EXISTENCE OF POSITIVE SOLUTIONS FOR A CLASS OF QUASILINEAR ELLIPTIC SYSTEM WITH CONCAVE-CONVEX NONLINEARITIES†

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Abstract. In this paper, our main purpose is to establish the existence of weak solutions of a class of \( p-q \)-Laplacian system involving concave-convex nonlinearities:

\[
\begin{align*}
-\Delta_p u - \Delta_q u &= \lambda V(x)|u|^{r-2}u + \frac{2\alpha}{\alpha + \beta}|u|^{\alpha-2}u|v|^\beta, \quad x \in \Omega \\
-\Delta_p v - \Delta_q v &= \theta V(x)|v|^{r-2}v + \frac{2\beta}{\alpha + \beta}|u|^\alpha|v|^{\beta-2}v, \quad x \in \Omega \\
\end{align*}
\]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^N \), \( \lambda, \theta > 0 \), and \( 1 < \alpha, \beta, \alpha + \beta = p^* = \frac{Np}{N-p} \) is the critical Sobolev exponent, \( \Delta_s u = \text{div}(|\nabla u|^{s-2}\nabla u) \) is the \( s \)-Laplacian of \( u \). when \( 1 < r < q < p < p^* \), we prove that there exist infinitely many weak solutions. We also obtain some results for the case \( 1 < q < p < r < p^* \). The existence results of solutions are obtained by variational methods.

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1. Introduction

In this paper, we are interested in finding multiple nontrivial weak solutions to the following nonlinear elliptic system of \( p-q \)-Laplacian type with concave-convex nonlinearities

\[
\begin{align*}
-\Delta_p u - \Delta_q u &= \lambda V(x)|u|^{r-2}u + \frac{2\alpha}{\alpha + \beta}|u|^{\alpha-2}u|v|^\beta, \quad x \in \Omega \\
-\Delta_p v - \Delta_q v &= \theta V(x)|v|^{r-2}v + \frac{2\beta}{\alpha + \beta}|u|^\alpha|v|^{\beta-2}v, \quad x \in \Omega \\
\end{align*}
\]

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where $\Omega$ is a bounded domain in $\mathbb{R}^N$, $\lambda, \theta > 0$, and $1 < r < q < p < N$, $1 < \alpha, \beta$, $\alpha + \beta = p^* = \frac{Np}{N-p}$ is the critical Sobolev exponent, $\triangle_s u = \text{div}(|\nabla u|^{s-2}\nabla u)$ is the $s$-Laplacian of $u$.

When $u = v$, $\alpha = \beta$ and $\lambda = \theta$, System (1.1) reduce to the $p$-$q$-Laplacian equations:

$$
\begin{cases}
-\Delta_p u - \Delta_q u = \lambda V(x)|u|^{r-2}u + |u|^{p^*-2}u, & x \in \Omega \\
u = 0, & x \in \partial \Omega
\end{cases}
$$

(1.2)

Problem (1.2) comes, for example, from a general reaction-diffusion system

$$
u_t = \text{div}[H(u)\nabla u] + c(x, u)$$

(1.3)

where $H(u) = |\nabla u|^{p-2} + |\nabla u|^{q-2}$. This system has a wide range of applications in physics and related science such as biophysics, plasma physics and chemical reaction design. Typically, in chemical and biological applications, the reaction term $c(x, u)$ has a polynomial form with respect to the concentration $u$.

Recently, the stationary solution of (1.3) was studied by many authors, that is many works considered the solutions of the following problem

$$
-\text{div}[H(u)\nabla u] = c(x, u).
$$

(1.4)

for example, see [6.19-21,26].

If $p = q = 2$, (1.2) can be reduced to

$$
\begin{cases}
-\Delta u = \lambda V(x)|u|^{r-2}u + |u|^{2^*-2}u, & x \in \Omega \\
u = 0, & x \in \partial \Omega
\end{cases}
$$

(1.5)

which is a normal Schrodinger equation and has been widely studied, see [10-12,23].

The solutions of problem (1.5) corresponds to the critical points of the energy functional

$$I(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \frac{\lambda}{r} \int_{\Omega} V(x)|u|^r dx - \frac{1}{2} \int_{\Omega} |u|^{2^*} dx$$

defined on $W^{1,2}_0(\Omega)$. When $r = 2$, the pioneer result of Brezis-Nirenberg [8] studied problem (1.5) and shows that under some suitable conditions, problem (1.5) possesses a positive solution in $W^{1,2}_0(\Omega)$. For more results see [9,17] and reference therein.

The typically difficulty in dealing with problem (1.5) is that the corresponding functional $I(u)$ doesn’t satisfy (PS) condition due to the lack of compactness of the embedding: $H^1_0 \hookrightarrow L^{2^*}(\Omega)$. Hence we couldn’t use the standard variational methods.

However, if $1 < r < 2$, the situation is quite different, see [5,25]. The main essence is that when $1 < r < 2$, the functional $I(u)$ is sublinear, when $\lambda$ is small enough, $I(u)$ satisfies (PS)$_c$ condition for $c < 0$, so we can look for critical points of negative critical values of $I(u)$. 
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For the general $p$-Laplacian problem

\[
\begin{aligned}
-\Delta_p u &= \lambda V(x)|u|^{q-2}u + |u|^{p^*-2}u, \quad x \in \Omega \\
u &= 0, \quad x \in \partial \Omega
\end{aligned}
\]  

(1.6)

which is a special case of (1.2) when $p = q$. Problem (1.6) was also studied by many authors, many results valid for problem (1.5) has been extended to problem (1.6). For example, see [4,18,27]. The main difficulty in extending the results for problem (1.5) to the corresponding results for problem (1.6) is that $W^{1,p}_0(\Omega)$ is not a Hilbert space in general, then more analysis is needed.

We recall some results about problem (1.4) now. In [26], M.Wu and Z.Yang proved the existence of a nontrivial solution to problem (1.4) with

\[c(x,u) = a(x)|u|^{p-2}u + b(x)|u|^{q-2}u - f(x,u)\]

in the whole space $\mathbb{R}^N$, where $a(x), b(x)$ are positive functions, also when $a(x) \equiv m, b(x) \equiv n$ are positive constants, it was proved in [19] that problem (1.4) has a nontrivial solution. Recently in [20], G.Li and G.Zhang studied problem (1.4) involving critical exponent with

\[c(x,u) = |u|^{p^*-2}u + \theta |u|^{r-2}u\]

by using Lusternik-Schnirelman’s theory (see also in [4]). Other results see [6,21] and reference therein.

At the same time, much attention has been paid to the existence of solutions for elliptic systems, especially for the following case

\[
\begin{align*}
-\Delta_p u &= \lambda |u|^{q-2}u + \frac{2\alpha}{\alpha + \beta} |u|^{\alpha-2}u|v|^{\beta}, \quad x \in \Omega \\
-\Delta_p v &= \theta |v|^{q-2}v + \frac{2\beta}{\alpha + \beta} |u|^{\alpha}|v|^{\beta-2}v, \quad x \in \Omega \\
u &= v = 0 & x \in \partial \Omega
\end{align*}
\]  

(1.7)

where $\alpha + \beta = p^*$. In fact system (1.7) is a special case of (1.1) when $p = q$. When $p = 2$ and $q = 2$, Alves et al [2] considered (1.7) and proved the existence of least energy solutions for any $\lambda, \theta \in (0, \lambda_1)$ and generalized the corresponding results of [8] to the case of system (1.7), here $\lambda_1$ denote the first eigenvalue of operator $-\Delta$. Subsequently, Han [14] considered the existence of multiple positive solutions for (1.7) and in [16] T.S.Hsu studied system (1.7) when $1 < q < p < N, \alpha + \beta = p^*$, more results see [15,24] etc..

However, as far as we know, there are few results on problem (1.1) with concave-convex nonlinearities. Motivated by [4,16,20], we shall extend the results of the above to problem (1.1). Let us denote the Banach space $H = W^{1,p}_0(\Omega) \times W^{1,p}_0(\Omega)$ in this paper, and for the functions $V(x)$, we add the following assumptions:

$(V_0)$ Suppose $V(x) \in L^{\frac{p^*}{p^*-r}}(\Omega)$ and $V(x) > \sigma > 0$ in $\Omega$.

Then we have the following results:
Theorem 1.1. Assume $1 < r < q < p < N$, and $(V_0)$ hold. Then there is a positive constant $\Lambda^*$ such that for any $0 < (\lambda + \theta) \leq \Lambda^*$, problem (1.1) possesses infinitely many weak solutions in $H$.

In the present paper, we also consider problem (1.1) for the case: $1 < q < p \leq r < p^*$, and obtain the following theorem:

Theorem 1.2. If $1 < q < p \leq r < p^*$ and $(V_0)$ hold, then there is a $\Lambda^* > 0$, such that for any $(\lambda + \theta) > \Lambda^*$, problem (1.1) has a nontrivial solution.

Remark 1.3. In [4], J.G.Azvoro and I.F.Aloson obtained that there exist a nontrivial solution for (1.6) with $V(x) \equiv 1$ by the Mountain Pass Lemma. In fact, Theorem 1.2 is an extension of Theorem 3.2 in [4] to $p$-q-Laplacian system (1.1).

The present paper is organized as follows, in section 2, we give some preliminary results; in section 3, we will prove the main result, Theorem 1.1.; and we will study (1.1) for the case $1 < q < p \leq r < p^*$, and prove Theorem 1.2 in section 4.

2. Preliminaries results

Let $H'$ be dual of $H$, $(\cdot, \cdot)$ the duality paring between $H'$ and $H$, the norm on $H$ is given by

$$\|z\|_p = \| (u, v) \|_p = (\|u\|^p_p + \|v\|^p_p)^{\frac{1}{p}}$$

and the norm on $L^p(\Omega) \times L^p(\Omega)$ is given by

$$|z|_p = |(u, v)|_p = (\|u\|^p_p + \|v\|^p_p)^{\frac{1}{p}}$$

where $z = (u, v) \in H$ and $\| \cdot \|_p$, $| \cdot |_p$ are the norm on $W^{1,p}_0(\Omega)$ and $L^p(\Omega)$ respectively, that is,

$$\|u\|_p = (\int_\Omega |\nabla u|^p dx)^{\frac{1}{2}}, \quad |u|_p = (\int_{\Omega} |u|^p dx)^{\frac{1}{2}}.$$

Throughout this paper, we denote weak converge by $\rightharpoonup$, and denote strong converge by $\to$, also we denote positive constants(possibly different) by $C_i$.

As usually, we also denote by

$$S_{\alpha + \beta} = \inf_{u \in W^{1,p}(\Omega) \setminus \{0\}} \frac{\|u\|^p_p}{(\int_{\Omega} |u|^{\alpha + \beta} dx)^{\frac{\alpha}{\alpha + \beta}}} \quad (2.1)$$

and

$$S_{\alpha, \beta} = \inf_{z \in H \setminus \{0\}} \frac{|z|_p^{p}}{\left( \int_{\Omega} |z|^{\alpha + \beta} dx \right)^{\frac{\alpha}{\alpha + \beta}}} \quad (2.2)$$

Easily, we have $\int_{\Omega} |u|^\alpha |v|^\beta dx \leq S_{\alpha, \beta}^{- \frac{\alpha + \beta}{\beta}} |z|^{\alpha + \beta}$ and

Lemma 2.1. Assume $1 < \alpha, \theta$ and $\alpha + \beta \leq p^*$, $\Omega \in \mathbb{R}^N (N \geq 3)$ be a domain (not necessarily bounded). Then we have

$$S_{\alpha, \beta} = \left[ (\frac{\alpha}{\beta})^{\frac{\alpha}{\alpha + \beta}} + (\frac{\alpha}{\beta})^{\frac{\beta}{\alpha + \beta}} \right] S_{\alpha + \beta}.$$
The proof of Lemma 2.1. is essentially given in [2] when \( p = 2 \), modifying the proof of [2], we can deduce our result. For the readers’ convenience, we give a sketch here.

Suppose \( \{w_n\} \) is a minimizing sequence for \( S_{\alpha+\beta} \), let \( u_n = sw_n, v_n = tw_n \), where \( s, t > 0 \) will be chosen later. Then from (2.2), we infer that

\[
S_{\alpha,\beta} \leq \frac{s^p + tp}{(s^\alpha t^\beta)^{\frac{p}{\alpha+\beta}}} \left( \frac{s^\alpha}{\alpha} + \frac{t^\beta}{\beta} \right)^{\frac{p}{\alpha+\beta}} \frac{\|w_n\|^p}{(\int_\Omega |w_n|^{\alpha+\beta}dx)^{\frac{p}{\alpha+\beta}}} (2.3)
\]

Define the function

\[
h(x) = x^{\frac{p}{\alpha+\beta}} + x^{-\frac{p}{\alpha+\beta}}, \quad x > 0.
\]

By a direct calculation, the minimum of the function \( h \) is achieved at the point \( x_0 = \left( \frac{\alpha}{\beta} \right)^{\frac{1}{p}} \) with the minimum value

\[
h(x_0) = \left( \frac{\alpha}{\beta} \right)^{\frac{p}{\alpha+\beta}} + \left( \frac{\alpha}{\beta} \right)^{-\frac{p}{\alpha+\beta}}.
\]

Thus, choosing \( s, t > 0 \) in (2.3) such that \( \frac{s}{t} = \left( \frac{\alpha}{\beta} \right)^{\frac{1}{p}} \), we obtain

\[
S_{\alpha,\beta} \leq \left[ \left( \frac{\alpha}{\beta} \right)^{\frac{p}{\alpha+\beta}} + \left( \frac{\alpha}{\beta} \right)^{-\frac{p}{\alpha+\beta}} \right] S_{\alpha+\beta}.
\]

To complete the proof, let \( z_n = (u_n, v_n) \) be a minimizing sequence for \( S_{\alpha,\beta} \). Define \( \omega_n = t_n v_n \) for some \( t_n > 0 \) such that

\[
\int_\Omega |u_n|^{\alpha+\beta}dx = \int_\Omega |\omega_n|^{\alpha+\beta}dx.
\]

Then we have

\[
\int_\Omega |u_n|^\alpha |\omega_n|^\beta dx \leq \frac{\alpha}{\alpha + \beta} \int_\Omega |u_n|^{\alpha+\beta}dx + \frac{\beta}{\alpha + \beta} \int_\Omega |\omega_n|^{\alpha+\beta}dx
\]

\[
= \int_\Omega |u_n|^{\alpha+\beta}dx = \int_\Omega |\omega_n|^{\alpha+\beta}dx.
\]

Therefore, we deduce from the above inequality that

\[
\frac{\|z_n\|^p}{(\int_\Omega |u_n|^\alpha |\omega_n|^\beta dx)^{\frac{p}{\alpha+\beta}}} = \left( \frac{s^\alpha}{\alpha} + \frac{t^\beta}{\beta} \right)^{\frac{p}{\alpha+\beta}} \frac{\|z_n\|^p}{\left(\int_\Omega |u_n|^{\alpha+\beta}dx\right)^{\frac{p}{\alpha+\beta}}}
\]

\[
\geq \left( \frac{s^\alpha}{\alpha} + \frac{t^\beta}{\beta} \right)^{\frac{p}{\alpha+\beta}} \frac{\|u_n\|^p}{\left(\int_\Omega |u_n|^{\alpha+\beta}dx\right)^{\frac{p}{\alpha+\beta}}} + \left( \frac{s^\alpha}{\alpha} + \frac{t^\beta}{\beta} \right)^{\frac{p}{\alpha+\beta}} \frac{\|\omega_n\|^p}{\left(\int_\Omega |\omega_n|^{\alpha+\beta}dx\right)^{\frac{p}{\alpha+\beta}}}
\]

\[
\geq h(t_n) S_{\alpha+\beta} \geq h(t_0) S_{\alpha+\beta}.
\]

Passing to the limit in the above inequality, we obtain

\[
S_{\alpha,\beta} \geq \left[ \left( \frac{\alpha}{\beta} \right)^{\frac{p}{\alpha+\beta}} + \left( \frac{\alpha}{\beta} \right)^{-\frac{p}{\alpha+\beta}} \right] S_{\alpha+\beta}.
\]

That’s end the proof of Lemma 2.1. □
Let’s study the energy functional associated with problem (1.1) defined by
\[
E(z) = E(u, v) = \frac{1}{p} \int_{\Omega} |\nabla u|^p + |\nabla v|^p \, dx + \frac{1}{q} \int_{\Omega} |\nabla u|^q + |\nabla v|^q \, dx
- \frac{1}{r} \int_{\Omega} \lambda V(x)|u|^r + \theta V(x)|v|^r \, dx
- \frac{1}{\alpha + \beta} \int_{\Omega} |u|^\alpha |v|^\beta \, dx.
\]
Obviously, \(E(z)\) is even and it is well known that \(E(z) \in C^1(H, \mathbb{R})\) and nontrivial critical points of \(E(z)\) are weak solutions of problem (1.1). By a weak solution of (1.1) we mean that \((u, v) \in H\) satisfying
\[
\int_{\Omega} (|\nabla u|^{p-2} \nabla u \nabla \varphi + |\nabla v|^{p-2} \nabla v \nabla \psi) \, dx + \int_{\Omega} (|\nabla u|^{q-2} \nabla u \nabla \varphi + |\nabla v|^{q-2} \nabla v \nabla \psi) \, dx
- \lambda \int_{\Omega} V(x)|u|^r \varphi \, dx - \theta \int_{\Omega} V(x)|v|^r \psi \, dx
- \frac{2\alpha}{\alpha + \beta} \int_{\Omega} |u|^\alpha |v|^\beta \varphi \, dx - \frac{2\beta}{\alpha + \beta} \int_{\Omega} |u|^\alpha |v|^\beta \psi \, dx = 0.
\]
for all \((\varphi, \psi) \in E\).

Now, we define the Palais-Smale(PS)-sequence, (PS)-value, and (PS)-conditions in \(H\) for \(E\) as follows.

**Definition 2.2.** (I) For \(c \in \mathbb{R}\), a sequence \(\{z_n\} \subset H\) is a (PS)\(_c\)-sequence for \(E\) if \(E(z_n) = c + o(1)\) and \(E'(z_n) = o(1)\) strongly in \(H'\) as \(n \to \infty\).

(II) \(c \in \mathbb{R}\) is a (PS)-value for \(E\) if there exists a (PS)\(_c\)-sequence in \(H\) for \(E\).

(III) \(E\) satisfies the (PS)\(_c\)-condition in \(H\) for \(E\) if every (PS)\(_c\)-sequence in \(H\) for \(E\) contains a convergent sub-sequence.

Now we give some results for the proof of Theorem 1.1.

**Lemma 2.3.** If \(\{z_n\} \subset H\) is a (PS)\(_c\)-sequence for \(E\), then \(\{z_n\}\) is bounded in \(H\).

**Proof.** Modified the proof of Lemma 2.3 in [16], we can obtain the results. □

**Lemma 2.4.** If \(\{z_n\} \subset H\) is a (PS)\(_c\)-sequence for \(E\), then there exists \(z \in H\) and \(M > 0\) such that
\[
E(z) \geq -M(\lambda + \theta)^{\frac{q}{q-r}},
\]
where \(M\) will be given later.

**Proof.** Similar to the proof of Lemma 2.2 in [16]. □

**Lemma 2.5.** \(E\) satisfies the (PS)\(_c\)-condition with \(c\) satisfying
\[
c \leq \frac{2}{N} \left( \frac{S_{\alpha, \beta}}{2} \right)^{\frac{q}{r}} - M(\lambda + \theta)^{\frac{q}{q-r}}.
\]
Proof. Suppose \( \{z_n\} \subset H \) is a \((PS)_c\) sequence of \( E \), i.e.,
\[
E(z_n) = c + o(1), \quad E'(z_n) = o(1),
\]
by Lemma 2.3, we may assume there exist a \( z \in H \), \( E(z) = 0 \), and extracting a subsequence such that \( z_n \to z \) in \( H \), Thus we have that
\[
u_n \to u, \quad v_n \to v \text{ in } L^s(\Omega), \quad 1 \leq s < p^*.
\]
and \( u_n \to u, v_n \to v \text{ a.e. on } \Omega \). Hence we have
\[
\int_{\Omega} \lambda V(x)|u_n|^p + \theta V(x)|v_n|^r \, dx = \int_{\Omega} \lambda V(x)|u|^p + \theta V(x)|v|^r \, dx + o(1).
\]
Let \( \tilde{v}_n = u_n - u, \tilde{v}_n = v_n - v \) and \( \tilde{z}_n = (\tilde{u}_n, \tilde{v}_n) \). Then by Brezis-Lieb’s Lemma (see [7]), we deduce that
\[
\|\tilde{z}_n\|_p^p = \|z_n\|_p^p - \|z\|_p^p + o(1), \quad \|\tilde{z}_n\|_q^q = \|z_n\|_q^q - \|z\|_q^q + o(1).
\]
By an argument of Han [15, Lemma 2.1], we obtain
\[
\int_{\Omega} |\tilde{u}_n|^\alpha |\tilde{v}_n|^\beta \, dx = \int_{\Omega} |u_n|^\alpha |v_n|^\beta \, dx - \int_{\Omega} |u|^\alpha |v|^\beta \, dx + o(1). \tag{2.6}
\]
Together with (2.4)-(2.6), we have that
\[
\frac{1}{p} \|\tilde{z}_n\|_p^p + \frac{1}{q} \|\tilde{z}_n\|_q^q - \frac{1}{r} \int_{\Omega} (\lambda V(x)|u|^p + \theta V(x)|v|^r) \, dx - \frac{2}{p^r} \int_{\Omega} |\tilde{u}_n|^\alpha |\tilde{v}_n|^\beta \, dx \\
+ \frac{1}{p} \|z\|_p^p + \frac{1}{q} \|z\|_q^q - \frac{2}{p^r} \int_{\Omega} |u|^\alpha |v|^\beta \, dx = c + o(1).
\]
and
\[
\|\tilde{z}_n\|_p^p + \|\tilde{z}_n\|_q^q - \int_{\Omega} (\lambda V(x)|u|^p + \theta V(x)|v|^r) \, dx - 2 \int_{\Omega} |\tilde{u}_n|^\alpha |\tilde{v}_n|^\beta \, dx \\
+ \|z\|_p^p + \|z\|_q^q - 2 \int_{\Omega} |u|^\alpha |v|^\beta \, dx = o(1).
\]
Or we have
\[
\frac{1}{p} \|\tilde{z}_n\|_p^p + \frac{1}{q} \|\tilde{z}_n\|_q^q - \frac{2}{p^r} \int_{\Omega} |\tilde{u}_n|^\alpha |\tilde{v}_n|^\beta \, dx = c - E(z) + o(1). \tag{2.7}
\]
and
\[
\|\tilde{z}_n\|_p^p + \|\tilde{z}_n\|_q^q - \frac{2}{p^r} \int_{\Omega} |\tilde{u}_n|^\alpha |\tilde{v}_n|^\beta \, dx = o(1). \tag{2.8}
\]
Hence, we may suppose that
\[
\|\tilde{z}_n\|_p^p \to a, \quad \|\tilde{z}_n\|_q^q \to b, \quad 2 \int_{\Omega} |\tilde{u}_n|^\alpha |\tilde{v}_n|^\beta \, dx \to l,
\]
if \( a = 0 \), then we have \( z_n \to z \) in \( H \), we complete the proof. On the contrary, we assume \( a > 0 \), then from (2.2) and (2.8), we obtain
\[
a \leq l \leq 2(S_{\alpha,\beta})^{-\frac{\alpha}{p^*}} a^\frac{\beta}{p^*},
\]
which implies that \( a \geq 2(S_{\alpha,\beta})^{-\frac{\alpha}{p^*}} \).
On the other hand, from (2.7) we have that
\[ c = \frac{a}{p} + \frac{b}{q} = \frac{1}{p^r} + E(z) \]
\[ = \left( \frac{1}{p} - \frac{1}{p^r} \right) \alpha + \left( \frac{1}{q} - \frac{1}{p^r} \right) b + E(z) \]
\[ > \frac{2}{N} \left( \frac{\alpha}{2} \right)^P - M(\lambda + \theta)^{\frac{P}{P^r}} \]
which contradicts \( c \leq \frac{2}{N} \left( \frac{\alpha}{2} \right)^P - M(\lambda + \theta)^{\frac{P}{P^r}} \).

The following is the classical Deformation Lemma:

**Lemma 2.6 (see [1]).** Let \( f \in C^1(X, R) \) and satisfy (PS) condition. If \( c \in R \) and \( N \) is any neighborhood of \( K_c = \{ u \in X | f(u) = c, f'(u) = 0 \} \), there exists \( \eta(t, x) \equiv \eta_i(x) \in C([0, 1] \times X, X) \) and constants \( \tau > \epsilon > 0 \) such that

1. \( \eta_0(x) = x \) for all \( x \in X \),
2. \( \eta_i(x) = x \) for all \( x \eta_i^{-1}[c-\tau, c+\tau] \),
3. \( \eta_i(x) \) is a homeomorphism of \( X \) onto \( X \) for all \( t \in [0, 1] \),
4. \( f(\eta_i(x)) \leq f(x) \) for all \( x \in X, t \in [0, 1] \),
5. \( \eta_i(A_{c+\epsilon} - N) \subset A_{c+\epsilon}, \) where \( A_c = \{ x \in X | f(x) \leq c \} \) for any \( c \in R \),
6. if \( K_c = \emptyset, \eta_i(A_{c+\epsilon}) \subset A_{c-\epsilon}, \)
7. if \( f \) is even, \( \eta_i \) is odd in \( x \).

**Remark 2.7.** Lemma 2.6 is also true if \( f \) satisfies (PS)_c condition for \( c < c_0 \) for some \( c_0 \in R \).

At the end of this section, we recall some concepts in minimax theory. Let \( X \) be a Banach space, and

\[ \Sigma = \{ A \subset X \setminus \{ 0 \} | A \ is \ closed, \ -A = A \}, \]

and

\[ \Sigma_k = \{ A \in \Sigma | \gamma(A) \geq k \}, \]

where \( \gamma(A) \) is the \( Z_2 \) genus of \( A \), that is

\[ \gamma(A) = \begin{cases} \inf \{ n : \text{there exist odd, continuous } h : A \to R^n \setminus \{ 0 \} \}, & \text{if it doesn’t exist odd, continuous } h : A \to R^n \setminus \{ 0 \}, \forall n \in Z_+, \\ +\infty, & \text{if } A = 0. \end{cases} \]

The main properties of genus are contained in the following lemma.

**Lemma 2.8 (see [22]).** Let \( A, B \in \Sigma \). Then

1. If there exists \( f \in C(A, B) \), odd, then \( \gamma(A) \leq \gamma(B) \).
2. If \( A \subset B \), then \( \gamma(A) \leq \gamma(B) \).
3. If there exists an odd homeomorphism between \( A \) and \( B \), then \( \gamma(A) = \gamma(B) \).
4. If \( S^{N-1} \) is the sphere in \( R^N \), then \( \gamma(S^{N-1}) = N \).
5. \( \gamma(A \cup B) \leq \gamma(A) + \gamma(B) \).
6. If \( \gamma(A) < \infty \), then \( \gamma(A - B) \geq \gamma(A) - \gamma(B) \).
7. If \( A \) is compact, then \( \gamma(A) < \infty \), and there exists \( \delta > 0 \) such that \( \gamma(A) = \gamma(N_{\delta}(A)) \), where \( N_{\delta}(A) = \{ x \in X | d(x, A) \leq \delta \} \).
(8) If $X_0$ is a subspace of $X$ with codimension $k$, and $\gamma(A) > k$, then $A \cap X_0 \neq \emptyset$.

3. Proof of Theorem 1.1

We will prove the existence of infinitely many solutions for system (1.1) in this section. We try to use Lusternik-Schnirelman’s theory for $Z_2$-invariant functional (see [22]). But since the functional $E(z)$ defined in section 2 is not bounded from below, so we following [4](or see [20]) to consider a truncated functional $E_\infty(z)$ which will be constructed later.

At first, let’s consider the functional $E(z)$, using the Sobolev’s inequality with the hypothesis $1 < r < q < p < N$, we obtain

$$E(z) \geq \frac{1}{p} ||z||_p^p - \frac{1}{r} \int_\Omega \lambda V(x)|u|^r + \theta V(x)|u|^r dx - \frac{2}{p^*} \int_\Omega |u|^\alpha |v|^\beta dx$$

$$\geq \frac{1}{p} ||z||_p^p - \frac{2}{p^* S_{\alpha,\beta}} \|z\|_p^{p^*} - \frac{1}{r} S_p^{-\frac{\sigma}{r}} |V(x)|_{\frac{\sigma}{p^*}} (\lambda + \theta) \|z\|_r^r$$

$$= C_3 \|z\|_p^p - C_4 \|z\|_{p^*}^{p^*} - C_5 (\lambda + \theta) \|z\|_r^r$$

where $C_3 = \frac{1}{p}, C_4 = \frac{2}{p^* S_{\alpha,\beta}}$, $C_5 = \frac{1}{r} S_p^{-\frac{\sigma}{r}} |V(x)|_{\frac{\sigma}{p^*}}$ are all positive constants.

We now consider function

$$h(x) = C_3 x^p - C_4 x^{p^*} - C_5 (\lambda + \theta) x^r, \ x > 0$$

by the hypothesis $1 < r < p < p^*$, we easily know that there exists a $\Lambda^* > 0$ such that for any $0 < (\lambda + \theta) \leq \Lambda^*$, we have the following results hold:

(a) $h(x)$ reaches its positive maximum;

(b) $\frac{2}{M} (\frac{S_{\alpha,\beta}}{2})^\frac{\gamma}{r} - M (\lambda + \theta) \frac{\sigma}{r} \geq 0$, where $M$ is given in Lemma 2.4.

From the structure of $h(x)$, we see that there are two positive solutions $R_1 < R_2$ of $h(x) = 0$. Then we can easily know that

$$h(x) \begin{cases} < 0, & x \in (0, R_1) \cup (R_2, \infty) \\ > 0, & x \in (R_1, R_2) \end{cases}$$

We let $\tau : R^+ \to [0, 1]$ be $C^\infty$ and nonincreasing function such that

$$\tau(x) = 1, \text{ if } x \in (0, R_1)$$

$$\tau(x) = 0, \text{ if } x \in (R_2, \infty).$$

Let $\varphi(u) = \tau(\|u\|_p)$, we consider the truncated functional

$$E_\infty(z) = \frac{1}{p} ||z||_p^p + \frac{1}{q} ||z||_q^q - \frac{1}{r} \int_\Omega \lambda V(x)|u|^r + \theta V(x)|v|^r dx - \frac{2}{p^*} \int_\Omega |u|^\alpha |v|^\beta \varphi(u) dx.$$

similar as above, we consider the function

$$\tilde{h}(x) = C_3 x^p - C_4 x^{p^*} \tau(x) - C_5 (\lambda + \theta) x^r,$$
We prove (1) by contradiction, assume for any \( c < k \), we will show that when \( 0 \leq \lambda + \theta \leq \Lambda^* \), \( E_{\infty}(z) \) satisfies the \((PS)_c\) condition for \( c < 0 \).

Proof. We prove (1) by contradiction, assume \( E_{\infty}(z) < 0 \) and \( \|z\|_p \in [R_1, \infty) \).

Then if \( \|z\|_p \in [R_1, R_2] \), by (3.1), (3.2), we see that

\[
E_{\infty}(z) \geq \overline{h}(\|z\|_p) \geq h(\|z\|_p) \geq 0.
\]

If \( \|z\|_p \in (R_2, \infty) \), by (3.2) and above analysis, we also have that

\[
E_{\infty}(z) \geq \overline{h}(\|z\|_p) \geq 0.
\]

Thus \( \|z\|_p \in (0, R_1) \), (1) holds.

Now, we prove (2), let \( \Lambda^* \) as above. If \( c < 0 \) and \( \{z_n\} \subset H \) is a \((PS)_c\) sequence of \( E_{\infty} \), then we may assume that \( E_{\infty}(z_n) < 0 \) and \( E^{\prime}_{\infty}(z_n) = o(1) \), by (1), \( \|z_n\|_p \in (0, R_1) \), hence \( E(z_n) = E_{\infty}(z_n) \) and \( E^{\prime}(z_n) = E^{\prime}_{\infty}(z_n) \). Since (b) hold when \( 0 < \lambda + \theta \leq \Lambda^* \), By Lemma 2.5, \( E(z) \) satisfies the \((PS)_c\) condition for \( c < 0 \). Thus \( E_{\infty}(z) \) satisfies the \((PS)_c\) condition for \( c < 0 \), (2) holds.

Now we prove our main result via genus.

Proof of Theorem 1.1. Let \( \Sigma = \{ A \in H - \{(0, 0)\} \} \), \( A \) is closed, \( A = -A, \gamma(A) \geq k \), \( c_k = \inf_{A \in \Sigma} \sup_{z \in A} E_{\infty}(z) \), \( K_c^* = \{ z \in H \mid E_{\infty}(z) = c, E^{\prime}_{\infty}(z) = 0 \} \), and suppose that \( 0 < \lambda + \theta \leq \Lambda^* \), \( \Lambda^* \) is as above.

We claim that if \( k, l \in N \) are such that \( c = c_k = c_{k+1} = \cdots = c_{k+l} \), then \( \gamma(K_c^*) \geq l + 1. \)

In fact, we assume

\[
E_{\infty}^{-\varepsilon} = \{ z \in H \mid E_{\infty}(z) \leq -\varepsilon \},
\]

we will show for any \( k \in N \), there exist an \( \varepsilon = \varepsilon(k) > 0 \), such that

\[
\gamma(E_{\infty}^{-\varepsilon}(z)) \geq k.
\]

Fix \( k \in N \), denote \( H_k \) be an \( k \)-dimensional subspace of \( H \), choose \( z = (u, v) \in H_k \), with \( \|z\|_p = 1 \), for \( 0 < \rho < R_1 \), we have

\[
E(\rho z) = E_{\infty}(\rho z) = \frac{1}{p} \rho^p + \frac{\rho^q}{q} \frac{\|z\|^q}{r} - \frac{\rho^r}{r} \int_{\Omega} \lambda V(x)|u|^r + \theta V(x)|v|^\theta \int_{\Omega} dx - \frac{2\rho^p}{p} \int_{\Omega} |u|^p |v|^\theta dx.
\]

(3.3)
Since $c < 0$, there exists an odd homeomorphism

By Lemma 2.8, there is a closed and symmetric set

the main reason that we consider

can define

which shows that

So we have proved our claim.

Thus we have

That's meaning

i.e.,

A $\subset E^{c+\delta}_\infty$, and

that's meaning

Again by Lemma 2.8, we have

Thus we have $\eta(A - U) \in \Sigma_k$ and $\sup_{z \in \eta(A - U)} E_\infty(z) \geq c_k = c$, which contradicts to (3.7). So we have proved our claim.

Now let's complete the proof of Theorem 1.1. If for all $k \in N$, we have

$\Sigma_{k+1} \subset \Sigma_k$, $c_k \leq c_{k+1} < 0$. If all $c_k$ are distinct, then $\gamma(K_{c_k}) \geq 1$, and we see that $\{c_k\}$ is a sequence of distinct negative critical values of $E_\infty$; if for some $k_0$, there is a $l \geq 1$ such that $c = c_{k_0} = c_{k_0+1} = \cdots = c_{k_1+l}$, then by the claim, we have

which shows that $K_c$ contains infinitely many distinct elements.
By Lemma 3.1, we know $E(z) = E_\infty(z)$ when $E_\infty(z) < 0$, so we show that there are infinitely many critical points of $E(z)$. Theorem 1.1 is proved. □

4. Proof of Theorem 1.2.

In this section, we will study problem (1.1) with $1 < q < p < r < p^*$, and will prove Theorem 1.2 by the following general version of the Mountain Pass Lemma (see [3]).

**Lemma 4.1.** Let $E$ be a functional on a Banach space $H$, $E \in C^1(H, \mathbb{R})$. Let us assume that there exists $\rho, R > 0$ such that

(i) $E(z) > \rho$, $\forall z \in H$ with $\|z\|_p = R$.
(ii) $E(0) = 0$, and $E(w_0) < \rho$ for some $w_0 \in H$, with $\|w_0\|_p > R$.

Let us define $\Gamma = \{ \gamma \in C([0,1], H) | \gamma(0) = 0, \gamma(1) = w_0 \}$, and

$$\mu = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} E(\gamma(t)).$$

(4.1)

Then there exists a sequence $\{z_n\} \subset H$, such that $E(z_n) \to \mu$, and $E'(z_n) \to 0$ in $H'$ (dual of $H$) as $n \to \infty$.

Now similar to Lemma 2.5 in section 2, we have the following result.

**Lemma 4.2.** Suppose $1 < q < p \leq r < p^*$ hold, then any $(PS)_c$ sequence $\{z_n\} \subset H$ of $E(z)$ contains a convergent subsequence when

$$c < \frac{2}{N} S_{\alpha,\beta}^\frac{q}{r}.$$  

(4.2)

Now we give the proof of Theorem 1.2.

**Proof of Theorem 1.2.** From (4.1) and (4.2), we only need to show

$$\mu < \frac{2}{N} S_{\alpha,\beta}^\frac{q}{r},$$

(4.3)

then Lemma 4.1 and Lemma 4.2 give the existence of the critical point of $E$.

To obtain (4.3), Let us choose $z_0 = (u_0, u_0) \in H$, with

$$|z_0|_{p^*} = 1, \lim_{t \to \infty} E(tz_0) = -\infty,$$

then there exists a $t_{\theta \lambda} > 0$ such that $\sup_{t \geq 0} E(tz_0) = E(t_{\theta \lambda}z_0)$ holds, and then $t_{\theta \lambda}$ satisfies

$$0 = t_{\theta \lambda}^{p-1} \|z_0\|_p^p + t_{\theta \lambda}^{q-1} \|z_0\|_q^q - (\lambda + \theta) t_{\theta \lambda}^{r-1} \int_{\Omega} V(x)|u_0|^r dx - t_{\theta \lambda}^{p-1}$$

then we get

$$(\lambda + \theta) \int_{\Omega} V(x)|u_0|^r dx = t_{\theta \lambda}^{p-r} \|z_0\|_p^p + t_{\theta \lambda}^{q-r} \|z_0\|_q^q - t_{\theta \lambda}^{p-1}$$
from 1 < q < p ≤ r < p*, we get \( t_{2\lambda} \to 0 \) as \( (\lambda + \theta) \to \infty \). Then there exists \( \Lambda_* > 0 \) such that for any \( (\lambda + \theta) > \Lambda_* \), we have

\[
\sup_{t \geq 0} E(tz_0) < \frac{2}{N} S_{\alpha, \beta}^N.
\]

Now we take \( w_0 = t_0 z_0 \) with \( t_0 \) large enough to verify \( E(w_0) < 0 \), we get

\[
\alpha \leq \max_{t \in [0, 1]} E(\gamma_0(t))
\]

where \( \gamma_0(t) = tw_0 \). Therefore,

\[
\mu \leq \sup_{t \geq 0} E(tw_0) < \frac{2}{N} S_{\alpha, \beta}^N.
\]

then we have proved (4.3), that’s complete the proof. \( \square \)

Now let’s assume 1 < q < \( \frac{N(p-1)}{N-1} < p \leq \max\{p, p^* - \frac{q}{p-1}\} < r < p^* \), and define, for \( \varepsilon > 0 \),

\[
u_{\varepsilon}(x) = \frac{\psi(x)}{(\varepsilon + |x|^{p-1})^{\frac{N-1}{p-1}}}, \quad \psi_{\varepsilon}(x) = \frac{u_{\varepsilon}(x)}{|u_{\varepsilon}(x)|^p},
\]

where \( \psi(x) \in C_0^\infty(B(0, 2R)) \) is such that 0 ≤ \( \psi(x) \leq 1 \), and \( \psi(x) \equiv 1 \) on \( B(0, R) \).

We obtain the following estimates (see [13]).

\[
\int_{\Omega} |u_{\varepsilon}|^t dx = \begin{cases} K_1 \varepsilon^{\frac{N(p-1)-(N-1)t}{p}} + O(1), & t > \frac{N(p-1)}{N-p} \\ K_1 |\ln \varepsilon| + O(1), & t = \frac{N(p-1)}{N-p} \\ O(1), & t < \frac{N(p-1)}{N-p} \end{cases} (4.4)
\]

\[
\int_{\Omega} |\nabla u_{\varepsilon}|^t dx = \begin{cases} K_2 \varepsilon^t + O(1), & t > \frac{N(p-1)}{N-1} \\ K_2 |\ln \varepsilon| + O(1), & t = \frac{N(p-1)}{N-1} \\ O(1), & t < \frac{N(p-1)}{N-1} \end{cases} (4.5)
\]

In particular, we have

\[
\int_{\Omega} |\nabla u_{\varepsilon}|^p dx = K_2 \varepsilon^{\frac{p-N}{p}} + O(1) (4.6)
\]

and

\[
(\int_{\Omega} |u_{\varepsilon}|^p dx)^\frac{p}{p} = K_2 \varepsilon^{\frac{p-N}{p}} + O(1) (4.7)
\]

\[
\int_{\Omega} |u_{\varepsilon}|^p dx = \begin{cases} K_1 \varepsilon^{\frac{p^2-N}{p^2}} + O(1), & p^2 < N \\ K_1 |\ln \varepsilon| + O(1), & p^2 = N \\ O(1), & p^2 > N \end{cases} (4.8)
\]

where \( K_1, K_2, K_3 \) are positive constants independent of \( \varepsilon \), and \( S = \frac{K_2}{K_3} \) is the best Sobolev constant given in section 2.

If we choosing \( w_0 \) in the proof of Theorem 1.2 carefully, we can prove the following stronger result.
Theorem 4.3. If $1 < q < \frac{N(p-1)}{N-r} < p \leq \max\{p, p^* - \frac{q}{p-r}\} < r < p^*$ and $V_0$ hold, then for any $\theta > 0$, $\lambda > 0$, problem (1.1) has a nontrivial solution.

Proof. Following [16], we can take $z_\varepsilon = (v_\varepsilon, v_\varepsilon)$, and

$$g(t) = E(tz_\varepsilon) = \frac{2t^p}{p} \|v_\varepsilon\|_p^p + \frac{2q}{q} \|v_\varepsilon\|_p^q - \frac{(\lambda + \theta)t^r}{r} \int_{\Omega} V(x)\|v_\varepsilon\|_r^r dx - \frac{2t^{p^*}}{p^*},$$

then there exists a $t_\varepsilon > 0$ such that $\sup_{t \geq 0} E(tz_\varepsilon) = E(t_\varepsilon z_\varepsilon)$ hold, and then $t_\varepsilon$ satisfies

$$g'(t_\varepsilon) = 2t^{p-1}\|v_\varepsilon\|_p^p + 2t^{q-1}\|v_\varepsilon\|_p^q - (\lambda + \theta)t^{r-1} \int_{\Omega} V(x)\|v_\varepsilon\|_r^r dx - 2t^{p^*-1} = 0. \quad (4.9)$$

then we have

$$\int_{\Omega} |\nabla v_\varepsilon|^p dx + t_\varepsilon^{q-p} \int_{\Omega} |\nabla v_\varepsilon|^q dx > t_\varepsilon^{p^*-p}$$

From (4.4)-(4.8) we can know

$$\int_{\Omega} |\nabla v_\varepsilon|^p dx = S + O(\varepsilon^{\frac{N-p}{p-r}}), \quad \int_{\Omega} |\nabla v_\varepsilon|^q dx = O(\varepsilon^{\frac{q(N-p)}{p-r}})$$

set $\varepsilon$ small enough, then we can have

$$t_\varepsilon^{p^*-p} \leq 2S,$$

here we use the fact that $t_\varepsilon \to t_0 = (\int_{\Omega} |\nabla v_\varepsilon|^p dx)^{\frac{1}{p-r}} > 0$ as $\varepsilon \to 0$, where $t_0$ will be given later.

Then from (4.9) we obtain

$$2\|v_\varepsilon\|_p^p < (\lambda + \theta)t_\varepsilon^{r-p}\|V(x)\|_\infty \|v_\varepsilon\|_r^r + 2t_\varepsilon^{p^*-p}. \quad (4.10)$$

From (4.4)-(4.8), and (4.10), choose $\varepsilon$ small enough, we have

$$t_\varepsilon^{p^*-p} \geq S \geq \frac{S}{2}$$

Now we consider

$$h(t) = \frac{t^p}{p} \int_{\Omega} |\nabla v_\varepsilon|^p dx - \frac{t^{p^*}}{p^*}$$

the function attains its maximum at $t_0 = (\int_{\Omega} |\nabla v_\varepsilon|^p dx)^{\frac{1}{p-r}}$, and again combine with (4.4)-(4.8), we have

$$g(t_\varepsilon) \leq 2h(t_\varepsilon) + \frac{t_\varepsilon^q}{q} \int_{\Omega} |\nabla v_\varepsilon|^q dx - \frac{(\lambda + \theta)t_\varepsilon^r}{r} \int_{\Omega} V(x)\|v_\varepsilon\|_r^r dx$$

$$\leq \frac{2S}{2} + C_\theta (\int_{\Omega} |\nabla v_\varepsilon|^p dx)^{\frac{1}{p-r}} \int_{\Omega} |\nabla v_\varepsilon|^q dx$$

$$\leq \frac{2}{N} S_{\varepsilon, \theta}^2 + C_\theta (\frac{q(N-p)}{p-r}) - C_\theta (\frac{q(N-p)}{p-r})^\frac{p-1}{p} (N-r \frac{N-p}{p-r})$$
where $C_6, C_7, C_8$ are positive constants independent with $\varepsilon$. Since $1 < q < \frac{N(p-1)}{N-1} < p \leq \max\{p, p^* - \frac{q}{p-1}\} < r < p^*$, we obtain that
\[ \frac{N-p}{p} > \frac{q(N-p)}{p^2} > \frac{p-1}{p}(N-r) \frac{N-p}{p}, \]
then we choose $\varepsilon$ small enough, by Lemma 2.1, we get $g(t_{\varepsilon}) = \sup_{t \geq 0} E(tv_{\varepsilon}) < 2^N S^\frac{N}{p} \leq 2^N S^\frac{N}{p_{\alpha, \beta}}$, by Lemma 4.1 and Lemma 4.2, we complete the proof. □

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