Bull. Korean Math. Soc. **49** (2012), No. 6, pp. 1131–1145
http://dx.doi.org/10.4134/BKMS.2012.49.6.1131

CONSTANT-SIGN SOLUTIONS OF p-LAPLACIAN TYPE OPERATORS ON TIME SCALES VIA VARIATIONAL METHODS

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Abstract. The purpose of this paper is to use an appropriate variational framework to discuss the boundary value problem with p-Laplacian type operators

$$\begin{cases} (\alpha(t, x^{\Delta}(t)))^{\Delta} - a(t)\phi_p(x^{\sigma}(t)) + f(\sigma(t), x^{\sigma}(t)) = 0, \ \Delta\text{-a.e. } t \in I \\ x^{\sigma}(0) = 0, \\ \beta_1 x^{\sigma}(1) + \beta_2 x^{\Delta}(\sigma(1)) = 0, \end{cases}$$

where $\beta_1, \beta_2 > 0$, $I = [0,1]^{k^2}$, $\alpha(\cdot, x(\cdot))$ is an operator of p-Laplacian type, $\mathbb{T}$ is a time scale. Some sufficient conditions for the existence of constant-sign solutions are obtained.

1. Introduction

The theory of time scales was introduced by Stefan Hilger in his PhD thesis in order to unify continuous and discrete analysis. By choosing the time scale to be the set of real numbers, the general result yields a result concerning an ordinary differential equation; by choosing the time scale to be the set of integers, the same general result yields a result for difference equations. Moreover, the time scales calculus has a tremendous potential for applications. For example, it can model insect populations that are continuous while in season, die out in winter, while their eggs are incubating or dormant, and then hatch in a new season, giving rise to a nonoverlapping population. Recently, the boundary value problems (short for BVPs) on time scales have received a lot of attention. Many works have been carried out to discuss the existence of at least one solutions, multiple solutions. The methods used therein mainly depend on the Leray-Schauder continuation theorem, the Mawhin continuation theorem, upper and lower techniques and monotone iteration. However, few people consider the existence of solutions for BVPs on time scales by critical

Received June 1, 2009.
2010 *Mathematics Subject Classification.* 34B24, 35A15.
Key words and phrases. p-Laplacian, time scale, variational, constant-sign.
Supported by Beijing Natural Science Foundation (1122016) and National Natural Science Foundation of China (10671012).

point theory. In [2], by using variational techniques and critical point theory, the authors considered the existence of positive solutions for the Dirichlet BVP

$$\begin{cases} -u^{\Delta\Delta} = f(t, u^{\sigma}(t)); \ t \in J \cap \mathbb{T}^{k^2} \\ u(a) = u(b) = 0, \end{cases}$$

where $J = \begin{cases} [a,b) \cap \mathbb{T}; \text{ if } a < \sigma(a), \\ (a,b) \cap \mathbb{T}; \text{ if } a = \sigma(a), \end{cases}$ $f(\cdot, x) \in C_{rd}(J)$ uniformly at $x \in [0,p]$ for all $p \in [0,+\infty)$ and $f(t,\cdot) \in C([0,+\infty))$ uniformly at $t \in [c,d] \cap \mathbb{T}$ for each $c, d \in J$. In [8], by the mountain pass lemma, the authors deal with the existence of weak solutions of the above BVP.

However, the BVP above is a special case of p-Laplacian BVPs which have many important applications such as non-Newtonian fluid theory and the turbulent flow of the gas in porous medium. Motivated by the work mentioned above, in this paper, by an appropriate variational framework in the critical point theory, we consider the existence of constant-sign solutions for the BVP

$$\begin{cases} (\alpha(t, x^{\Delta}(t)))^{\Delta} - a(t)\phi_p(x^{\sigma}(t)) + f(\sigma(t), x^{\sigma}(t)) = 0, \ \Delta\text{-a.e. } t \in I \\ x^{\sigma}(0) = 0, \\ \beta_1 x^{\sigma}(1) + \beta_2 x^{\Delta}(\sigma(1)) = 0, \end{cases} \tag{1.1}$$

where $p > 1$, $\alpha(t,x)$ is an operator of p-Laplacian type, $\beta_1, \beta_2 > 0$, $[0,1]_{\mathbb{T}} = [0,1] \cap \mathbb{T}$, $\phi_p(x) = |x|^{p-2}x$, $\sigma(0) = 0$. Throughout, we assume $a(t) \in C([0,1]_{\mathbb{T}}; (0,\infty))$, $f(t,0) \not\equiv 0$ on any subset $E \subset I$ with $\mu_{\Delta} E \neq 0$, $f(\cdot, x)$ is Δ-measurable on $[0,1]_{\mathbb{T}}$ for each $x \in \mathbb{R}$ and $f(t,\cdot)$ is continuous for Δ-a.e. $t \in [0,1]_{\mathbb{T}}$, $F(t,x) = \int_0^x f(t,u)du$. Δ-a.e. means there exists a set $E \subset [0,1]_{\mathbb{T}}$ with null Lebesgue Δ-measure such that this property holds for every $t \in [0,1]_{\mathbb{T}} \setminus E$.

Definition 1.1. $\alpha(t,x)$ is called an operator of p-Laplacian type, that is, $\alpha(t,x)$ satisfies the following conditions:

(1) $G(t,x) = \int_0^x \alpha(t,u)du$, $\alpha(t,x)x \geq pG(t,x)$ and there exist $c_1 \geq c_2 > 0$, $N > 0$, $b(t) \in C([0,\sigma(1)]_{\mathbb{T}}, (0,\infty))$ such that

$$c_2\phi_p(x) - b(t) \leq \alpha(t,x) \leq c_1\phi_p(x) + b(t), \ x \geq N;$$

$$c_1\phi_p(x) - b(t) \leq \alpha(t,x) \leq c_2\phi_p(x) + b(t), \ x \leq -N;$$

(2) $G(t,x) : [0,\sigma(1)]_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists a $c_0 > 0$ such that $G(t,x) \geq c_0|x|^p$ for all $x \in \mathbb{R}$, $t \in [0,\sigma(1)]_{\mathbb{T}}$, $G(t,\cdot) : \mathbb{R} \to \mathbb{R}$ is strictly convex and continuously differentiable for any $t \in [0,\sigma(1)]_{\mathbb{T}}$.

For example, $\alpha(t,x) = \beta(t)|x|^{p-2}x$ $(p > 1)$, then, $G(t,x) = \frac{\beta(t)}{p}|x|^p$. Here, $\beta(t) \in C([0,\sigma(1)]_{\mathbb{T}}, [0,+\infty))$.

Our aim is to apply critical point theory to BVP (1.1) and prove the existence of at least one constant-sign solutions. The paper is organized as follows. In the forthcoming section, we present some properties of time scales that will be used in the paper. We refer the reader to [4] and [5] for a broad introduction to dynamic equations on time scales. The aim of Section 3 is to prove the existence of at least one positive solutions and in Section 4, we give some results of the existence of at least one negative solutions. Moreover, the conditions on f are easily verified.

2. Preliminary

Assume $p > 1$ and $L^p_\Delta[0,1] = \{f : [0,\sigma(1)]_{\mathbb{T}} \to \mathbb{R} : \ |f|^p$ is Lebesgue delta integrable on $[0,\sigma(1)) \cap \mathbb{T}\}$. Then, $L^p_\Delta[0,1]$ is a complete linear space with the norm $\|x\|_p = (\int_0^{\sigma(1)} |x(t)|^p \Delta t)^{\frac{1}{p}}$.

The Sobolev space $W^{1,p}_\Delta[0,1]$ is defined by $W^{1,p}_\Delta[0,1] = \{x : [0,\sigma(1)]_{\mathbb{T}} \to \mathbb{R} : \ x$ is absolutely continuous on $[0,\sigma(1)]_{\mathbb{T}},\ x(0)=0,$ and $x^\Delta \in L^p_\Delta[0,1]\}$ and is endowed with the norm

$$\|x\| = \left(\int_0^{\sigma(1)} |x^\sigma(t)|^p \Delta t + \int_0^{\sigma(1)} |x^\Delta(t)|^p \Delta t \right)^{\frac{1}{p}}.$$

Obviously, $W^{1,p}_\Delta[0,1]$ is a Banach space. To prove the main results in this paper, we employ several definitions and lemmas.

Definition 2.1. If $x(t) \geq 0$ for Δ-a.e. $t \in [0,1]_{\mathbb{T}}$, $x(t) \not\equiv 0$ and satisfies BVP (1.1), then x is a positive solution of BVP (1.1).

Lemma 2.1. *For $x \in X$, assume $x^\pm = \max\{\pm x, 0\}$. Then we have the following results*:

(i) $x = x^+ - x^-$;

(ii) $\|x^+\|_X \leq \|x\|_X$;

(iii) $(\phi_p(x), x^+) = |x^+|^p$, $(\phi_p(x), x^-) = -|x^-|^p$;

(iv) *If $(x_n)_{n\in\mathbb{N}}$ uniformly converges to x in $C[0,1]$, then $(x_n^\pm)_{n\in\mathbb{N}}$ uniformly converges to $x^\pm$ in $C[0,1]$.*

Lemma 2.2 ([10])**.** *There exists a positive constant c_p such that*

$$(|x|^{p-2}x - |y|^{p-2}y, x-y) \geq \begin{cases} c_p|x-y|^p, & p \geq 2, \\ c_p \frac{|x-y|^2}{(|x|+|y|)^{2-p}}, & 1 < p < 2 \end{cases} \tag{2.1}$$

for any $x, y \in \mathbb{R}^N$, $|x|+|y| \neq 0$. Here, $(x,y) = x \cdot y^T$.

Lemma 2.3 (Hölder Inequality, [4])**.** *Let $f, g \in C_{rd}[0,1]$, $p > 1$ and q be the conjugate number of p. Then*

$$\int_0^1 |f(t)g(t)|\Delta t \leq \left(\int_0^1 |f(t)|^p \Delta t \right)^{\frac{1}{p}} \left(\int_0^1 |g(t)|^q \Delta t \right)^{\frac{1}{q}}.$$

Lemma 2.4 (Arzelà Ascoli Theorem, [1])**.** *Let X be a subset of $C[0,1]_{\mathbb{T}}$ satisfying the following conditions*:

(i) *X is bounded*;

(ii) *For any $\varepsilon > 0$, there exists $\delta > 0$ such that $t_1, t_2 \in [0,1]_{\mathbb{T}}$, $|t_1 - t_2| < \delta$ implies $|f(t_1) - f(t_2)| < \varepsilon$ for all $f \in X$. Then X is relatively compact.*

For $x \in C[0, \sigma(1)]_{\mathbb{T}}$, suppose

$$\|x\|_\infty = \max_{t\in[0,\sigma(1)]_{\mathbb{T}}} |x(t)|, \quad |x|_m = \min_{t\in[0,\sigma(1)]_{\mathbb{T}}} |x(t)|.$$

Lemma 2.5. *The immersion $W^{1,p}_\Delta[0, 1] \hookrightarrow C[0, \sigma(1)]_{\mathbb{T}}$ is compact.*

Proof. Assume $x \in W^{1,p}_\Delta[0, 1]$, $t_1, t_2 \in [0, \sigma(1)]_{\mathbb{T}}$ with $t_1 \le t_2$. Then

$$|x(t_1) - x(t_2)| = \left|\int_{t_1}^{t_2} x^\Delta(s)\Delta s\right| \le \int_{t_1}^{t_2} |x^\Delta(s)|\Delta s \le |t_2 - t_1|^{\frac{1}{q}}\|x\|. \tag{2.2}$$

Hence, $W^{1,p}_\Delta[0, 1] \subset C[0, \sigma(1)]_{\mathbb{T}}$. Moreover, let $t_1 = t, t_2 = \sigma(0)$, one has

$$|x(t)| \le t^{\frac{1}{q}}\|x\| \le (\sigma(1))^{\frac{1}{q}}\|x\|.$$

Then, $W^{1,p}_\Delta$ is embedded into $C[0, \sigma(1)]_{\mathbb{T}}$.

Let B be the unit ball in $W^{1,p}_\Delta[0, 1]$ defined by $B = \{x \in W^{1,p}_\Delta[0, 1] | \, \|x\| \le 1\}$, we need to show that B is relatively compact in $C[0, \sigma(1)]_{\mathbb{T}}$. It is obvious $B \subset C[0, \sigma(1)]_{\mathbb{T}}$ and is bounded in $C[0, \sigma(1)]_{\mathbb{T}}$. According to the Arzelà-Ascoli Theorem, it suffices to show that the functions in B are equicontinuous. In fact, for all $x \in B$, from (2.2), we can easily obtain the functions in B are equicontinuous. Hence, B is relatively compact. Then, $W^{1,p}_\Delta[0, 1] \hookrightarrow C[0, \sigma(1)]_{\mathbb{T}}$ is compact. □

From the proof of Lemma 2.5, we can easily obtain:

Lemma 2.6. *If $x \in W^{1,p}_\Delta[0, 1]$, then $\|x\|_\infty \le (\sigma(1))^{\frac{1}{q}}\|x\|$.*

In the following, let H be a Banach space, $\varphi : H \to \mathbb{R}$ differentiable. We state the condition (C) ([9]):

(C) Every sequence $(x_n)_{n\in\mathbb{N}} \subset H$ such that the following conditions hold:

(i) $(\varphi(x_n))_{n\in\mathbb{N}}$ is bounded,

(ii) $(1 + \|x_n\|_H)\|\varphi'(x_n)\|_{H^*} \to 0$ as $n \to \infty$

has a subsequence which converges strongly in H.

This condition is weaker than the usual Palais-Smale condition, but can be used in place of it when constructing deformations of sublevel sets via negative pseudo-gradient flows, and therefore also in minimax theorems such as the mountain pass lemma, the saddle point theorem.

3. Existence of positive solutions

Consider the following BVP

$$\begin{cases} (\alpha(t, x^\Delta(t)))^\Delta - a(t)\phi_p(x^\sigma(t)) + f(\sigma(t), (x^+)^\sigma) = 0, \ \Delta\text{-a.e. } t \in I \\ x^\sigma(0) = 0, \\ \beta_1 x^\sigma(1) + \beta_2 x^\Delta(\sigma(1)) = 0, \end{cases} \tag{3.1}$$

and assume $(x, y) = xy$ for $x, y \in W^{1,p}_\Delta[0, 1]$.

Lemma 3.1. *Assume*

(A_1) $f(t,x):[0,\sigma(1)]_{\mathbb{T}}\times[0,\infty)\to[0,\infty)$,

and $x\in W_{\Delta}^{1,p}[0,1]$ *is a solution of BVP* (3.1). *Then* $x(t)\geq 0$, $x(t)\not\equiv 0$ *for a.e.* $t\in I$.

Proof. Let $x\in W_{\Delta}^{1,p}[0,1]$ be a solution of BVP (3.1). If there exists a subset $E_0\subset I$, $\mu_\Delta(E_0)\neq 0$, $x(t)\equiv 0$ for $t\in E_0$, then from BVP (3.1), one has $f(t,0)\equiv 0$ for $t\in E_0$ which contradicts with the assumptions. Moreover, we know that x^- is an absolutely continuous function on I, and so, the Fundamental Theorem of Calculus ensures the existence of a set $E_1\subset I$ such that $\mu_\Delta(I\setminus E_1)=0$ and x^- is Δ-differentiable on E_1, $(x^-)^\Delta\in L_\Delta^1[0,1]$. Let $E_2=\{t\in I: x(t)<0\}$. Then

$$
\begin{aligned}
0&=\int_0^{\sigma(1)}((\alpha(t,x^\Delta(t)))^\Delta-a(t)\phi_p(x^\sigma(t))+f(\sigma(t),(x^+)^\sigma),(x^-)^\sigma)\Delta t\\
&\geq\int_0^{\sigma(1)}((\alpha(t,x^\Delta(t)))^\Delta,(x^-)^\sigma)\Delta t-\int_0^{\sigma(1)}(a(t)\phi_p(x^\sigma(t)),(x^-)^\sigma)\Delta t\\
&=x^-(\sigma(1))\alpha^\sigma(1,x^\Delta(1))-\int_{E_1}(\alpha(t,x^\Delta(t)),(x^-)^\Delta)\Delta t\\
&\quad-\int_0^{\sigma(1)}(a(t)\phi_p(x^\sigma(t)),(x^-)^\sigma)\Delta t\\
&=x^-(\sigma(1))\alpha^\sigma(1,x^\Delta(1))+\int_{E_2}(\alpha(t,x^\Delta(t)),x^\Delta)\Delta t\\
&\quad-\int_0^{\sigma(1)}(a(t)\phi_p(x^\sigma(t)),(x^-)^\sigma))\Delta t\\
&=\begin{cases}\int_{E_2}(\alpha(t,x^\Delta(t)),x^\Delta)\Delta t-\int_0^{\sigma(1)}(a(t)\phi_p(x^\sigma(t)),(x^-)^\sigma))dt, & x^\sigma(1)\geq 0\\[2mm] \frac{\beta_2x^\Delta(\sigma(1))}{\beta_1}\alpha^\sigma(1,x^\Delta(1))+\int_{E_2}(\alpha(t,x^\Delta(t)),x^\Delta)\Delta t\\ -\int_0^{\sigma(1)}(a(t)\phi_p(x^\sigma(t)),(x^-)^\sigma))\Delta t, & x^\sigma(1)<0\end{cases}\\
&\geq\int_{E_2}(\alpha(t,x^\Delta(t)),x^\Delta)\Delta t+\int_0^{\sigma(1)}a(t)|(x^-)^\sigma|^p\Delta t.
\end{aligned}
$$

Therefore, for Δ-a.e. $t\in I$, $(x^-)^\sigma=0$. Thus $x^-(t)=0$, Δ-a.e. $t\in I$. Otherwise, if there exists a subset $E_0'\subset I$, $\mu_\Delta E_0'\neq 0$, $x^-(t)\neq 0$, $t\in E_0'$. Without loss of generality, we may assume $E_0'=[a,b]\subset I$, $x^-(\sigma(t))\neq 0$ for $t\in(a,b)$ which contradicts $(x^-)^\sigma=0$ for Δ-a.e. $t\in I$. Hence $x(t)\geq 0$ for a.e. $t\in I$ and $x(t)\not\equiv 0$. □

Lemma 3.2. *Assume* (A_1) *and*

(A_2) *there exist* $e_1(t),e_2(t)$ *satisfying* $e_1^\sigma(t)$, $e_2^\sigma(t)\in L_\Delta^1[0,1]$ *and a positive constant* $\gamma<p$ *such that*

$$f(t,x)\leq e_1(t)x^{\gamma-1}+e_2(t),\ t\in[0,\sigma(1)]_{\mathbb{T}},\ x\in[0,\infty),$$

hold. Then $x(t) \in W_{\Delta}^{1,p}[0,1]$ *is a solution of BVP* (3.1) *if and only if* $x(t)$ *is a critical point of the functional*

$$\begin{aligned}\varphi(x) = & \int_0^{\sigma(1)} G(t,x^{\Delta})\Delta t + \frac{1}{p}\int_0^{\sigma(1)} a(t)|x^{\sigma}|^p\Delta t \\ & -\int_0^{\sigma(1)} [F(\sigma(t),(x^+)^{\sigma}(t)) - (f(\sigma(t),0),(x^-)^{\sigma}(t))]\Delta t \\ & + \frac{\beta_2}{\beta_1}G(\sigma(1),-\frac{\beta_1 x^{\sigma}(1)}{\beta_2}).\end{aligned} \tag{3.2}$$

Proof. Obviously, φ is continuously differentiable on $W_{\Delta}^{1,p}[0,1]$ and by computation, one has

$$\begin{aligned}\langle\varphi'(x),y\rangle = & \int_0^{\sigma(1)} (\alpha(t,x^{\Delta}),y^{\Delta})\Delta t + \int_0^{\sigma(1)} (a(t)\phi_p(x^{\sigma}),y^{\sigma})\Delta t \\ & -\int_0^{\sigma(1)} (f(\sigma(t),(x^+)^{\sigma}(t)),y^{\sigma})\Delta t \\ & -\alpha(\sigma(1),-\frac{\beta_1 x^{\sigma}(1)}{\beta_2})y^{\sigma}(1),\ x,y\in W_{\Delta}^{1,p}[0,1].\end{aligned} \tag{3.3}$$

Moreover,

$$\int_0^{\sigma(1)} (\alpha(t,x^{\Delta}),y^{\Delta})\Delta t = \alpha(\sigma(1),x^{\Delta}(\sigma(1)))y^{\sigma}(1) - \int_0^{\sigma(1)} ((\alpha(t,x^{\Delta}))^{\Delta},y^{\sigma})\Delta t. \tag{3.4}$$

Obviously, (3.3) and (3.4) hold for $\forall y \in W_{\Delta}^{1,p}[0,1]$ satisfying $y^{\sigma}(1)=0$. Then, if $x \in W_{\Delta}^{1,p}[0,1]$ is a critical point of φ,

$$\begin{aligned}0 &= \langle\varphi'(x),y\rangle \\ &= -\int_0^{\sigma(1)} ((\alpha(t,x^{\Delta}))^{\Delta},y^{\sigma})\Delta t + \int_0^{\sigma(1)} (a(t)\phi_p(x),y^{\sigma})\Delta t \\ &\quad -\int_0^{\sigma(1)} (f(\sigma(t),(x^+)^{\sigma}(t)),y^{\sigma})\Delta t \\ &\quad (\forall y\in W_{\Delta}^{1,p}[0,1],\ y^{\sigma}(1)=0).\end{aligned}$$

That is, x satisfies the equation of BVP (3.1). In the following, we will show that x satisfies the boundary conditions of BVP (3.1). Since x satisfies the equation of BVP (3.1), from (3.3) and (3.4), we have

$$\alpha(\sigma(1),x^{\Delta}(\sigma(1)))y^{\sigma}(1) - \alpha(\sigma(1),-\frac{\beta_1 x^{\sigma}(1)}{\beta_2})y^{\sigma}(1) = 0 \tag{3.5}$$

holds for all $y \in W_{\Delta}^{1,p}[0,1]$. Suppose x does not satisfy the boundary conditions of BVP (3.1). Without loss of generality, assume $\beta_1 x^{\sigma}(1) + \beta_2 x^{\Delta}(\sigma(1)) > 0$, that is $x^{\Delta}(\sigma(1)) > -\frac{\beta_1 x^{\sigma}(1)}{\beta_2}$. The strict convexity means $\alpha(x, y)$ is strictly increasing in y. Then,

$$\alpha(\sigma(1), x^{\Delta}(\sigma(1))) > \alpha(\sigma(1), -\frac{\beta_1 x^{\sigma}(1)}{\beta_2}).$$

Let $y^{\sigma}(t) = t$, one has

$$\begin{aligned} &\alpha(\sigma(1), x^{\Delta}(\sigma(1)))y^{\sigma}(1) - \alpha(\sigma(1), -\frac{\beta_1 x^{\sigma}(1)}{\beta_2})y^{\sigma}(1) \\ &= \alpha(\sigma(1), x^{\Delta}(\sigma(1))) - \alpha(\sigma(1), -\frac{\beta_1 x^{\sigma}(1)}{\beta_2}) > 0, \end{aligned} \tag{3.6}$$

which contradicts with (3.5). Similarly, we have $\beta_1 x^{\sigma}(1) + \beta_2 x^{\Delta}(\sigma(1)) < 0$ does not hold either. Hence, $\beta_1 x^{\sigma}(1) + \beta_2 x^{\Delta}(\sigma(1)) = 0$. Therefore, x is a solution of BVP (3.1).

Conversely, if x is a solution of BVP (3.1), multiplying $y \in W_{\Delta}^{1,p}[0, 1]$ on both sides of the equation and integrating from 0 to $\sigma(1)$, we have $\langle \varphi'(x), y\rangle = 0$ by the boundary conditions. Then, x is a critical point of φ. □

Remark 3.1. From Lemma 3.1 and Lemma 3.2, to prove the existence of positive solutions for BVP (1.1), we only need to prove the existence of the critical points of the functional φ.

Lemma 3.3. *Assume* (A_1) *and* (A_2) *hold. Then the functional* φ *satisfies the condition* (C).

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $W_{\Delta}^{1,p}[0, 1]$ such that $\varphi(x_n)$ is bounded and $\|\varphi'(x_n)\|_{(W_{\Delta}^{1,p})^*} \times (1 + \|x_n\|) \to 0$ as $n \to \infty$. Then, there exists a constant C_1 such that

$$|\varphi(x_n)| \le C_1,\ \|\varphi'(x_n)\|_{(W_{\Delta}^{1,p})^*}(1 + \|x_n\|) \le C_1 \tag{3.7}$$

for all $n \in \mathbb{N}$. Thus,

$$\begin{aligned} (p + 1)C_1 &= C_1 + pC_1 \\ &\ge \|\varphi'(x_n)\|_{(W_{\Delta}^{1,p})^*}(1 + \|x_n\|) - p\varphi(x_n) \\ &\ge \|\varphi'(x_n)\|_{(W^{1,p})^*}\|x_n\| - p\varphi(x_n) \\ &\ge \langle\varphi'(x_n), x_n\rangle - p\varphi(x_n) \\ &= \int_0^{\sigma(1)} (\alpha(t, x_n^{\Delta}), x_n^{\Delta})\Delta t - p\int_0^{\sigma(1)} G(t, x_n^{\Delta})\Delta t \\ &\quad - \alpha^{\sigma}(1, -\frac{\beta_1 x_n(1)}{\beta_2})x_n^{\sigma}(1) - p\frac{\beta_2}{\beta_1}G^{\sigma}(1, -\frac{\beta_1 x_n(1)}{\beta_2})) \\ &\quad + p\int_0^{\sigma(1)} F^{\sigma}(t, x_n^{+})\Delta t - \int_0^{\sigma(1)} (f^{\sigma}(t, x_n^{+}), x_n^{\sigma})\Delta t \end{aligned}$$

$$
\begin{aligned}
& - p\int_0^{\sigma(1)}(f^\sigma(t,0),(x_n^-)^\sigma)\Delta t \\
= \ & \int_0^{\sigma(1)}(\alpha(t,x_n^\Delta),x_n^\Delta)\Delta t - p\int_0^{\sigma(1)}G(t,x_n^\Delta)\Delta t \\
& - \alpha^\sigma(1,-\frac{\beta_1 x_n(1)}{\beta_2})x_n^\sigma(1) - p\frac{\beta_2}{\beta_1}G^\sigma(1,-\frac{\beta_1 x_n(1)}{\beta_2}) \\
& + p\int_0^{\sigma(1)}F^\sigma(t,x_n^+)\Delta t - \int_0^{\sigma(1)}(f^\sigma(t,x_n^+),(x_n^+)^\sigma)\Delta t \\
& + (1-p)\int_0^{\sigma(1)}(f^\sigma(t,0),(x_n^-)^\sigma)\Delta t \\
= \ & \int_0^{\sigma(1)}(\alpha(t,x_n^\Delta),x_n^\Delta)\Delta t - p\int_0^{\sigma(1)}G(t,x_n^\Delta)\Delta t \\
& - \alpha^\sigma(1,-\frac{\beta_1 x_n(1)}{\beta_2})x_n^\sigma(1) - p\frac{\beta_2}{\beta_1}G^\sigma(1,-\frac{\beta_1 x_n(1)}{\beta_2}) \\
& + p\left(\int_0^{\sigma(1)}F^\sigma(t,x_n^+)\Delta t - \int_0^{\sigma(1)}(f^\sigma(t,0),(x_n^-)^\sigma)\Delta t\right) \\
& + \int_0^{\sigma(1)}(f^\sigma(t,0),(x_n^-)^\sigma)\Delta t - \int_0^{\sigma(1)}(f^\sigma(t,x_n^+),(x_n^+)^\sigma)\Delta t \\
\geq \ & \int_0^{\sigma(1)}(\alpha(t,x_n^\Delta),x_n^\Delta)\Delta t - p\int_0^{\sigma(1)}G(t,x_n^\Delta)\Delta t \\
& - p\frac{\beta_2}{\beta_1}G^\sigma(1,-\frac{\beta_1 x_n(1)}{\beta_2}) \\
& + p\left(\int_0^{\sigma(1)}F^\sigma(t,x_n^+)\Delta t - \int_0^{\sigma(1)}(f^\sigma(t,0),(x_n^-)^\sigma)\Delta t\right) \\
& + \int_0^{\sigma(1)}(f^\sigma(t,0),(x_n^-)^\sigma)\Delta t - \int_0^{\sigma(1)}(f^\sigma(t,x_n^+),(x_n^+)^\sigma)\Delta t \\
\geq \ & - p\frac{\beta_2}{\beta_1}G^\sigma(1,-\frac{\beta_1 x_n(1)}{\beta_2}) \\
& + p\left(\int_0^{\sigma(1)}F^\sigma(t,x_n^+)\Delta t - \int_0^{\sigma(1)}(f^\sigma(t,0),(x_n^-)^\sigma)\Delta t\right) \\
& + \int_0^{\sigma(1)}(f^\sigma(t,0),(x_n^-)^\sigma)\Delta t - \int_0^{\sigma(1)}(f^\sigma(t,x_n^+),(x_n^+)^\sigma)\Delta t. \qquad (3.8)
\end{aligned}
$$

Hence,

$$
(p+1)C_1 - \int_0^{\sigma(1)}(f^\sigma(t,0),(x_n^-)^\sigma)\Delta t + \int_0^{\sigma(1)}(f^\sigma(t,x_n^+),(x_n^+)^\sigma)\Delta t
$$

$$\geq p\left(\int_0^{\sigma(1)} F^\sigma(t,x_n^+)\Delta t-\int_0^{\sigma(1)}(f^\sigma(t,0),(x_n^-)^\sigma)\Delta t\right)-\frac{\beta_2}{\beta_1}G^\sigma(1,\frac{\beta_1 x_n(1)}{\beta_2}).$$

Moreover, by Lemma 2.4,

$$\begin{aligned}\left|\int_0^{\sigma(1)}(f^\sigma(t,0),(x_n^-)^\sigma)\Delta t\right| &\leq \int_0^{\sigma(1)} |(f^\sigma(t,0),(x_n^-)^\sigma)|\Delta t\\
&\leq \int_0^{\sigma(1)} f^\sigma(t,0)|x_n^\sigma|\Delta t\\
&= \int_0^{1} f^\sigma(t,0)|x_n^\sigma|\Delta t+\int_1^{\sigma(1)} f^\sigma(t,0)|x_n^\sigma|\Delta t\\
&= \int_0^{1} f^\sigma(t,0)|x_n^\sigma|\Delta t+f^\sigma(1,0)|x_n^\sigma(1)|\mu(1)\\
&\leq \|x_n\|_\infty\left(\int_0^1 f^\sigma(t,0)\Delta t+f^\sigma(1,0)\mu(1)\right)\\
&= \|x_n\|_\infty\int_0^{\sigma(1)} f^\sigma(t,0)\Delta t\\
&\leq (\sigma(1))^{\frac{1}{q}}\|x_n\|\int_0^{\sigma(1)} f^\sigma(t,0)\Delta t, \qquad (3.9)\end{aligned}$$

$$\begin{aligned}&\int_0^{\sigma(1)}(f^\sigma(t,x_n^+),(x_n^+)^\sigma)\Delta t\\
\leq\ & \int_0^{\sigma(1)} e_1^\sigma(t)|(x_n^+)^\sigma|^\gamma\Delta t+\int_0^{\sigma(1)} e_2^\sigma(t)|(x_n^+)^\sigma|\Delta t\\
\leq\ & \int_0^{\sigma(1)} e_1^\sigma(t)|x_n^\sigma|^\gamma\Delta t+\int_0^{\sigma(1)} e_2^\sigma(t)|x_n^\sigma|\Delta t\\
\leq\ & \|x_n\|_\infty^\gamma\int_0^{\sigma(1)} e_1^\sigma(t)\Delta t+\|x_n\|_\infty\int_0^{\sigma(1)} e_2^\sigma(t)\Delta t\\
\leq\ & (\sigma(1))^{\frac{\gamma}{q}}\|x_n\|^\gamma\int_0^{\sigma(1)} e_1^\sigma(t)\Delta t+(\sigma(1))^{\frac{1}{q}}\|x_n\|\int_0^{\sigma(1)} e_2^\sigma(t)\Delta t. \qquad (3.10)\end{aligned}$$

From (3.9)-(3.10), one has

$$\begin{aligned}&(p+1)C_1+(\sigma(1))^{\frac{1}{q}}\|x_n\|\left(\int_0^{\sigma(1)} f^\sigma(t,0)\Delta t+\int_0^{\sigma(1)} e_2^\sigma(t)\Delta t\right)\\
&+(\sigma(1))^{\frac{\gamma}{q}}\int_0^{\sigma(1)} e_1^\sigma(t)dt\|x_n\|^\gamma\\
\geq\ & p\left(\int_0^{\sigma(1)} F^\sigma(t,x_n^+)\Delta t-\int_0^{\sigma(1)}(f^\sigma(t,0),(x_n^-))\Delta t-\frac{\beta_2}{\beta_1}G^\sigma(1,\frac{\beta_1 x_n(1)}{\beta_2})\right).\end{aligned}$$

Then,

$$\begin{aligned}\varphi(x_n) \geq\ & \min\{\frac{1}{p}|a(t)|_m, c_0\}\|x_n\|^p - \frac{1}{p}[(p+1)C_1 \\ & + (\sigma(1))^{\frac{1}{q}}\|x_n\|\left(\int_0^{\sigma(1)} f^\sigma(t,0)\Delta t + \int_0^{\sigma(1)} e_2^\sigma(t)\Delta t\right) \\ & + (\sigma(1))^{\frac{\gamma}{q}}\int_0^{\sigma(1)} e_1^\sigma(t)\Delta t\|x_n\|^\gamma].\end{aligned}$$

Hence,

$$\begin{aligned}C_1 + \frac{1}{p}(p+1)C_1 \geq\ & \min\{\frac{1}{p}|a(t)|_m, c_0\}\|x_n\|^p - \frac{1}{p}[((\sigma(1))^{\frac{1}{q}}\int_0^{\sigma(1)} f^\sigma(t,0)\Delta t \\ & + (\sigma(1))^{\frac{1}{q}}\int_0^{\sigma(1)} e_2^\sigma(t)\Delta t)\|x_n\| \\ & + (\sigma(1))^{\frac{\gamma}{q}}\int_0^{\sigma(1)} e_1^\sigma(t)\Delta t\|x_n\|^\gamma].\end{aligned}$$

Therefore, there exists a constant $C_2 > 0$ such that $\|x_n\| \leq C_2$. That is $(x_n)_{n\in\mathbb{N}}$ is uniformly bounded in $W_\Delta^{1,p}[0,1]$. By the compactness of the embedding $W_\Delta^{1,p}[0,1] \hookrightarrow C[0,\sigma(1)]_{\mathbb{T}}$, the sequence $(x_n)_{n\in\mathbb{N}}$ has a subsequence, again denoted by $(x_n)_{n\in\mathbb{N}}$, such that

$$x_n \rightharpoonup x \text{ weakly in } W_\Delta^{1,p}[0,1], \tag{3.11}$$

$$x_n \to x \text{ strongly in } C[0,\sigma(1)]_{\mathbb{T}}. \tag{3.12}$$

From (3.3), one has

$$\begin{aligned}& \langle\varphi'(x_n) - \varphi'(x_m), x_n - x_m\rangle \\ =\ & \int_0^{\sigma(1)}(\alpha(t,x_n^\Delta) - \alpha(t,x_m^\Delta), x_n^\Delta - x_m^\Delta)\Delta t \\ & + \int_0^{\sigma(1)}(a(t)(\phi_p^\sigma(x_n) - \phi_p^\sigma(x_m)), x_n^\sigma - x_m^\sigma)\Delta t \\ & - \int_0^{\sigma(1)}(f^\sigma(t,x_n^+) - f^\sigma(t,x_m^+), x_n^\sigma - x_m^\sigma)\Delta t \\ & - (\alpha^\sigma(1, -\frac{\beta_1 x_n(1)}{\beta_2}) - \alpha^\sigma(1, -\frac{\beta_1 x_m(1)}{\beta_2}))(x_n^\sigma(1) - x_m^\sigma(1)).\end{aligned}$$

Since $x_n \to x$ in $C[0,\sigma(1)]_{\mathbb{T}}$, $\alpha(t,\cdot)$ is continuous, then, $(\alpha^\sigma(1, \frac{\beta_1 x_n(1)}{\beta_2}) - \alpha^\sigma(1, \frac{\beta_1 x_m(1)}{\beta_2}))(x_n^\sigma(1) - x_m^\sigma(1)) \to 0$ as $n \to \infty$. Moreover, with the similar discussion as (3.9), we have

$$\left|\int_0^{\sigma(1)}(a(t)(\phi_p^\sigma(x_n) - \phi_p^\sigma(x_m)), x_n^\sigma - x_m^\sigma)\Delta t\right|$$

$$\le \|a(t)\|_\infty \|x_n - x_m\|_\infty \int_0^{\sigma(1)} |\phi_p^\sigma(x_n) - \phi_p^\sigma(x_m)|\Delta t$$
$$\to 0 \text{ as } n, m \to \infty,$$

$$\left|\int_0^{\sigma(1)} (f^\sigma(t, x_n^+) - f^\sigma(t, x_m^+), x_n^\sigma - x_m^\sigma)\Delta t\right|$$
$$\le \Big((\sigma(1))^{\frac{\gamma-1}{q}} \int_0^{\sigma(1)} e_1^\sigma(t)\Delta t(\|x_m\|^{\gamma-1} + \|x_n\|^{\gamma-1})$$
$$+ \int_0^{\sigma(1)} e_2^\sigma(t)\Delta t\Big)\|x_n - x_n\|_\infty \to 0 \text{ as } n, m \to \infty.$$

From $|\langle\varphi'(x_n) - \varphi'(x_m), x_n - x_m\rangle| \le (\|\varphi'(x_n)\| + \|\varphi'(x_m)\|) \cdot (\|x_n\| + \|x_m\|)$ and $\|x_n\| + \|x_m\|$ is bounded in $W_\Delta^{1,p}[0,1]$, $\|\varphi'(x_n)\| \to 0$, $\|\varphi'(x_m)\| \to 0$ as $m, n \to \infty$, one has $\langle\varphi'(x_n) - \varphi'(x_m), x_n - x_m\rangle \to 0$. Hence

$$\int_0^{\sigma(1)} (\alpha(t, x_n^\Delta) - \alpha(t, x_m^\Delta), x_n^\Delta - x_m^\Delta)\Delta t \to 0 \text{ as } n, m \to \infty. \tag{3.13}$$

From the definition of $\alpha(t,x)$, there exists a constant $M > 0$ such that

$$c_2\phi_p(x) - b(t) - M \le \alpha(t,x) \le c_1\phi_p(x) + b(t) + M,\ x \ge 0,\ t \in [0,\sigma(1)]_{\mathbb{T}},$$
$$c_1\phi_p(x) - b(t) - M \le \alpha(t,x) \le c_2\phi_p(x) + b(t) + M,\ x \le 0,\ t \in [0,\sigma(1)]_{\mathbb{T}}.$$

In order to obtain

$$\int_0^{\sigma(1)} (\phi_p(x_n^\Delta) - \phi_p(x_m^\Delta), x_n^\Delta - x_m^\Delta)\Delta t \to 0 \text{ as } n, m \to \infty, \tag{3.14}$$

we must take in account several cases. Without loss of generality, we assume $x_n^\Delta \ge x_m^\Delta \ge 0$, $t \in [0,\sigma(1)]_{\mathbb{T}}$, then, we have

$$\int_0^{\sigma(1)} (\alpha(t, x_n^\Delta) - \alpha(t, x_m^\Delta), x_n^\Delta - x_m^\Delta)\Delta t$$
$$\ge c_2 \int_0^{\sigma(1)} (\phi_p(x_n^\Delta) - \phi_p(x_m^\Delta), x_n^\Delta - x_m^\Delta)\Delta t + (c_2 - c_1)$$
$$\int_0^{\sigma(1)} (\phi_p(x_m^\Delta), x_n^\Delta - x_m^\Delta)\Delta t - 2(M + \|b(t)\|_\infty) \int_0^{\sigma(1)} (x_n^\Delta - x_m^\Delta)\Delta t$$
$$= c_2 \int_0^{\sigma(1)} (\phi_p(x_n^\Delta) - \phi_p(x_m^\Delta), x_n^\Delta - x_m^\Delta)\Delta t + (c_2 - c_1)$$
$$\int_0^{\sigma(1)} (\phi_p(x_m^\Delta), x_n^\Delta - x_m^\Delta)\Delta t - 2(M + \|b(t)\|_\infty)(x_n^\sigma(1) - x_m^\sigma(1)).$$

Moreover, from the boundedness of x_m in $W_\Delta^{1,p}[0,1]$, we have x_m^Δ is bounded for Δ-a.e. $t \in [0,\sigma(1)]_{\mathbb{T}}$, then, $\int_0^{\sigma(1)} (\phi_p(x_m^\Delta), x_n^\Delta - x_m^\Delta)\Delta t \to 0$ as $n, m \to \infty$. Hence, we obtain (3.14). Other cases can be discussed similarly.

If $p \geq 2$, from Lemma 2.2, there exists a positive constant c_p such that

$$\int_0^{\sigma(1)} (\phi_p(x_n^\Delta) - \phi_p(x_m^\Delta), x_n^\Delta - x_m^\Delta)\Delta t \geq c_p \int_0^1 |x_n^\Delta - x_m^\Delta|^p \Delta t. \tag{3.15}$$

If $p < 2$, by Lemma 2.2, Hölder inequality and the boundedness of $(x_n)_{n\in\mathbb{N}}$ in $W_\Delta^{1,p}$, one has

$$\begin{aligned}
&\int_0^{\sigma(1)} |x_n^\Delta - x_m^\Delta|^p \Delta t \\
&= \int_0^{\sigma(1)} \frac{|x_n^\Delta - x_m^\Delta|^p}{(|x_n^\Delta| + |x_m^\Delta|)^{\frac{p(2-p)}{2}}} (|x_n^\Delta| + |x_m^\Delta|)^{\frac{p(2-p)}{2}} \Delta t \\
&\leq \left(\int_0^{\sigma(1)} \frac{|x_n^\Delta - x_m^\Delta|^2}{(|x_n^\Delta| + |x_m^\Delta|)^{2-p}} \Delta t \right)^{\frac{p}{2}} \left(\int_0^{\sigma(1)} (|x_n^\Delta| + |x_m^\Delta|)^p \Delta t \right)^{\frac{2-p}{2}} \\
&\leq c_p^{-\frac{p}{2}} \left(\int_0^{\sigma(1)} (\phi_p(x_n^\Delta) - \phi_p(x_m^\Delta), x_n^\Delta - x_m^\Delta)\Delta t \right)^{\frac{p}{2}} 2^{\frac{(p-1)(2-p)}{2}} \\
&\quad \left(\int_0^{\sigma(1)} (|x_n^\Delta|^p + |x_m^\Delta|^p) \Delta t \right)^{\frac{2-p}{2}} \\
&\leq c_p^{-\frac{p}{2}} \left(\int_0^{\sigma(1)} (\phi_p(x_n^\Delta) - \phi_p(x_m^\Delta), x_n^\Delta - x_m^\Delta)\Delta t \right)^{\frac{p}{2}} 2^{\frac{(p-1)(2-p)}{2}} \\
&\quad (\|x_n\|^p + \|x_m\|^p)^{\frac{2-p}{2}}.
\end{aligned} \tag{3.16}$$

From (3.15) and (3.16), we have $\int_0^1 |x_n^\Delta - x_m^\Delta|^p \Delta t \to 0$ as $n, m \to \infty$. Then, $\|x_n - x_m\| \to 0$, that is, $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $W_\Delta^{1,p}[0,1]$. By the completeness of $W_\Delta^{1,p}[0,1]$, we have $x_n \to x$ in $W_\Delta^{1,p}[0,1]$. □

Theorem 3.1. *Assume* (A_1), (A_2) *hold. Then, BVP* (1.1) *has at least one positive solution.*

Proof. Obviously,

$$\begin{aligned}
\int_0^{\sigma(1)} F^\sigma(t, x^+)\Delta t &= \int_0^{\sigma(1)} \int_0^{(x^+)^\sigma} f(\sigma(t), s) ds \Delta t \\
&\leq \int_0^{\sigma(1)} \int_0^{(x^+)^\sigma} (|s|^{\gamma-1} e_1^\sigma(t) + e_2^\sigma(t)) ds \Delta t \\
&\leq \frac{1}{\gamma} \int_0^{\sigma(1)} |(x^+)^\sigma|^\gamma e_1^\sigma(t) \Delta t + \int_0^{\sigma(1)} e_2^\sigma(t) |(x^+)^\sigma| \Delta t \\
&\leq \frac{(\sigma(1))^{\frac{\gamma}{q}}}{\gamma} \|x\|^\gamma \int_0^{\sigma(1)} e_1^\sigma(t)\Delta t + (\sigma(1))^{\frac{1}{q}} \|x\| \int_0^{\sigma(1)} e_2^\sigma(t) \Delta t.
\end{aligned}$$

Then,

$$\begin{aligned}
\varphi(x) \geq\ & \min\{\frac{1}{p}|a|_m, c_0\}\|x\|^p - \int_0^{\sigma(1)} F^\sigma(t,x^+)\Delta t \\
& + \int_0^{\sigma(1)} (f^\sigma(t,0),(x^-)^\sigma)\Delta t \\
\geq\ & \min\{\frac{1}{p}|a|_m, c_0\}\|x\|^p - \frac{(\sigma(1))^{\frac{\gamma}{q}}}{\gamma}\|x\|^\gamma \int_0^{\sigma(1)} e_1^\sigma(t)\Delta t \\
& - (\sigma(1))^{\frac{1}{q}} \int_0^{\sigma(1)} e_2^\sigma(t)dt\|x\| \\
\geq\ & \min\{\frac{1}{p}|a|_m, c_0\}\|x\|^p - \frac{(\sigma(1))^{\frac{\gamma}{q}}}{\gamma}\|x\|^\gamma \int_0^{\sigma(1)} e_1^\sigma(t)\Delta t \\
& - (\sigma(1))^{\frac{1}{q}} \int_0^{\sigma(1)} e_2^\sigma(t)dt\|x\|. \qquad (3.17)
\end{aligned}$$

Since $\gamma < p$, one has φ is bounded from blow. Hence, φ has a critical point, that is BVP (1.1) has at least one positive solution. □

With the similar discussion, we have the following theorem.

Theorem 3.2. *Assume* (A_1) *and*
(A_2') *there exists a function* $\tilde{e}_1^\sigma(t)$, $\tilde{e}_2^\sigma(t) \in L_\Delta^1[0,1]$ *satisfying* $\min\{|a(t)|_m, pc_0\} > (\sigma(1))^{\frac{p}{q}} \int_0^{\sigma(1)} \tilde{e}_1^\sigma(t)\Delta t$ *such that*

$$f(t,x) \leq |x|^{p-1}\tilde{e}_1(t) + \tilde{e}_2(t),\ t \in [0,\sigma(1)]_{\mathbb{T}},\ x \in [0,\infty)$$

hold. Then BVP (1.1) *has at least one positive solution.*

4. Existence of negative solutions

Consider the following BVP

$$\begin{cases} (\alpha(t,x^\Delta(t)))^\Delta - a(t)\phi_p(x^\sigma(t)) + f^\sigma(t,-x^-) = 0,\ \Delta\text{-}a.e.\ t \in I \\ x(0) = 0, \\ \beta_1 x^\sigma(1) + \beta_2 x^\Delta(\sigma(1)) = 0. \end{cases} \qquad (4.1)$$

Similarly, we obtain the following lemmas and theorems.

Lemma 4.1. *Assume*
(B_1) $f(t,x) : [0,\sigma(1)]_{\mathbb{T}} \times (-\infty,0] \to (-\infty,0]$,
and $x \in W_\Delta^{1,p}[0,1]$ *is a solution of BVP* (4.1). *Then* $x(t) \leq 0$, $x(t) \not\equiv 0$, $t \in I$.

Lemma 4.2. $x(t) \in W_\Delta^{1,p}[0,1]$ *is a solution of BVP* (4.1) *if and only if* $x(t)$ *is a critical point of the functional*

$$\varphi(x) = \int_0^{\sigma(1)} G(t,x^\Delta)\Delta t + \frac{1}{p}\int_0^{\sigma(1)} a(t)|x^\sigma|^p\Delta t$$

$$-\int_0^{\sigma(1)}[F^\sigma(t,-x^-(t))+(f^\sigma(t,0),(x^+)^\sigma(t))]\Delta t$$
$$+\frac{\beta_2}{\beta_1}G^\sigma(1,-\frac{\beta_1 x(1)}{\beta_2}). \tag{4.2}$$

Theorem 4.1. *Assume* (B_1) *and the following condition*

(B_2) *there exist* $l_1^\sigma(t)$, $l_2^\sigma(t)\in L^1_\Delta[0,1]$ *and a positive constant* $\gamma'<p$ *such that*

$$f(t,x)\geq -|x|^{\gamma'-1}l_1(t)-l_2(t),\ t\in[0,\sigma(1)]_{\mathbb{T}},\ x\in(-\infty,0],$$

hold. Then BVP (1.1) *has at least one negative solution.*

Theorem 4.2. *Assume* (B_1) *and*

(B_2') *there exists a function* $\tilde{l}_1^\sigma(t)$, $\tilde{l}_2^\sigma(t)\in L^1_\Delta[0,1]$ *satisfying* $\min\{|a(t)|_m, pc_0\}>(\sigma(1))^{\frac{p}{q}}\int_0^{\sigma(1)}\tilde{l}_1^\sigma(t)\Delta t$ *such that*

$$f(t,x)\geq -|x|^{p-1}\tilde{l}_1(t)-\tilde{l}_2(t),\ t\in[0,\sigma(1)]_{\mathbb{T}},\ x\in(-\infty,0],$$

hold. Then BVP (1.1) *has at least one negative solution.*

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