of N and C are calculated by the formula

$$P(N = n) = \sum_{k=0}^{c} \sum_{j=0}^{n} x_{nkj},$$
$$P(C = k) = \sum_{n=0}^{\infty} \sum_{j=0}^{n} x_{nkj}.$$

The formula $R^{(k)}$ in (4) is given by the unknown term $R^{(k+1)}$. So we have to truncate the matrix Q by a finite level, say K. Once a truncation level K is determined, the original system Q is approximated by one that has finite orbit size K and new arrivals are lost when the system is full, that is, $N(t) = K$ and $C(t) = c$. The generator corresponding to the truncated system is given by

$$Q_K = \begin{pmatrix} Q_1^{(0)} & Q_0^{(0)} & & & \\ Q_2^{(1)} & Q_1^{(1)} & Q_0^{(1)} & & \\ & \ddots & \ddots & \ddots & \\ & & Q_2^{(K-1)} & Q_1^{(K-1)} & Q_0^{(K-1)} \\ & & & Q_2^{(K)} & \tilde{Q}_1^{(K)} \end{pmatrix}, \tag{9}$$

where the block (c, c)-matrix component of $\tilde{Q}_1^{(K)}$ is $B_c^{(n)} + \lambda I_{K+1}$ and other components are the same as those of $Q_1^{(K)}$. Once a truncation level K is determined, approximating $R^{(K)}$ by

$$\tilde{R}^{(K)} = Q_0^{(K-1)}(-\tilde{Q}_1^{(K)})^{-1} \tag{10}$$

and the $R^{(n)}$, $n = K-1, K-2, \cdots, 1$ are determined by the formula (4). The stationary distribution $\boldsymbol{x}^K = (\boldsymbol{x}_n^K, n = 0, 1, \cdots, K)$ of $\hat{Q}_K$ is given by (6) and (7) with (7).

The matrices $\tilde{R}^{(n)}$ can be effectively obtained using the following well known results for the inverse of a partitioned matrix (e.g. see [9]) and the method for computing the fundamental matrix of the transient quasi-birth-and death process in Shin [11].

Lemma 3.1. *Let a square matrix A be partitioned as*

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where A_{11} and A_{22} are also square matrices. Then A^{-1} is given by

$$A^{-1} = \begin{pmatrix} (A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1} & A_{11}^{-1}A_{12}(A_{21}A_{11}^{-1}A_{12} - A_{22})^{-1} \\ (A_{21}A_{11}^{-1}A_{12} - A_{22})^{-1}A_{21}A_{11}^{-1} & -(A_{21}A_{11}^{-1}A_{12} - A_{22})^{-1} \end{pmatrix},$$

provided that all the relevant inverse exists.

The following lemma is immediate from [11, Proposition 2.1].

Lemma 3.2. *Let $Q_{11}^{(n)}$ be $(k+1)c \times (k+1)c$ matrix consisting of the north-left c blocks of $Q_1^{(n)}$ denote the inverse $(-Q_{11}^{(n)})^{-1}$ by the block partitioned form $(-Q_{11}^{(n)})^{-1} = (X_{ij}^{(n)})_{0\le i,j\le c-1}$, where $X_{ij}^{(n)}$ is the $(n+1)\times(n+1)$ matrix. Let*

$$S_{c-1} = [-B_{c-1}^{(n)}]^{-1},$$
$$S_i = [-(B_i^{(n)} + (i+1)\mu\lambda S_{i+1})]^{-1},\ i = c-2, c-3, \cdots, 0.$$

Then $X_{ij}^{(n)}$ are given as follows:

(1) The first row and column blocks : $X_{00}^{(n)} = S_0$ and

$$X_{0j}^{(n)} = \lambda X_{0,j-1}^{(n)} S_j,\ X_{j0}^{(n)} = i\mu S_j X_{j-1,0}^{(n)},\ j = 1, \cdots, c-1$$

(2) The i-th $(1 \le i \le c-2)$ row and column blocks : $X_{ii}^{(n)} = (I + \lambda X_{i,i-1}^{(n)})S_i$ and

$$X_{ij}^{(n)} = \lambda X_{i,j-1}^{(n)} S_j,\ X_{ji}^{(n)} = j\mu S_j X_{j-1,i}^{(n)},\ j = i+1, i+2, \cdots, c-1.$$

(3) The $(c-1, c-1)$-block component :

$$X_{c-1,c-1}^{(n)} = (I + \lambda X_{c-1,c-2}^{(n)})S_{c-1},$$

Proposition 3.3. *Write the $\tilde{Q}_1^{(n)}$ in the block partitioned form*

$$\tilde{Q}_1^{(n)} = \begin{pmatrix} Q_{11}^{(n)} & Q_{12}^{(n)} \\ Q_{21}^{(n)} & \tilde{Q}_{22}^{(n)} \end{pmatrix},$$

where $\tilde{Q}_{22}^{(n)} = B_c^{(n)} + \lambda I_{n+1}$ and let

$$\tilde{R}^{(n)} = Q_0^{(n-1)}(-\tilde{Q}_1^{(n)})^{-1} = \begin{pmatrix} & & O & \\ \tilde{R}_0^{(n)} & \tilde{R}_1^{(n)} & \cdots & \tilde{R}_c^{(n)} \end{pmatrix},$$

where $\tilde{R}_j^{(n)}$ is the $n \times (n+1)$ matrix. Then

$$\tilde{R}_j^{(n)} = \begin{cases} \lambda c\mu A^{(n-1)}\tilde{D}^{(n)} X_{c-1,j}^{(n)}, & j = 0, 1, \cdots, c-1 \\ \lambda A^{(n-1)}\tilde{D}^{(n)}, & j = c, \end{cases} \tag{11}$$

where

$$\tilde{D}^{(n)} = [-(B_c^{(n)} + \lambda I_{n+1}) - \lambda c\mu X_{c-1,c-1}^{(n)}]^{-1}.$$

Proposition 3.4. *Writing $\hat{Q}^{(n)} = Q_1^{(n)} + R^{(n+1)}Q_2^{(n+1)}$ in the block partitioned form*

$$\hat{Q}^{(n)} = \begin{pmatrix} Q_{11}^{(n)} & Q_{12}^{(n)} \\ \hat{Q}_{21}^{(n)} & \hat{Q}_{22}^{(n)} \end{pmatrix},$$

where $\hat{Q}_{22}^{(n)} = B_c^{(n)} + R_{c-1}^{(n+1)}C^{(n+1)}$ is square matrix of size $n+1$. Then

$$R_j^{(n)} = \lambda A^{(n-1)}D^{(n)}\left[\sum_{i=0}^{c-2} R_i^{(n+1)}C^{(n+1)}X_{i+1,j}^{(n)} + c\mu X_{c-1,j}^{(n)}\right],\ 0 \le j \le c-1 \tag{12}$$
$$R_c^{(n)} = \lambda A^{(n-1)}D^{(n)}, \tag{13}$$

where

$$D^{(n)} = -\left[B_c^{(n)} + R_{c-1}^{(n+1)}C^{(n+1)} + \lambda\sum_{j=0}^{c-2} R_j^{(n+1)}C^{(n+1)}X_{j+1,c-1}^{(n)} + \lambda c\mu X_{c-1,c-1}^{(n)}\right]^{-1}$$

Proof. It follows from (4) and (3.1) that

$$[R_0^{(k)}, R_1^{(k)}, \cdots, R_{c-1}^{(k)}] = \lambda A^{(k-1)}D^{(n)}\hat{Q}_{21}^{(n)}(Q_{11}^{(n)})^{-1}$$
$$R_c^{(k)} = \lambda A^{(k-1)}D^{(n)},$$

where

$$D^{(n)} = [(-\hat{Q}_{22}^{(n)}) - \hat{Q}_{21}^{(n)}(-Q_{11}^{(n)})^{-1}Q_{12}^{(n)}]^{-1}.$$

Noting

$$\begin{aligned}\hat{Q}_{21}^{(n)} &= \left[0, R_0^{(n+1)}C^{(n+1)}, \cdots, R_{c-3}^{(n+1)}C^{(n+1)}, R_{c-2}^{(n+1)}C^{(n+1)} + c\mu I\right],\\ \hat{Q}_{22}^{(n)} &= B_c^{(n)} + R_{c-1}^{(n+1)}C^{(n+1)}\end{aligned}$$

and

$$(-Q_{11}^{(n)})^{-1}Q_{12}^{(n)} = \lambda\begin{pmatrix} X_{0,c-1}^{(n)} \\ X_{1,c-1}^{(n)} \\ \vdots \\ X_{c-1,c-1}^{(n)}\end{pmatrix},$$

the formulae (12) and (13) are immediately obtained. □

Now we propose a criteria for selecting a truncation level K. Let N^K and C^K be the number of customers in orbit and service facility in stationary state, respectively. Enlarge K so that the distributions $\boldsymbol{y}^K = (y_j^K, j = 0, 1, \cdots)$ with $y_j^K = P(N^K = j)$ and $\boldsymbol{z}^K$ of C^K do not vary significantly as K increases. That is, for given tolerance $\epsilon > 0$, choose K such that

$$TOL = \max(||\boldsymbol{y}^K - \boldsymbol{y}^{K-1}||_\infty, ||\boldsymbol{z}^K - \boldsymbol{z}^{K-1}||_\infty) < \epsilon. \quad (14)$$

As the level K varies until K satisfies (14), $R^{(n)}$, $n = K, K-1, \cdots, 1$ should be computed repeatedly for computing $\boldsymbol{x}^K$. The algorithm needs to compute many inverses of matrices. To avoid the repeating computation, it is necessary to choose an appropriate initial truncation level K_0. Many numerical experiments show that the truncation level mainly depend on the traffic intensity ρ and the first moment m_1 of retrial time. So it is recommended to determine the initial level K_0 by the system with exponential retrial time with tolerance ϵ_0 larger than ϵ. The following algorithm summarizes the results above.

Algorithm

1. Determine the initial truncation level K_0 and let $K = K_0$.
   Set $\boldsymbol{x}^{\text{OLD}} = 0$.
2. Compute $R^{(K)}$ using (11).
3. Compute $R^{(n)}$, $n = K-1, \cdots, 1$ using (12) and (13).
4. Compute $\boldsymbol{x}^K$ using (6). Set $\boldsymbol{x}^{\text{NEW}} = \boldsymbol{x}^K$.

5. Check the stopping criterion.
   IF $(TOL < \epsilon)$ **THEN** STOP;
   ELSE $K := K + 1$, $\boldsymbol{x}^{\text{OLD}} := \boldsymbol{x}^{\text{NEW}}$ and GO TO 2. **ENDIF**;

4. Numerical results

We consider $M/M/5$ retrial queue with arrival rate $\mu = 1.0$. Let T be the retrial time and $m_i = \mathbb{E}[T^i]$, $i = 1, 2, 3$ and the squared coefficient of variation $C_T^2 = \frac{m_2 - m_1^2}{m_1^2}$ of T. The main objectives of numerical experiments are to investigate the feasibility of the algorithm. It follows from Little's formula that the mean number of busy servers $\mathbb{E}[C] = \frac{\lambda}{\mu}$ is independent of the distribution of retrial time and the numerical results for $\mathbb{E}[C]$ are omitted in the tables below.

Numerical results with truncation level K for the system with Erlang distribution E_2 of order 2 and hyperexponential distribution H_2 of retrial time of order 2 with squared coefficient of variation $C_T^2 = 2.0$ are presented in Table 1. The truncation level K is chosen by the stopping criteria $TOL < \epsilon = 10^{-6}$. The hyperexponential distribution H_2 of order 2 has the probability density function of the form $f(t) = p\gamma_1 e^{-\gamma_1 t} + (1-p)\gamma_2 e^{-\mu_2 t},\ t \geq 0$. The parameters used in Table 1 are as follows $p = \frac{1}{2}\left(1 + \sqrt{(C_T^2 - 1)/(C_T^2 + 1)}\right)$, $\gamma_1 = \frac{2p}{m_1}$ and $\gamma_2 = \frac{2(1-p)}{m_1}$. Table 1 shows that the truncation level mainly depends on the traffic intensity ρ and the first moment m_1 of retrial time.

TABLE 1. $M/M/5$ retrial queue with PH_2 retrial time

Retrial time			E_2			$H_2(C_T^2 = 2.0)$			$H_2(C_T^2 = 10.0)$	
ρ	m_1	K	P_B	$\mathbb{E}[N]$	K	P_B	$\mathbb{E}[N]$	K	P_B	$\mathbb{E}[N]$
0.3	0.1	9	0.0191	0.0108	9	0.0188	0.0121	9	0.0187	0.0150
	1.0	9	0.0163	0.0300	9	0.0166	0.0352	10	0.0169	0.0399
	5.0	10	0.0152	0.1179	10	0.0155	0.1275	11	0.0156	0.1327
	10.0	11	0.0150	0.2305	11	0.0152	0.2416	12	0.0153	0.2468
	20.0	13	0.0150	0.4574	12	0.0151	0.4695	13	0.0151	0.4747
0.5	0.1	17	0.1218	0.1561	17	0.1196	0.1691	17	0.1190	0.2018
	1.0	18	0.1003	0.3644	18	0.1016	0.4221	18	0.1045	0.4895
	5.0	23	0.0910	1.301	22	0.0928	1.415	22	0.0941	1.492
	10.0	31	0.0897	2.502	26	0.0909	2.634	25	0.0917	2.713
	20.0	34	0.0891	4.923	32	0.0898	5.069	31	0.0903	5.148
0.8	0.1	56	0.5248	2.519	56	0.5192	2.619	54	0.5171	2.873
	1.0	62	0.4545	5.003	62	0.4572	5.466	63	0.4646	6.204
	5.0	90	0.4245	16.15	90	0.4276	17.00	88	0.4315	17.99
	10.0	120	0.4198	30.27	119	0.4217	31.22	115	0.4241	32.27
	20.0	167	0.4174	58.61	166	0.4185	59.62	160	0.4198	60.70

5. Conclusions

One of the main difficulties for computing the stationary distribution of LDQBD process is to calculate the rate matrices $R^{(n)}$. The number of matrices to be calculated their inverses $(-\tilde{Q}_1^{(n)})^{-1}$ and $(-\hat{Q}_1^{(n)})^{-1}$ for $R^{(n)}$, $n = 1, 2, \cdots, K$ is the same as the truncation level K. As the traffic intensity ρ and mean retrial time m_1 increase, the truncation level K should be enlarged rapidly for satisfactory accuracy. Main contribution of this paper is to present an effective algorithm for $(-\tilde{Q}_1^{(n)})^{-1}$ and $(-\hat{Q}_1^{(n)})^{-1}$ that computes the $c+1$ inverses of the matrices of size $n+1$ instead of the matrix of size $(n+1)(c+1)$ for computing $R^{(n)}$. Thus the algorithm presented in this paper can be applied to systems with high traffic intensity and large mean retrial time.

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