Bull. Korean Math. Soc. **49** (2012), No. 6, pp. 1163–1178
http://dx.doi.org/10.4134/BKMS.2012.49.6.1163

EINSTEIN HALF LIGHTLIKE SUBMANIFOLDS WITH SPECIAL CONFORMALITIES

Dae Ho Jin

Abstract. In this paper, we study the geometry of Einstein half lightlike submanifolds M of a semi-Riemannian space form $\bar{M}(c)$ subject to the conditions: (a) M is screen conformal, and (b) the coscreen distribution of M is a conformal Killing one. The main result is a classification theorem for screen conformal Einstein half lightlike submanifolds of a Lorentzian space form with a conformal Killing coscreen distribution.

1. Introduction

A submanifold M of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is called a *lightlike submanifold* of $\bar{M}$ if its radical distribution $Rad(TM) = TM \cap TM^{\perp}$ is a vector subbundle of the tangent bundle TM, of rank $r(> 0)$. A codimension 2 lightlike submanifold M is called a *half lightlike submanifold* if $\text{rank}(Rad(TM)) = 1$. Then there exists two complementary non-degenerate distributions $S(TM)$ and $S(TM^{\perp})$ of $Rad(TM)$ in TM and $TM^{\perp}$ respectively, which called the *screen* and *coscreen distribution* on M, such that

$$TM = Rad(TM) \oplus_{orth} S(TM)\,,\; TM^{\perp} = Rad(TM) \oplus_{orth} S(TM^{\perp}), \tag{1.1}$$

where the symbol $\oplus_{orth}$ denotes the orthogonal direct sum. We denote such a half lightlike submanifold by $M = (M, g, S(TM))$. Denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $F(M)$ module of smooth sections of any vector bundle E over M. Then there exist a non-null section u on $S(TM^{\perp})$ and a null section ξ on $Rad(TM)$ such that

$$\bar{g}(u, u) = \epsilon, \quad \bar{g}(\xi, v) = 0, \ \forall\, v \in \Gamma(TM^{\perp}),$$

where $\epsilon = \pm 1$. Consider the orthogonal complementary distribution $S(TM)^{\perp}$ to $S(TM)$ in $T\bar{M}$. Certainly ξ and u belong to $\Gamma(S(TM)^{\perp})$. Thus we have

$$S(TM)^{\perp} = S(TM^{\perp}) \oplus_{orth} S(TM^{\perp})^{\perp},$$

Received June 18, 2009.
2010 *Mathematics Subject Classification.* Primary 53C25, 53C40, 53C50.
Key words and phrases. half lightlike submanifold, screen conformal, conformal Killing distribution.

where $S(TM^\perp)^\perp$ is the orthogonal complementary to $S(TM^\perp)$ in $S(TM)^\perp$. For any null section ξ of $Rad(TM)$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a uniquely defined vector field $N \in \Gamma(ltr(TM))$ [4] satisfying

$$\bar{g}(\xi, N) = 1, \ \bar{g}(N, N) = \bar{g}(N, X) = \bar{g}(N, u) = 0, \ \forall \ X \in \Gamma(S(TM)). \tag{1.2}$$

We call $ltr(TM)$, N and $tr(TM) = S(TM^\perp) \oplus_{orth} ltr(TM)$ the *lightlike transversal vector bundle, lightlike transversal vector field* and *transversal vector bundle* of M with respect to $S(TM)$ respectively. Then the tangent bundle $T\bar{M}$ of the ambient manifold $\bar{M}$ is decomposed as follows:

$$\begin{aligned} T\bar{M} &= TM \oplus tr(TM) = \{Rad(TM) \oplus tr(TM)\} \oplus_{orth} S(TM) \\ &= \{Rad(TM) \oplus ltr(TM)\} \oplus_{orth} S(TM^\perp) \oplus_{orth} S(TM). \end{aligned} \tag{1.3}$$

Example 1. Suppose M is a surface M of R^4_1 given by the equations

$$x_3 = \sqrt{x_1^2 - x_2^2}, \quad x_4 = \sqrt{1 + x_1^2}.$$

Then we derive $TM = Span\{\xi, U\}$ and $TM^\perp = Span\{\xi, u\}$, where

$$\begin{aligned} &U = x_3x_4\partial_1 + x_1x_4\partial_3 + x_1x_3\partial_4, \\ &\xi = x_1\partial_1 + x_2\partial_2 + x_3\partial_3, \qquad u = x_1\partial_1 + x_4\partial_4. \end{aligned}$$

It follows that $Rad(TM)$ is a distribution on M of rank 1 spanned by ξ. Hence M is a half-lightlike submanifold of R^4_1 such that $S(TM) = Span\{U\}$ and $S(TM^\perp) = Span\{u\}$. Then the lightlike transversal bundle $ltr(TM)$ and the transversal bundle $tr(TM)$ with respect to the screen distribution $S(TM)$ are given by $ltr(TM) = Span\{N\}$ and $tr(TM) = Span\{N, u\}$, where

$$N = -\frac{1}{2x_1^2}(x_1\partial_1 - x_2\partial_2 - x_3\partial_3).$$

The classification of Einstein hypersurfaces M in Euclidean spaces $\mathbb{R}^{n+1}$ was first studied by Fialkow [7] and Thomas [14] in the middle of 1930's. It was proved that if M is a connected Einstein hypersurface ($n \geq 3$) such that $Ric = \gamma g$ for some constant γ, then γ is non-negative. Moreover,

(1) if $\gamma > 0$, then M is contained in an n-sphere and

(2) if $\gamma = 0$, then M is locally isometric to $\mathbb{R}^n$.

The objective of this paper is the study of half lightlike version of above classical results. For this reason, we consider only screen conformal half lightlike submanifolds with a conformal Killing coscreen distribution. In Section 2, we investigate geometric properties for screen conformal half lightlike submanifolds M of a semi-Riemannian space form $(\bar{M}^{m+3}(c), \bar{g})$, $m > 2$, with a conformal Killing coscreen distribution. In the last Section 3, we prove our main classification theorem for screen conformal Einstein half lightlike submanifolds M of a Lorentzian space form with a conformal Killing coscreen distribution (Theorem 3.2). Recall the following structure equations.

Let $\bar{\nabla}$ be the Levi-Civita connection of $\bar{M}$ and P the projection morphism of TM on $S(TM)$ with respect to the decomposition (1.1). Then the local Gauss and Weingartan formulas M and $S(TM)$ are given respectively by

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y)N + D(X, Y)u, \tag{1.4}$$

$$\bar{\nabla}_X N = -A_N X + \tau(X)N + \rho(X)u, \tag{1.5}$$

$$\bar{\nabla}_X u = -A_u X + \phi(X)N; \tag{1.6}$$

$$\nabla_X PY = \nabla^*_X PY + C(X, PY)\xi, \tag{1.7}$$

$$\nabla_X \xi = -A^*_\xi X - \tau(X)\xi, \tag{1.8}$$

for any $X, Y \in \Gamma(TM)$, where ∇ and ∇^* are induced linear connections on TM and $S(TM)$ respectively, the bilinear forms B and D on TM are called the *local lightlike* and *screen second fundamental forms* of M respectively, C is called the *local radical second fundamental form* on $S(TM)$. A_N, A^*_ξ and A_u are linear operators on $\Gamma(TM)$ and τ, ρ and ϕ are 1-forms on TM.

Since $\bar{\nabla}$ is torsion-free, ∇ is also torsion-free, and B and D are symmetric. From the facts $B(X, Y) = \bar{g}(\bar{\nabla}_X Y, \xi)$ and $D(X, Y) = \epsilon\bar{g}(\bar{\nabla}_X Y, u)$, we know that B and D are independent of the choice of a screen distribution and satisfy

$$B(X, \xi) = 0, \quad D(X, \xi) = -\epsilon\phi(X), \quad \forall\, X \in \Gamma(TM). \tag{1.9}$$

The induced connection ∇ on M is not metric and satisfies

$$(\nabla_X g)(Y, Z) = B(X, Y)\,\eta(Z) + B(X, Z)\,\eta(Y) \tag{1.10}$$

for all $X, Y, Z \in \Gamma(TM)$, where η is a 1-form on TM such that

$$\eta(X) = \bar{g}(X, N), \ \forall\, X \in \Gamma(TM). \tag{1.11}$$

But we show that ∇^* is metric. The above three local second fundamental forms on TM and $S(TM)$ are related to their shape operators by

$$B(X, Y) = g(A^*_\xi X, Y), \qquad \bar{g}(A^*_\xi X, N) = 0, \tag{1.12}$$

$$C(X, PY) = g(A_N X, PY), \quad \bar{g}(A_N X, N) = 0, \tag{1.13}$$

$$\epsilon D(X, PY) = g(A_u X, PY), \quad \bar{g}(A_u X, N) = \epsilon\rho(X), \tag{1.14}$$

$$\epsilon D(X, Y) = g(A_u X, Y) - \phi(X)\eta(Y). \tag{1.15}$$

From (1.12), A^*_ξ is $S(TM)$-valued and self-adjoint on $\Gamma(TM)$ such that

$$A^*_\xi \xi = 0. \tag{1.16}$$

We denote by $\bar{R}$, R and R^* the curvature tensors of the Levi-Civita connection $\bar{\nabla}$ of $\bar{M}$, the induced connection ∇ of M and the induced connection ∇^* on $S(TM)$ respectively. Using the Gauss-Weingarten equations for M and $S(TM)$, we obtain the Gauss-Codazzi equations for M and $S(TM)$:

$$\begin{aligned} \bar{g}(\bar{R}(X,Y)Z, PW) = g(R(X,Y)Z, PW) & \\ + B(X,Z)C(Y,PW) - B(Y,Z)C(X,PW) & \\ + \epsilon\{D(X,Z)D(Y,PW) - D(Y,Z)D(X,PW)\}, & \end{aligned} \tag{1.17}$$

$$\bar{g}(\bar{R}(X,Y)Z,\,\xi) = (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) + B(Y,Z)\tau(X) - B(X,Z)\tau(Y) + D(Y,Z)\phi(X) - D(X,Z)\phi(Y), \tag{1.18}$$

$$\bar{g}(\bar{R}(X,Y)Z,\,N) = \bar{g}(R(X,Y)Z,\,N) + \epsilon\{D(X,Z)\rho(Y) - D(Y,Z)\rho(X)\}, \tag{1.19}$$

$$\bar{g}(\bar{R}(X,Y)\xi,\,N) = g(A^*_\xi X,\,A_{_N} Y) - g(A^*_\xi Y,\,A_{_N} X) + \rho(X)\phi(Y) - \rho(Y)\phi(X) - 2d\tau(X,Y), \tag{1.20}$$

$$\bar{g}(\bar{R}(X,Y)Z,\,u) = \epsilon\{(\nabla_X D)(Y,Z) - (\nabla_Y D)(X,Z) + B(Y,Z)\rho(X) - B(X,Z)\rho(Y)\}, \tag{1.21}$$

$$\bar{g}(R(X,Y)PZ,\,PW) = g(R^*(X,Y)PZ,\,PW) + C(X,PZ)B(Y,PW) - C(Y,PZ)B(X,PW), \tag{1.22}$$

$$g(R(X,Y)PZ,\,N) = (\nabla_X C)(Y,PZ) - (\nabla_Y C)(X,PZ) + C(X,PZ)\tau(Y) - C(Y,PZ)\tau(X) \tag{1.23}$$

for all $X,\,Y,\,Z \in \Gamma(TM)$. The Ricci curvature tensor $\bar{Ric}$ of $\bar{M}$ and the induced Ricci type tensor $R^{(0,\,2)}$ of M are defined by

$$\bar{Ric}(X,Y) = trace\{Z \to \bar{R}(Z,X)Y\}, \quad \forall\, X,\,Y \in \Gamma(T\bar{M}), \tag{1.24}$$

$$R^{(0,\,2)}(X,Y) = trace\{Z \to R(Z,X)Y\}, \quad \forall\, X,\,Y \in \Gamma(TM). \tag{1.25}$$

Consider the induced quasi-orthonormal frame fields $\{\xi; W_a\}$ on M such that $Rad\,TM = Span\{\xi\}$ and $S(TM) = Span\{W_a\}$ and let $E = \{\xi, W_a; u, N\}$ be the corresponding frame fields on $\bar{M}$. Let $\epsilon_a = g(W_a, W_a)$ be the sign of W_α. Using this quasi-orthonormal frame, (1.24) and (1.25) reduce respectively to

$$\bar{Ric}(X,Y) = \sum_{a=1}^{m} \epsilon_a\, \bar{g}(\bar{R}(W_a,X)Y,\,W_a) + \bar{g}(\bar{R}(\xi,X)Y,\,N) + \epsilon\, \bar{g}(\bar{R}(u,X)Y,\,u) + \bar{g}(\bar{R}(N,X)Y,\,\xi), \tag{1.26}$$

$$R^{(0,\,2)}(X,Y) = \sum_{a=1}^{m} \epsilon_a\, g(R(W_a,X)Y,\,W_a) + \bar{g}(R(\xi,X)Y,\,N) \tag{1.27}$$

for any $X,\,Y \in \Gamma(TM)$. Substituting (1.17) and (1.19) in (1.26) and then, using (1.12), (1.13) and (1.27), we obtain

$$\begin{aligned} R^{(0,\,2)}(X,Y) = \bar{Ric}(X,Y) &+ B(X,Y)trA_{_N} + D(X,Y)trA_u \\ &- g(A_{_N}X, A^*_\xi Y) - \epsilon\, g(A_uX, A_uY) + \rho(X)\phi(Y) \\ &- \bar{g}(\bar{R}(\xi,Y)X,\,N) - \epsilon\, \bar{g}(\bar{R}(u,Y)X,\,u) \end{aligned} \tag{1.28}$$

for any $X,\,Y \in \Gamma(TM)$. A tensor field $R^{(0,\,2)}$ of M is called its *induced Ricci tensor* if it is symmetric. A symmetric $R^{(0,\,2)}$ tensor will be denoted by *Ric*.

Note 1. Using (1.20), (1.28) and the first Bianchi's identity, we obtain

$$R^{(0,2)}(X,Y) - R^{(0,2)}(Y,X) = 2d\tau(X,Y), \quad \forall\, X,Y \in \Gamma(TM).$$

It follow that $R^{(0,2)}$ is a symmetric, if and only if, each 1-form τ is closed, i.e., $d\tau = 0$, on any $\mathcal{U} \subset M$ [6]. Therefore, suppose $R^{(0,2)}$ is symmetric, then there exists a smooth function f on $\mathcal{U}$ such that $\tau = df$. Consequently we get $\tau(X) = X(f)$. If we take $\bar{\xi} = \alpha\xi$, it follows that $\tau(X) = \bar{\tau}(X) + X(\ln\alpha)$. Setting $\alpha = \exp(f)$ in this equation, we get $\bar{\tau}(X) = 0$ for any $X \in \Gamma(TM_{|\mathcal{U}})$. In the sequel, we call the pair $\{\xi,\, N\}$ on $\mathcal{U}$ such that the corresponding 1-form τ vanishes the *canonical null pair* [9] of M.

2. Screen conformal submanifolds

Definition. A half lightlike submanifold $(M, g, S(TM))$ of $\bar{M}$ is said to be *screen conformal* [1] if there exists a non-vanishing smooth function φ on a neighborhood $\mathcal{U}$ in M such that $A_N = \varphi\, A^*_\xi$, or equivalently,

$$C(X,PY) = \varphi\, B(X,Y), \; \forall\, X,\, Y \in \Gamma(TM). \tag{2.1}$$

In general, $S(TM)$ is not necessarily integrable. From (1.7) and (1.13), we get $g(A_{N}X,Y) - g(X,A_{N}Y) = C(X,Y) - C(Y,X) = \eta([X,Y])$ for all $X,Y \in \Gamma(S(TM))$. Thus A_{N} is self-adjoint on $S(TM)$ with respect to g if and only if C is symmetric on $S(TM)$ if and only if $\eta([X,Y]) = 0$ for all $X,Y \in \Gamma(S(TM))$, i.e., $S(TM)$ is integrable [4].

Note 2. For a screen conformal M, since C is symmetric on $S(TM)$, the screen distribution $S(TM)$ is integrable. Thus M is locally a product manifold $L \times M^*$ where L is a null curve and M^* is a leaf of $S(TM)$ [5].

Example 2. Consider a surface M in R^5_2 given by the equation

$$x_4 = \sqrt{x_1^2 + x_2^2}, \quad x_5 = \sqrt{1 - x_3^2}.$$

Then we have $TM = Span\{\xi, U, V\}$ and $TM^\perp = Span\{\xi, u\}$, where

$$U = x_4\partial_1 + x_1\partial_4, \qquad V = x_5\partial_3 - x_3\partial_5,$$
$$\xi = x_1\partial_1 + x_2\partial_2 + x_4\partial_4, \qquad u = x_3\partial_3 + x_5\partial_5.$$

By direct calculations we check that $Rad(TM)$ is a distribution on M of rank 1 spanned by ξ. Hence M is a half-lightlike submanifold of R^5_2 such that $S(TM) = Span\{U,V\}$ and $S(TM^\perp) = Span\{u\}$. Then the lightlike transversal bundle $ltr(TM)$ of the screen $S(TM)$ is given by

$$ltr(TM) = Span\left\{N = \frac{1}{2x_2^2}(x_1\partial_1 - x_2\partial_2 + x_4\partial_4)\right\},$$

and the transversal bundle $tr(TM)$ is given by $tr(TM) = Span\{N, u\}$.

Denote by $\bar{\nabla}$ the Levi-Civita connection on R_2^5. By straightforward calculations, we obtain

$$\bar{\nabla}_U U = \xi + 2x_2^2 N, \quad \bar{\nabla}_U V = 0, \quad \bar{\nabla}_U \xi = 2U + \frac{x_1 x_4}{2x_2^2}\xi - x_1 x_4 N,$$
$$\bar{\nabla}_U N = \frac{1}{2x_2^2}U - 2\frac{x_1 x_4}{x_2^2}N + \frac{x_1 x_4}{x_2^2}\xi, \quad \bar{\nabla}_U u = 0,$$
$$\bar{\nabla}_V U = 0, \quad \bar{\nabla}_V V = -2u, \quad \bar{\nabla}_V \xi = 0, \quad \bar{\nabla}_V N = 0, \quad \bar{\nabla}_V u = 2V,$$
$$\bar{\nabla}_\xi U = U + \frac{x_4}{2x_1}\xi + \frac{x_2^2 x_4}{x_1}N, \quad \bar{\nabla}_\xi \xi = \frac{x_4}{x_1}U + \left(\frac{3}{2} + \frac{x_1^2}{2x_2^2}\right)\xi - x_4^2 N,$$
$$\bar{\nabla}_\xi V = 0, \quad \bar{\nabla}_\xi N = -N, \quad \bar{\nabla}_\xi u = 0.$$

Then taking into account of Gauss and Weingarten formulas infer

$$A_\xi^* U = -U, \quad A_\xi^* V = 0, \quad A_N U = -\frac{1}{2x_4^2}U, \quad A_N V = 0, \quad A_N \xi = 0,$$
$$\tau(U) = \tau(V) = \tau(\xi) = 0, \qquad \rho(U) = \rho(V) = \rho(\xi) = 0.$$

Thus $A_N X = (1/2x_4^2)A_\xi^* X$ for any $X \in \Gamma(TM)$ and M is a screen conformal half-lightlike submanifold of R_2^5 with a conformal factor $\varphi = 1/2x_2^2$.

Definition. A vector field X on $\bar{M}$ is said to be a *conformal Killing* [15] if $\bar{\mathcal{L}}_X \bar{g} = -2\delta\bar{g}$, where δ is a non-vanishing smooth function on $\bar{M}$ and $\bar{\mathcal{L}}_X$ denotes the Lie derivative with respect to X. In particular, if $\delta = 0$, then X is called a *Killing*. A distribution $\mathcal{G}$ on $\bar{M}$ is said to be a *conformal Killing* (*Killing*) if each vector field belonging to $\mathcal{G}$ is a conformal Killing (Killing).

Theorem 2.1. *Let $(M, g, S(TM))$ be a half lightlike submanifold of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then the coscreen distribution is a conformal Killing if and only if $D(X,Y) = \epsilon\delta\, g(X,Y)$ for any X, $Y \in \Gamma(TM)$.*

Proof. By straightforward calculations and use (1.6) and (1.15), we have

$$(\bar{\mathcal{L}}_u \bar{g})(X,Y) = \bar{g}(\bar{\nabla}_X u, Y) + \bar{g}(X, \bar{\nabla}_Y u),$$
$$\bar{g}(\bar{\nabla}_X u, Y) = -g(A_u X, Y) + \phi(X)\eta(Y) = -\epsilon D(X,Y)$$

for any X, $Y \in \Gamma(TM)$. Therefore, we obtain $(\bar{\mathcal{L}}_u \bar{g})(X,Y) = -2\epsilon D(X,Y)$. □

Let $(M, g, S(TM))$ be a screen conformal half lightlike submanifold of a semi-Riemannian space form $(\bar{M}(c), \bar{g})$ with a conformal Killing coscreen. For all X, Y, Z, $W \in \Gamma(TM)$, by (1.9), (1.14) and (1.15), we have

$$D(X,Y) = \epsilon\delta\, g(X,Y), \quad \phi(X) = 0, \quad A_u X = \delta\, PX + \epsilon\rho(X)\xi. \tag{2.2}$$

Using (2.1) and (2.2), the Gauss equations (1.17) and (1.22) reduce to

$$\begin{aligned} & g(R(X,Y)Z,\, PW) \\ &= (c + \epsilon\delta^2)\{g(Y,Z)g(X,PW) - g(X,Z)g(Y,PW)\} \\ &\quad + \varphi\{B(Y,Z)B(X,PW) - B(X,Z)B(Y,PW)\}, \end{aligned} \tag{2.3}$$

$$\begin{aligned}(2.4)\qquad & g(R^*(X,Y)PZ,\, PW)\\ &= (c+\epsilon\delta^2)\{g(Y,PZ)g(X,PW) - g(X,PZ)g(Y,PW)\}\\ &\quad + 2\varphi\{B(Y,PZ)B(X,PW) - B(X,PZ)B(Y,PW)\}\end{aligned}$$

respectively. From (1.18) with $\phi = 0$ and (1.21), we have

$$(2.5)\qquad (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) = B(X,Z)\tau(Y) - B(Y,Z)\tau(X),$$

$$(2.6)\qquad (\nabla_X D)(Y,Z) - (\nabla_Y D)(X,Z) = B(X,Z)\rho(Y) - B(Y,Z)\rho(X).$$

Differentiating the first equation of (2.2) and using (2.6), we have

$$\begin{aligned}(2.7)\qquad & \{\delta\eta(X) - \epsilon\rho(X)\}B(Y,Z) - \{\delta\eta(Y) - \epsilon\rho(Y)\}B(X,Z)\\ &= X[\delta]\, g(Y,Z) - Y[\delta]\, g(X,Z).\end{aligned}$$

Replacing Y by ξ in the last equation and using (1.9), we obtain

$$(2.8)\qquad \{\delta - \epsilon\rho(\xi)\}B(X,Z) = \xi[\delta]\, g(X,Z).$$

Using (1.19), (1.23), (2.1) and (2.5), we obtain

$$\begin{aligned}(2.9)\qquad & \{X[\varphi] - 2\varphi\tau(X)\}B(Y,PZ) - \{Y[\varphi] - 2\varphi\tau(Y)\}B(X,PZ)\\ &= \{c\,\eta(X) + \delta\rho(X)\}g(Y,PZ) - \{c\,\eta(Y) + \delta\rho(Y)\}g(X,PZ).\end{aligned}$$

Replacing Y by ξ in the last equation and using (1.9), we obtain

$$(2.10)\qquad \{\xi[\varphi] - 2\varphi\tau(\xi)\}B(X,PZ) = (c + \delta\rho(\xi))g(X,PZ).$$

Theorem 2.2. *Let $(M, g, S(TM))$ be a screen conformal half lightlike submanifold of a semi-Riemannian space form $(\bar{M}^{m+3}(c),\, \bar{g})$, $m > 2$, with a conformal Killing coscreen distribution. Then we have $c + \delta\rho(\xi) = 0$.*

Proof. Assume that $c+\delta\rho(\xi) \neq 0$. Then we have $\xi[\varphi] - 2\varphi\tau(\xi) \neq 0$ and $B \neq 0$ by virtue of (2.10). Thus, from (1.9), (2.1) and (2.10), we have

$$(2.11)\qquad B(X,Y) = \sigma\, g(X,Y),\ C(X,PY) = \varphi\sigma\, g(X,Y),\ \forall\, X,\, Y \in \Gamma(TM),$$

where $\sigma = (c + \delta\rho(\xi))(\xi[\varphi] - 2\varphi\tau(\xi))^{-1} \neq 0$. From the first equation of (2.2) and (2.11), M is totally umbilical in $\bar{M}$ and $S(TM)$ is also totally umbilical in M and $\bar{M}$. As $\bar{M}$ has a constant curvature c, from (2.4) and (2.11), we have

$$R^*(X,Y)Z = (c + 2\varphi\sigma^2 + \epsilon\delta^2)\{g(Y,Z)X - g(X,Z)Y\}$$

for all $X, Y, Z \in \Gamma(S(TM))$. Let M^* be the leaf of $S(TM)$ and Ric^* be the Ricci tensor of M^*. Then, from the last equation, we have

$$Ric^*(X,Y) = (c + 2\varphi\sigma^2 + \epsilon\delta^2)(m-1)g(X,Y), \quad \forall\, X, Y \in \Gamma(S(TM)).$$

Thus M^* is Einstein. As $m > 2$, $(c + 2\varphi\sigma^2 + \epsilon\delta^2)$ is a constant and M^* is a space of constant curvature $(c + 2\varphi\sigma^2 + \epsilon\delta^2)$. Differentiating the first equation of (2.11) and using (1.10) and (2.5), we have

$$\{X[\sigma] + \sigma\tau(X) - \sigma^2\eta(X)\}g(Y,Z) = \{Y[\sigma] + \sigma\tau(Y) - \sigma^2\eta(Y)\}g(X,Z)$$

for all $X,\, Y,\, Z \in \Gamma(TM)$. Replacing Y by ξ in this equation, we have $\xi[\sigma] = \sigma^2 - \sigma\tau(\xi)$. From (2.8) and (2.11), we have $\xi[\delta] = \sigma\delta - \epsilon\sigma\rho(\xi)$. Since $(c+2\varphi\sigma^2+$

$\epsilon\delta^2$) is a constant, we have $\xi[c+2\varphi\sigma^2+\epsilon\delta^2] = 2\sigma(c+2\varphi\sigma^2+\epsilon\delta^2) = 0$. Therefore, as $\sigma \neq 0$, we have $c + 2\varphi\sigma^2 + \epsilon\delta^2 = 0$ and consequently we get $R^* = 0$. Thus M^* is a semi-Euclidean space. As the second fundamental form of the totally umbilical semi-Euclidean space M^* as a submanifold of the semi-Riemannian space form $\bar{M}(c)$ vanishes [3, Section 2.3], we get $C = 0$. Consequently, from (2.1), we get $B = 0$ and $c + \delta\rho(\xi) = 0$ due to (2.10). It is a contradiction to $c + \delta\rho(\xi) \neq 0$. Thus we have $c + \delta\rho(\xi) = 0$. □

Corollary 2.3 ([10]). *Let $(M, g, S(TM))$ be a screen conformal half lightlike submanifold of a semi-Riemannian space form $(\bar{M}^{m+3}(c), \bar{g})$, $m > 2$, with a Killing coscreen distribution. Then we have $c = 0$ and $\delta = 0$.*

Theorem 2.4. *Let $(M, g, S(TM))$ be a screen conformal Einstein half lightlike submanifold of a semi-Riemannian space form $(\bar{M}^{m+3}(c), \bar{g})$, $m > 2$, with a conformal Killing coscreen distribution of conformal factor δ. Then the leaf M^* of $S(TM)$ is an Einstein manifold and δ is a constant.*

Proof. From (2.3) and (2.4), we show that

$$\begin{aligned} &2g(R(X,Y)PZ,\, PW) \\ &= g(R^*(X,Y)PZ,\, PW) + (c+\epsilon\delta^2)\{g(Y,PZ)g(X,PW) - g(X,PZ)g(Y,PW)\} \end{aligned} \tag{2.12}$$

for all $X, Y, Z, W \in \Gamma(TM)$. Using the equations (1.27), (2.12) and the fact that $\bar{g}(R(\xi,X)Y,N) = (c + \delta\rho(\xi))g(X,Y) = 0$, we get

$$2R^{(0,2)}(X,Y) = Ric^*(X,Y) + (m-1)(c+\epsilon\delta^2)g(X,Y).$$

This shows that the induced tensor $R^{(0,2)}$ on M is symmetric. Thus M admits a symmetric Ricci tensor and $R^{(0,2)} = Ric$. Since M is Einstein, i.e., $Ric = \gamma g$, where γ is a constant if $m > 2$, the last equation reduces to

$$Ric^*(X,Y) = \{2\gamma - (m-1)(c+\epsilon\delta^2)\}g(X,Y),\ \forall X, Y \in \Gamma(TM). \tag{2.13}$$

Thus M^* is also Einstein. Since $m > 2$, the function $\{2\gamma - (m-1)(c+\epsilon\delta^2)\}$ is a constant. Therefore, the conformal factor δ is a constant. □

Theorem 2.5. *Let $(M, g, S(TM))$ be a screen conformal Einstein half lightlike submanifold of a semi-Riemannian space form $(\bar{M}^{m+3}(c), \bar{g})$, $m > 2$, with a conformal Killing coscreen distribution of conformal factor δ. If either $\gamma \neq (m-1)(c+\epsilon\delta^2)$ or $\operatorname{rank} A^*_\xi > 0$, then we have $c + \epsilon\delta^2 = 0$.*

Proof. Since M is Einstein, the conformal factor δ is a constant by Theorem 2.4. From (2.8) with $c + \delta\rho(\xi) = 0$, we get $\{c + \epsilon\delta^2\}B(Y,Z) = 0$, or equivalently, $\{c + \epsilon\delta^2\}A^*_\xi X = 0$ for any $X, Y \in \Gamma(TM)$. First, if $\operatorname{rank} A^*_\xi > 0$, we get $c + \epsilon\delta^2 = 0$. Next, if $c + \epsilon\delta^2 \neq 0$, then, since $(c + \epsilon\delta^2)$ is a constant, we have $B(X,Y) = 0$ for any $X, Y \in \Gamma(TM)$. Thus, from (1.27), (2.3) and the fact that $\bar{g}(R(\xi,X)Y,N) = (c + \delta\rho(\xi))g(X,Y) = 0$, we have $\gamma = (m-1)(c + \epsilon\delta^2)$. This implies that if $\gamma \neq (m-1)(c + \epsilon\delta^2)$, then we get $c + \epsilon\delta^2 = 0$. □

Recall the following notion of *null sectional curvature* [2, 5, 6, 8]. Let $x \in M$ and ξ be a null vector of T_xM. A plane H of T_xM is called a *null plane* directed by ξ if it contains ξ, $g_x(\xi,\, W) = 0$ for any $W \in H$ and there exists $W_o \in H$ such that $g_x(W_o,\, W_o) \neq 0$. Then, the null sectional curvature of H, with respect to ξ and ∇, is defined as a real number

$$K_\xi(H) = \frac{g_x(R(\xi, W)W,\, \xi)}{g_x(W, W)},$$

where $W \neq 0$ is any vector in H independent with ξ. It is easy to see that $K_\xi(H)$ is independent of W but depends in a quadratic fashion on ξ. An $n(\geq 3)$-dimensional Lorentzian manifold is of constant curvature if and only if its null sectional curvatures are everywhere zero [12].

Theorem 2.6. *Let $(M, g, S(TM))$ be a screen conformal half lightlike submanifold of a semi-Riemannian space form $(\bar{M}^{m+3}(c),\, \bar{g})$, $m > 2$, with a conformal Killing coscreen distribution. Then every null plane H of T_xM directed by ξ has everywhere zero null sectional curvatures.*

Proof. From (1.9), (1.19) and (2.3), we show that $g(R(\xi, X)Y, PW) = 0$ and $g(R(\xi, X)Y, N) = (c + \delta\rho(\xi))g(X, Y) = 0$ for any $X,\, Y \in \Gamma(TM)$. Thus the curvature tensor R of M satisfies $R(\xi, X)Y = 0$ for any $X,\, Y \in \Gamma(TM)$. Thus $K_\xi(H) = \frac{g_x(R(\xi,W)W,\,\xi)}{g_x(W,W)} = 0$ for any null plane H of T_xM directed by ξ. □

3. Einstein submanifolds

In this section, let $(M, g, S(TM))$ be a screen conformal half lightlike submanifold of a Lorentzian space form $(\bar{M}(c), \bar{g})$ with a conformal Killing coscreen distribution. Then $\epsilon = 1, \phi = 0$ and $S(TM)$ is a Riemannian and integrable vector bundle. As $\bar{M}$ is a Lorentzian space form, then $\bar{R}(\xi, Y)X = c\bar{g}(X, Y)\xi$, $\bar{R}(u, X)Y = c\bar{g}(X, Y)u$ and $\bar{Ric}(X, Y) = (m + 2)c\,\bar{g}(X, Y)$. Thus the equation (1.28) reduces to

$$\text{(3.1)} \quad \begin{aligned} Ric(X, Y) = mcg(X, Y) + B(X, Y)\, trA_{_N} + D(X, Y)\, trA_u \\ - \varphi g(A^*_\xi X, A^*_\xi Y) - g(A_u X, A_u Y), \quad \forall\, X,\, Y \in \Gamma(TM). \end{aligned}$$

From (1.16), ξ is an eigenvector field of A^*_ξ corresponding to the eigenvalue 0. Since A^*_ξ is $\Gamma(S(TM))$-valued real self-adjoint operator on $\Gamma(TM)$ with respect to g, A^*_ξ have m real orthonormal eigenvector fields in $S(TM)$ and is diagonalizable. Consider a frame field of eigenvectors $\{\xi, E_1, \ldots, E_m\}$ of A^*_ξ such that $\{E_1, \ldots, E_m\}$ is an orthonormal frame field of $S(TM)$. Then

$$A^*_\xi E_i = \lambda_i E_i, \quad 1 \leq i \leq m.$$

Let M be an Einstein manifold. Then $Ric = \gamma g$ and (3.1) reduces to

$$\text{(3.2)} \qquad g(A^*_\xi X,\, A^*_\xi Y) - sg(A^*_\xi X,\, Y) + Fg(X, Y) = 0,$$

where $s = trA^*_\xi$ is the trace of A^*_ξ and $F = \varphi^{-1}\{\gamma - mc - \delta\rho(\xi) + (1-m)\delta^2\}$ is a smooth function. In case $m > 2$, we show that $F = \varphi^{-1}\{\gamma - (m-1)(c+\delta^2)\}$. Put $X = Y = E_i$ in (3.2), the eigenvalue λ_i is a solution of

$$x^2 - sx + F = 0. \tag{3.3}$$

The equation (3.3) has at most two distinct solutions. Assume that there exists $p \in \{0, 1, \ldots, m\}$ such that $\lambda_1 = \cdots = \lambda_p = \alpha$ and $\lambda_{p+1} = \cdots = \lambda_m = \beta$, by renumbering if necessary. From (3.3), we have

$$s = \alpha + \beta = p\alpha + (m-p)\beta, \qquad \alpha\beta = F. \tag{3.4}$$

Although $S(TM)$ is not unique, it is canonically isomorphic to the factor vector bundle $S(TM)^\sharp = TM/Rad\,TM$ considered by Kupeli [11]. Thus all $S(TM)$ are isomorphic. For this reason, in the sequel, let $(M, g, S(TM))$ be a screen conformal Einstein half lightlike submanifold equipped with the canonical null pair $\{\xi, N\}$ of a Lorentzian space form $(\bar{M}^{m+3}(c), \bar{g})$, $m > 2$, with a conformal Killing coscreen distribution.

Theorem 3.1. *Let $(M, g, S(TM))$ be a screen conformal Einstein half lightlike submanifold of a Lorentzian space form $(\bar{M}^{m+3}(c), \bar{g}), m > 2$, with a conformal Killing coscreen distribution. Then M is locally a product manifold $L \times M_\alpha \times M_\beta$, where L is a null curve and M_α and M_β are totally umbilical leaves of some distributions of M.*

Proof. If (3.3) has only one solution α, then, since M is screen conformal, $M = L \times M^* \cong L \times M^* \times \{x\}$ for any $x \in \bar{M}$, where $M_\alpha = M^*$ is a leaf of $S(TM)$ and $M_\beta = \{x\}$ is a leaf of the trivial vector bundle $\{0\}$. Since $B(X, Y) = g(A^*_\xi X, Y) = \alpha g(X, Y)$ for all $X, Y \in \Gamma(TM)$, we get $C(X, Y) = \varphi\alpha g(X, Y)$ for all $X, Y \in \Gamma(TM)$ by (2.1). Thus M^* is totally umbilical and $\{x\}$ is also totally umbilical. In this case, our assertion is true.

Assume that (3.3) has exactly two distinct solutions α and β. If $p = 0$ or $p = m$, then we also show that $M = L \times M^* \cong L \times M^* \times \{x\}$ for any $x \in \bar{M}$, and $M^* = M_\alpha$ and $M_\beta = \{x\}$ (if $p = m$) or M_β and $M_\alpha = \{x\}$ (if $p = 0$). In these cases, M^* is totally umbilical. If $0 < p < m$. Consider the following four distributions D_α, D_β, D^s_α and D^s_β on M:

$$\Gamma(D_\alpha) = \{X \in \Gamma(TM) \mid A^*_\xi X = \alpha PX\}, \quad D^s_\alpha = PD_\alpha;$$
$$\Gamma(D_\beta) = \{U \in \Gamma(TM) \mid A^*_\xi U = \beta PU\}, \quad D^s_\beta = PD_\beta.$$

Then $D_\alpha \cap D_\beta = Rad(TM)$ and $D^s_\alpha \cap D^s_\beta = \{0\}$.

Since $A^*_\xi PX = A^*_\xi X = \alpha PX$ for all $X \in \Gamma(D_\alpha)$ and $A^*_\xi PU = A^*_\xi U = \beta PU$ for all $U \in \Gamma(D_\beta)$, PX and PU are eigenvector fields of the real symmetric operator A^*_ξ corresponding to the different eigenvalues α and β respectively. Thus $PX \perp_g PU$ and $g(X, U) = g(PX, PU) = 0$, that is, $D_\alpha \perp_g D_\beta$. Also, since $B(X, U) = g(A^*_\xi X, U) = \alpha g(PX, PU) = 0$, we show that $D_\alpha \perp_B D_\beta$.

For any $x \in M$, since $\{E_i\}_{1\le i\le p}$ and $\{E_a\}_{p+1\le a\le m}$ are p and $(m-p)$ smooth linearly independent vector fields of D^s_α and D^s_β respectively, D^s_α and D^s_β are smooth distributions. Also, as $\{\xi,\, E_i\}_{1\le i\le p}$ and $\{\xi,\, E_a\}_{p+1\le a\le m}$ are $(p+1)$ and $(m-p+1)$ smooth linearly independent vector fields of D_α and D_β respectively, D_α and D_β are also smooth distributions on M. Thus D^s_α and D^s_β are orthogonal vector subbundle of $S(TM)$, D^s_α and D^s_β are non-degenerate distributions of rank p and rank $(m-p)$ respectively. Thus $S(TM) = D^s_\alpha \oplus_{orth} D^s_\beta$. Consequently, $TM = Rad(TM) \oplus_{orth} D^s_\alpha \oplus_{orth} D^s_\beta$.

From (3.2), we show that $(A^*_\xi)^2 - (\alpha+\beta)A^*_\xi + \alpha\beta P = 0$. Let $Y \in Im(A^*_\xi - \alpha P)$, then there exists $X \in \Gamma(TM)$ such that $Y = (A^*_\xi - \alpha P)X$. Then $(A^*_\xi - \beta P)Y = 0$ and $Y \in \Gamma(D_\beta)$. Thus $Im(A^*_\xi - \alpha P) \subset \Gamma(D_\beta)$. Since the morphism $A^*_\xi - \alpha P$ maps $\Gamma(TM)$ onto $\Gamma(S(TM))$, we have $Im(A^*_\xi - \alpha P) \subset \Gamma(D^s_\beta)$. By duality, we also have $Im(A^*_\xi - \beta P) \subset \Gamma(D^s_\alpha)$.

For $X,\, Y \in \Gamma(D_\alpha)$ and $U \in \Gamma(D_\beta)$, we have

$$(\nabla_X B)(Y,U) = -g((A^*_\xi - \alpha P)\nabla_X Y, U) + \alpha^2 g(X,Y)\eta(U)$$

and $(\nabla_X B)(Y,U) = (\nabla_Y B)(X,U)$ due to (2.5). Thus $g((A^*_\xi - \alpha P)[X,Y],U) = 0$. As D^s_β is non-degenerate and $Im(A^*_\xi - \alpha P) \subset \Gamma(D^s_\beta)$, we have $(A^*_\xi - \alpha P)[X,Y] = 0$. Thus $[X,Y] \in \Gamma(D_\alpha)$ and D_α is integrable. By duality, D_β is also integrable. Since $S(TM)$ is integrable, for any $X,\, Y \in \Gamma(D^s_\alpha)$, we have $[X,Y] \in \Gamma(D_\alpha)$ and $[X,Y] \in \Gamma(S(TM))$. Thus $[X,Y] \in \Gamma(D^s_\alpha)$ and D^s_α is integrable. So is D^s_β.

For $X,\, Y \in \Gamma(D_\alpha)$ and $Z \in \Gamma(TM)$, we show that

$$\begin{aligned}(\nabla_X B)(Y,Z) = &-g((A^*_\xi - \alpha P)\nabla_X Y, Z) + \alpha^2 g(X,Y)\eta(Z)\\ &+ (X\alpha)\, g(Y,Z) + \alpha^2\eta(Y)\, g(X,Z).\end{aligned}$$

Using this equation and the facts that $(\nabla_X B)(Y,Z) = (\nabla_Y B)(X,Z)$ due to (2.5) and $(A^*_\xi - \alpha P)[X,\, Y] = 0$ for any $X,\, Y \in \Gamma(D_\alpha)$, we have

$$\{X\alpha - \alpha^2\eta(X)\}g(Y,Z) = \{Y\alpha - \alpha^2\eta(Y)\}g(X,Z),\ \forall X,\, Y \in \Gamma(D_\alpha).$$

Therefore, for $X,\, Y \in \Gamma(D^s_\alpha)$ and $Z \in \Gamma(S(TM))$, we obtain $(X\alpha)g(Y,Z) = (Y\alpha)g(X,Z)$. Since $S(TM)$ is non-degenerate, we have $d\alpha(X)Y = d\alpha(Y)X$. Suppose there exists a vector field $X_o \in \Gamma(D^s_\alpha)$ such that $d\alpha(X_o)_x \neq 0$ at each point $x \in M$, then $Y = fX_o$ for any $Y \in \Gamma(D^s_\alpha)$, where f is a smooth function. It follows that all vectors from the fiber $(D^s_\alpha)_x$ are colinear with $(X_o)_x$. It is a contradiction as $\dim((D^s_\alpha)_x) = p > 1$. Thus we have $d\alpha|_{D^s_\alpha} = 0$. By duality, we also have $d\beta|_{D^s_\beta} = 0$. Thus α is a constant along D^s_α and β is a constant along D^s_β. From the first equation of (3.4), we have $(p-1)\alpha = -(m-p-1)\beta$. Thus both α and β are constants along $S(TM)$.

Using (2.9) with $c + \delta\rho(\xi) = 0$ and $\tau = 0$, we have

$$(X\varphi)B(Y,Z) - (Y\varphi)B(X,Z) = \delta\{\rho(PX)g(Y,Z) - \rho(PY)g(X,Z)\} \tag{3.5}$$

for any $X, Y, Z \in \Gamma(TM)$. Take $X, Y, Z \in \Gamma(D^s_\alpha)$, then (3.5) reduces to

$$\{\alpha(X\varphi) - \delta\rho(X)\}Y = \{\alpha(Y\varphi) - \delta\rho(Y)\}X.$$

Since $\dim (D^s_\alpha)_x > 1$, we have $(X\varphi)\alpha = \delta\rho(X)$ for all $X \in \Gamma(D^s_\alpha)$. While, take $X \in \Gamma(D^s_\beta)$ and $Y, Z \in \Gamma(D^s_\alpha)$ in (3.5), we have $(X\varphi)\alpha = \delta\rho(X)$ for all $X \in \Gamma(D^s_\beta)$. Consequently, we obtain $(X\varphi)\alpha = \delta\rho(X)$ for all $X \in \Gamma(S(TM))$. By duality, we get $(X\varphi)\beta = \delta\rho(X)$ for all $X \in \Gamma(S(TM))$. Thus we have $(X\varphi)\alpha = (X\varphi)\beta$ for all $X \in \Gamma(S(TM))$. Since $\alpha \neq \beta$, we have $X\varphi = 0$ for all $X \in \Gamma(S(TM))$, that is, φ is a constant along $S(TM)$. Take $X, Y \in \Gamma(D^s_\alpha)$ in (2.10), we have $\xi[\varphi]\alpha = 0$. Also, take $X, Y \in \Gamma(D^s_\beta)$ in (2.10), we have $\xi[\varphi]\beta = 0$. Since $(\alpha, \beta) \neq (0, 0)$, we have $\xi[\varphi] = 0$. Thus we have $X\varphi = 0$ for all $X \in \Gamma(TM)$, i.e., φ is a constant on M.

For all $X \in \Gamma(D^s_\alpha)$ and $U \in \Gamma(D^s_\beta)$, since $(\nabla_X B)(U, Z) = (\nabla_U B)(X, Z)$,

$$g(\{(A^*_\xi - \beta P)\nabla_X U - (A^*_\xi - \alpha P)\nabla_U X\}, Z) \;=\; 0, \;\forall Z \in \Gamma(S(TM)).$$

As $S(TM)$ is non-degenerate, we get $(A^*_\xi - \beta P)\nabla_X U = (A^*_\xi - \alpha P)\nabla_U X$. Since the left term of the last equation is in $\Gamma(D^s_\alpha)$ and the right term is in $\Gamma(D^s_\beta)$ and $D^s_\alpha \cap D^s_\beta = \{0\}$, we have $(A^*_\xi - \beta P)\nabla_X U = 0$ and $(A^*_\xi - \alpha P)\nabla_U X = 0$. This imply that $\nabla_X U \in \Gamma(D_\beta)$ and $\nabla_U X \in \Gamma(D_\alpha)$. On the other hand, $\nabla_X U = \nabla^*_X U$ and $\nabla_U X = \nabla^*_U X$ due to $D_\alpha \perp_B D_\beta$, we have

$$\nabla_X U \in \Gamma(D^s_\beta), \quad \nabla_U X \in \Gamma(D^s_\alpha), \quad \forall\, X \in \Gamma(D^s_\alpha),\, U \in \Gamma(D^s_\beta). \tag{3.6}$$

For $X, Y \in \Gamma(D^s_\alpha)$ and $U, V \in \Gamma(D^s_\beta)$, since $g(X, U) = 0$, we have

$$g(\nabla_Y X, U) + g(X, \nabla_Y U) = 0, \quad g(\nabla_V U, X) + g(U, \nabla_V X) = 0.$$

Using (3.6), we have $g(X, \nabla_Y U) = g(U, \nabla_V X) = 0$. Thus we show that

$$g(\nabla_Y X, U) = 0\,, \qquad g(X, \nabla_V U) = 0. \tag{3.7}$$

Since the leaf M^* of $S(TM)$ is a semi-Riemannian manifold and $S(TM) = D^s_\alpha \oplus_{orth} D^s_\beta$, where D^s_α and D^s_β are integrable and parallel distributions with respect to the induced connection ∇^* on M^* due to (3.7), by the decomposition theorem of de Rham [13], we have $M^* = M_\alpha \times M_\beta$, where M_α and M_β are some leaves of D^s_α and D^s_β respectively. Thus we have our theorem. □

Theorem 3.2. *Let $(M, g, S(TM))$ be a screen conformal half lightlike submanifold of a Lorentzian space form $(\bar{M}^{m+3}(c), \bar{g})$, $m > 2$, with a conformal Killing coscreen distribution. If M is Einstein, i.e., $Ric = \gamma g$, then M is locally a product manifold $L \times M_\alpha \times M_\beta$, where L is a null curve and M_α and M_β are totally umbilical leaves of some distributions of M:*

1. *If $\gamma \neq (m-1)(c + \delta^2)$, then either M_α or M_β is an m-dimensional Einstein Riemannian space form which is isometric to a sphere $(\gamma > 0)$ or a hyperbolic space $(\gamma < 0)$ and the other is a point on M.*

(2) *If* $\gamma = (m-1)(c+\delta^2)$, *then* M_α *is an* $(m-1)$ *or* m*-dimensional Einstein Riemannian space form which is isometric to a sphere* $(\gamma > 0)$ *or a hyperbolic space* $(\gamma < 0)$ *or a Euclidean space* $(\gamma = 0)$ *and* M_β *is a spacelike curve or a point on* M.

Proof. First, we prove that $\gamma = 0$ and $\alpha\beta = 0$ if $0 < p < m$: If $0 < p < m$, then, since $\operatorname{rank} A^*_\xi > 0$, we have $c + \delta^2 = 0$ by Theorem 2.5. If $p = 1$ or $p = m-1$, then, from the facts that $(p-1)\alpha + (m-p-1)\beta = 0$ and $m > 2$, we show that if $p = 1$, then $\beta = 0$ and if $p = m-1$, then $\alpha = 0$. Thus $\gamma = \varphi\alpha\beta = 0$. If $1 < p < m-1$, then, from (3.7), we know that $\nabla_U U$ has no component of D_α. Since the projection morphism P maps $\Gamma(D_\beta)$ onto $\Gamma(D^s_\beta)$ and $S(TM) = D^s_\alpha \oplus_{orth} D^s_\beta$,

$$\nabla_U U = P(\nabla_U U) + \eta(\nabla_U U)\xi\,, \quad P(\nabla_U U) \in \Gamma(D^s_\beta).$$

It follows that

$$\begin{aligned} g(\nabla_X \nabla_U U, X) &= g(\nabla_X P(\nabla_U U), X) + \eta(\nabla_U U) g(\nabla_X \xi, X) \\ &= -\alpha\, \eta(\nabla_U U) g(X, X). \end{aligned}$$

As $\eta(\nabla_U U) = -\bar{g}(U, \bar{\nabla}_U N) = g(U, A_{_N} U) = \varphi g(U, A^*_\xi U) = \varphi\beta\, g(U, U)$, we get

$$g(R(X,U)U, X) = -\,\varphi\alpha\beta g(X,X) g(U,U).$$

While, from the Gauss equation (2.3), we have

$$g(R(X,U)U, X) = \varphi\alpha\beta g(X,X) g(U,U),$$

due to $c + \delta^2 = 0$. From the last two equations, we get $\gamma = \varphi\alpha\beta = 0$.

(1) Let $\gamma \neq (m-1)(c+\delta^2)$: In this case, we have $c + \delta^2 = 0$. First, in case $s^2 \neq 4F$. The equation (3.3) has two non-vanishing distinct solutions α and β. If $0 < p < m$, then $\gamma = 0$. This implies that $\gamma = (m-1)(c+\delta^2)$. Therefore, we have $p = 0$ or $p = m$. If $p = 0$, then $M = L \times M^* = L \times \{x\} \times M^*$ and $B(X,Y) = g(A^*_\xi X, Y) = \beta g(X,Y)$ for any $X, Y \in \Gamma(TM)$. From this and (2.1), we show that $C(X,Y) = \varphi\beta g(X,Y)$ for all $X, Y \in \Gamma(TM)$. Thus M^* is totally umbilical. From (2.4) and (2.13), we have

$$\begin{aligned} &R^*(X,Y)Z = 2\varphi\beta^2 \{g(Y,Z)X - g(X,Z)Y\}, \\ &Ric^*(X,Y) = 2\varphi\beta^2(m-1)\, g(X,Y),\ \forall X,Y,Z \in \Gamma(S(TM)). \end{aligned}$$

Thus M^* is Einstein and $2\varphi\beta^2$ is a constant due to $m > 2$. By (2.13), we have $2\gamma = 2\varphi\beta^2$. Therefore, M^* is an Einstein space of constant curvature 2γ. By duality, if $p = m$, then $M = L \times M^* = L \times M^* \times \{x\}$ and $B(X,Y) = \alpha g(X,Y)$ for any $X, Y \in \Gamma(TM)$. Thus M is totally umbilical and M^* is a totally umbilical Einstein space of constant curvature $2\gamma = 2\varphi\alpha^2$. In case $s^2 = 4F$. The equation (3.3) has only one non-vanishing solution, named by α and α is a unique eigenvalue of A^*_ξ. In this case, the first equation of (3.4) reduces to $2\alpha = m\alpha$. This implies $m = 2$. Thus this case is an impossible one.

(2) Let $\gamma = (m-1)(c+\delta^2)$: The equation (3.3) reduces to $x(x-s) = 0$. In case $s \neq 0$. Let $\alpha = 0$ and $\beta = s$. Then we have $s = \beta = (m-p)\beta$, i.e., $(m-p-1)\beta = 0$. So $p = m-1$. Thus M_α is a totally geodesic $(m-1)$-dimensional Riemannian manifold and M_β is a spacelike curve in M. In the sequel, let X, Y, $Z \in \Gamma(D^s_\alpha)$ and $U \in \Gamma(D^s_\beta)$. From (2.4), we have

$$R^*(X,Y)Z = (c+\delta^2)\{g(Y,Z)X - g(X,Z)Y\},$$
$$Ric^*(X,Y) = (c+\delta^2)(m-1)g(X,Y).$$

Thus $g(R^*(X,Y)Z,\, U) = 0$. This implies $\pi_\alpha R^*(X,Y)Z = R^*(X,Y)Z$, where π_α is the projection morphism of $\Gamma(S(TM))$ on $\Gamma(D^s_\alpha)$ and $\pi_\alpha R^*$ is the curvature tensor of D^s_α. Thus M_α is an Einstein manifold of a constant curvature $(c+\delta^2)$. Therefore, M is locally a product $L \times M_\alpha \times M_\beta$, where M_α is an $(m-1)$-dimensional Einstein Riemannian space form of a constant curvature $(c+\delta^2)$ and M_β is a spacelike curve in $\bar{M}$. In case $s = 0$, we get $\alpha = \beta = 0$, $A^*_\xi = B = 0$ and $D^s_\alpha = D^s_\beta = S(TM)$. Since M is screen conformal, we also have $C = A_{_N} = 0$. Thus M^* is totally geodesic. Using (2.4), we have

$$R^*(X,Y)Z = (c+\delta^2)\{g(Y,Z)X - g(X,Z)Y\}$$

for all $X, Y, Z \in \Gamma(S(TM))$. Thus M is locally a product $L \times M^* \times \{x\}$, where M^* is an m-dimensional Einstein Riemannian space form of a constant curvature $(c+\delta^2)$ and $\{x\}$ is a point. In these cases, since $(c+\delta^2) = \frac{\gamma}{m-1}$, we have $\mathrm{sgn}(c+\delta^2) = \mathrm{sgn}\,\gamma$. Thus M_α and M^* are isometric to spheres (if $\gamma > 0$) or hyperbolic spaces (if $\gamma < 0$) or Euclidean spaces (if $\gamma = 0$). □

Corollary 3.3. *Let $(M, g, S(TM))$ be a screen conformal Einstein half lightlike submanifold of a Lorentzian space form $(\bar{M}^{m+3}(c), \bar{g})$, $m > 2$, with a Killing coscreen distribution. Then M is locally a product manifold $L \times M_\alpha \times M_\beta$, where L is a null curve and M_α and M_β are totally umbilical leaves of some distributions of M:*

1. *If $\gamma \neq 0$, either M_α or M_β is an m-dimensional Riemannian space form which is isometric to a sphere ($\gamma > 0$) or a hyperbolic space ($\gamma < 0$) and the other is a point in M.*
2. *If $\gamma = 0$, M_α is an $(m-1)$ or m-dimensional Euclidean space and M_β is a spacelike curve or a point in M.*

Proof. (1) Let $\gamma \neq 0$: In case $s^2 \neq 4F$. If $0 < p < m$, then $\gamma = 0$. Thus $p = 0$ or $p = m$. Either M_α or M_β is a totally umbilical Riemannian manifold M^* of constant curvature $2\varphi\alpha^2$ or $2\varphi\beta^2$ respectively due to $\delta = c = 0$. Thus M is locally a product manifold $L \times M^* \times \{x\}$ or $L \times \{x\} \times M^*$, where M^* is an m-dimensional totally umbilical Riemannian manifold of constant curvature $2\gamma = 2\varphi\beta^2$ or $2\gamma = 2\varphi\alpha^2$ which is isometric to a sphere or a hyperbolic space according to the sign of γ and $\{x\}$ is a point. The case $s^2 = 4F$ is not appear because $m > 2$.

(2) Let $\gamma = 0$: In case $s \neq 0$. Then $\alpha = 0$ and $\beta = s$. Since $p = m-1$, M_α is an $(m-1)$-dimensional Riemannian manifold of curvature $c + \delta^2 = 0$ and M_β is a spacelike curve. Thus M is locally a product manifold $L \times M_\alpha \times M_\beta$, where M_α is an $(m-1)$-dimensional Euclidean space and M_β is a spacelike curve in M. In case $s = 0$. Then $\alpha = \beta = 0$ and $D^s_\alpha = D^s_\beta = S(TM)$. Thus M^* is an m-dimensional Riemannian manifold of curvature $c + \delta^2 = 0$. Thus M is locally a product $L \times M^* \times \{x\}$ where M^* is an m-dimensional Euclidean space, L is a null curve and $\{x\}$ is a point. □

Example 3. Consider a surface M in R^4_2 given by the equations

$$x_3 = \frac{1}{\sqrt{2}}(x_1 + x_2), \qquad x_4 = \frac{1}{2}\ln(1 + (x_1 - x_2)^2).$$

Then $TM = Span\{U, V\}$ and $TM^\perp = Span\{\xi, u\}$, where we set

$$\begin{aligned}
&U = \sqrt{2}(1 + (x_1 - x_2)^2)\partial_1 + (1 + (x_1 - x_2)^2)\partial_3 + \sqrt{2}(x_1 - x_2)\partial_4,\\
&V = \sqrt{2}(1 + (x_1 - x_2)^2)\partial_2 + (1 + (x_1 - x_2)^2)\partial_3 - \sqrt{2}(x_1 - x_2)\partial_4,\\
&\xi = \partial_1 + \partial_2 + \sqrt{2}\partial_3,\\
&u = 2(x_2 - x_1)\partial_2 + \sqrt{2}(x_2 - x_1)\partial_3 + (1 + (x_1 - x_2))\partial_4.
\end{aligned}$$

By direct calculations we check that $Rad(TM)$ is a distribution on M of rank 1 spanned by ξ. Hence M is a half-lightlike submanifold of R^4_2. Choose $S(TM)$ and $S(TM^\perp)$ spanned by V and u which are timelike and spacelike respectively. We obtain the lightlike transversal vector bundle

$$ltr(TM) = Span\left\{N = -\frac{1}{2}\partial_1 + \frac{1}{2}\partial_2 + \frac{1}{\sqrt{2}}\partial_3\right\},$$

and the transversal bundle $tr(TM) = Span\{N, u\}$. Denote by $\bar{\nabla}$ the Levi-Civita connection on R^4_2 and by straightforward calculations we obtain

$$\begin{aligned}
&\bar{\nabla}_V V = 2(1 + (x_1 - x_2)^2)\left\{2(x_2 - x_1)\partial_2 + \sqrt{2}(x_2 - x_1)\partial_3 + \partial_4\right\},\\
&\bar{\nabla}_\xi V = 0, \qquad \bar{\nabla}_X \xi = \bar{\nabla}_X N = 0, \quad \forall\, X \in \Gamma(TM).
\end{aligned}$$

Taking into account of Gauss and Weingarten formulae, we infer

$$\begin{aligned}
&B = 0, \quad A^*_\xi = 0, \quad A_N = 0, \quad \nabla_X \xi = 0, \quad \tau(X) = \rho(X) = 0,\\
&[D(X, \xi) = 0, \quad D(V, V) = 2, \quad \nabla_X V = \frac{2\sqrt{2}(x_2 - x_1)^3}{1 + (x_1 - x_2)^2} X^2 V
\end{aligned}$$

for any $X = X^1\xi + X^2 V$ tangent to M. As $A^*_\xi X = A_N X = 0$ for any $X \in \Gamma(TM)$, M is a trivial screen conformal half lightlike submanifold of R^4_2. Since $g(V, V) = -(1 + (x_1 - x_2)^4)$ we have

$$D(V, V) = \delta g(V, V), \quad \text{where} \quad \delta = -\frac{2}{(1 + (x_1 - x_2)^4)}.$$

Therefore M is a screen conformal half lightlike submanifold of R^4_2 with a conformal Killing coscreen distribution $S(TM^\perp)$. Thus M is locally a product manifold $M = L_1 \times L_2$, where L_1 is a null curve tangent to $Rad(TM)$ and L_2 is a timelike curve tangent to $S(TM)$.

References

[1] C. Atindogbe and K. L. Duggal, *Conformal screen on lightlike hypersurfaces*, Int. J. Pure Appl. Math. **11** (2004), no. 4, 421–442.

[2] J. K. Beem, P. E. Ehrlich, and K. L. Easley, *Global Lorentzian Geometry*, Marcel Dekker, Inc. New York, Second Edition, 1996.

[3] B. Y. Chen, *Geometry of Submanifolds*, Marcel Dekker, New York, 1973.

[4] K. L. Duggal and A. Bejancu, *Lightlike submanifolds of codimension two*, Math. J. Toyama Univ. **15** (1992), 59–82.

[5] ———, *Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications*, Kluwer Acad. Publishers, Dordrecht, 1996.

[6] K. L. Duggal and D. H. Jin, *Half-lightlike submanifolds of codimension two*, Math. J. Toyama Univ. **22** (1999), 121–161.

[7] A. Fialkow, *Hypersurfaces of a space of constant curvature*, Ann. of Math. (2) **39** (1938), no. 4, 762–785.

[8] S. G. Harris, *A triangle comparison theorem for Lorentz manifolds*, Indiana Univ. Math. J. **31** (1982), no. 3, 289–308.

[9] D. H. Jin, *Einstein half lightlike submanifolds with a Killing co-screen distribution*, Honam Math. J. **30** (2008), no. 3, 487–504.

[10] ———, *A characterization of screen conformal half lightlike submanifolds*, Honam Math. J. **31** (2009), no. 1, 17–23.

[11] D. N. Kupeli, *Singular Semi-Riemannian Geometry*, Mathematics and Its Applications, Kluwer Acad. Publishers, Dordrecht, 1996.

[12] B. O'Neill, *Semi-Riemannian Geometry with Applications to Relativity,* Academic Press, 1983.

[13] G. de Rham, *Sur la réductibilité d'un espace de Riemannian*, Comment. Math. Helv. **26** (1952), 328–344.

[14] T. Y. Thomas, *On closed spaces of constant mean curvature*, Amer. J. Math. **58** (1936), no. 4, 702–704.

[15] K. Yano, *Differential Geometry on Complex and Almost Complex Spaces*, The Macmillan Company, 1965.

Department of Mathematics
Dongguk University
Kyongju 780-714, Korea
E-mail address: jindh@dongguk.ac.kr