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DOUBLY NONLINEAR PARABOLIC EQUATIONS INVOLVING p-LAPLACIAN OPERATORS VIA TIME-DISCRETIZATION METHOD

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Abstract. In this paper, we consider a doubly nonlinear parabolic partial differential equation $\frac{\partial\beta(u)}{\partial t} - \Delta_p u + f(x,t,u) = 0$ in $\Omega \times [0,T]$, with Dirichlet boundary condition and initial data given. We prove the existence of a discrete approximate solution by means of the Rothe discretization in time method under some conditions on β, f and p.

1. Introduction

In this paper, we study a doubly nonlinear parabolic partial differential equation involving the p-Laplacian operator. More precisely, we are interested in the existence and uniqueness of the solution of problem

$$(1)\qquad \begin{cases} \frac{\partial\beta(u)}{\partial t} - \Delta_p u + f(x,t,u) = 0 & \text{in} \quad \Omega \times [0,T], \\ u = 0 & \text{on} \quad \partial\Omega \times [0,T], \\ u(\cdot,0) = u^0 & \text{in} \quad \Omega, \end{cases}$$

where $\Delta_p u$=div$(|\nabla u|^{p-2}\nabla u)$, $1 < p < \infty$, β is a nonlinearity of porous medium type and f is a nonlinearity of reaction diffusion type. Let Ω be a regular open bounded subset of finite dimensional space $\mathbb{R}^d$ $(d \geq 3)$ and $\partial\Omega$ be its smooth boundary. These problems arise in many applications in the fields of mechanics, physics and biology (non Newtonian fluids, gas flow in porous media, spread of biological populations, etc).

Equations of the form (1) for $p = 2$ has been motivated by the following two papers. The first one due to M. Gurtin [9] gives a non phenomenological derivation of the generalized Allen-Cahn equation. This equation describes some particular aspects of isothermal phase separation process. Performing a clear distinction between thermodynamical laws and constitutive equations, which is in the first part of [9], M. Gurtin propose the following generalization

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of the Allen-Cahn equation

$$a(u, \nabla u, u_t)u_t - \Delta u + f(t, x, u) = 0 \tag{2}$$

with $a \geq 0$. So, this equation can be degenerate and if the coefficient a depends only on u, we may rewrite (2) under the form of equation (1) for $p = 2$. Next, in [13], A. Miranvile and G. Schimperna use Gurtin's approach to model non-isothermal phase transition problem. They end up with the following system of PDE's

$$(u^2)_t - \Delta u = f + u\chi\chi_t + (\chi_t)^2, \qquad \chi_t - \Delta\chi + g(\chi) = -u\chi,$$

where the unknowns are the absolute temperature u and the phase field χ. If χ is given, then the above equation takes the form (1) if $\beta(u) = u^2$ for positive u. Regarding the potential f, we assume that it satisfies some sign condition which is often used in phase transition problems [7]. For instance, $f(t, x, u)$ may be equal to $f(u) = u^p - u^2$ with $p \geq 3$ odd. Uniqueness of the solution to (2) for $p = 2$ satisfying homogeneous Dirichlet boundary conditions is a simple consequence of a deep result of Otto [14].

In case of $p = 2$ of (1), M. Schatzman, A. Eden, B. Michaux, J. M. Rakotoson, A. Rougirel, J. I. Diaz and J. F. Padial [4, 5, 6, 15, 17] dedicated to the existence of solutions and to the large time behavior of these equations in a lot of works. M. Schatzman [17] considered for the problem in which $\beta(u) = u$ for $p = 2$ of (1) and then the problem reduces the reaction-diffusion equations. A. Eden, B. Michaux and J. M. Rakotoson studied the existence of solutions using the method of semi-discretization [5] as well as the method of Galerkin approximation [6]. A. Rougirel [15] studied the solution of asymptotic behavior for $|\frac{\partial f}{\partial t}(t, x, u)| \leq C_M$, $(C_M > 0)$. And J. I. Diaz and J. F. Padial [4] are considered the existence of solution in $BV_t(Q)$ space using $\beta(u_t)$ instead of $\frac{\partial \beta(u)}{\partial t}$.

In case of $p > 1$ of (1), A. Bensoussan, L. Boccardo and F. Murat [3] studied the existence of solution for $\beta = 0$ and A. El Hachimi and H. El Ouardi [8] studied the existence and regularity of this equation under the fact that f is differentiable by the method of Galerkin approximation.

Various abstract evolution equations have been considered using Rothe time-discretization method, see, for instance, Kartsatos and Parrott [10] and references cited therein. In the paper, the authors solved the initial-boundary value problem for the time-dependent functional equation by the time-discretization method.

This is the plan of paper. We recall our assumptions and state main results in Section 2. In Section 3, we show the existence of discrete scheme. And, after showing some estimates on the approximations, the passage to the limit and the existence results are given in Section 4.

2. Assumptions and main results

We let $||\cdot||_p$, $||\cdot||_{1,p}$ and $||\cdot||_{-1,p}$ denote the norm in $L^p(\Omega)$, $W_0^{1,p}(\Omega)$ and $W^{-1,p}(\Omega)$ for $1<p<\infty$, respectively. And $\langle\cdot,\cdot\rangle$ denotes the duality between $W_0^{1,p}(\Omega)$ and $W^{-1,p}(\Omega)$ or denotes inner product of $L^2(\Omega)$. For $1<p<\infty$, we define the conjugate p' of p by $1/p+1/p'=1$. In this paper, C_i and C will denote positive constants and λ_i the imbedding constants such that

$$\begin{cases} ||\cdot||_p \le \lambda_1 ||\cdot||_{1,p} & \text{if } \frac{2d}{2+d} \le p < \infty, \\ ||\cdot||_2 \le \lambda_2 ||\cdot||_{1,p} & \text{if } \frac{2d}{2+d} \le p < 2, \\ ||\cdot||_1 \le \lambda_3 ||\cdot||_p \le \lambda_4 ||\cdot||_{1,p} & \text{if } p \ge 2 \end{cases}$$

(cf. [1]).

We define ψ by $\psi(t)=\int_0^t \beta(s)ds$ for $t\in\mathbb{R}$ and a continuous function β with $\beta(0)=0$. Then, the Legendre transform of ψ is also defined by $\psi^*(\tau)=\sup_{s\in\mathbb{R}}\{\tau s-\psi(s)\}$.

Now, we present our assumptions which are used throughout this paper.

We suppose $d^*\le p<\infty$ where $d^*=(2d)/(d+2)$, $u^0\in L^\infty(\Omega)$ with $u^0=0$ on $\partial\Omega$ and the followings:

(H1) The function $\beta:\mathbb{R}\to\mathbb{R}$ is increasing and continuous with $\beta(0)=0$.

(H2) For $\xi\in\mathbb{R}$, the map $(x,t)\mapsto f(x,t,\xi)$ is measurable and $\xi\mapsto f(x,t,\xi)$ is continuous a.e. in $\Omega\times[0,T]$. Furthermore, we assume that there exits $C_1>0$ such that $\operatorname{sign}\xi\, f(x,t,\xi)\ge -C_1$ for a.e. $(x,t)\in\Omega\times[0,T]$.

(H3) For all $M>0$, there exists $C_M>0$ such that, if $|\xi|+|\xi'|\le M$, then

$$|f(x,t,\xi)-f(x,t,\xi')|^\alpha \le C_M(\beta(\xi)-\beta(\xi'))(\xi-\xi'),$$

where $\alpha=\begin{cases} 2 & \text{if } 1<p<2, \\ p' & \text{if } p\ge 2. \end{cases}$

(H3)$'$ There is $C_2>0$ such that $\xi\mapsto f(x,t,\xi)+C_2\beta(\xi)$ is increasing for almost $(x,t)\in\Omega\times[0,T]$.

(H4) For almost every $x\in\Omega$ and for all $M>0$, there exists $\widetilde{C}_M>0$ such that, if $t+t'+|\xi|\le M$, then

$$|f(x,t,\xi)-f(x,t',\xi)|\le \widetilde{C}_M|t-t'|^{1/\alpha},$$

where α is same as in (H3).

Definition ([2]). Let X be a reflexive Banach space and $A:X\to X'$. We say that A is *monotone* if $\langle Ay-Az,y-z\rangle\ge 0$ for all $y,z\in X$, and A is *hemicontinuous* if for each y, z, $w\in X$ the real-valued function $t\to\langle A(y+tz),w\rangle$ is continuous.

Lemma 2.1 (Minty's Theorem [18]). *Let X be a reflexive Banach space. If $A:X\to X'$ is monotone and hemicontinuous, then*

$$Ay=f \quad \textit{if and only if} \quad \langle f-Az,y-z\rangle\ge 0$$

for all $z\in X$.

Lemma 2.2 ([3]). *Let Ω be a bounded set in $\mathbb{R}^d$. Let $1 < p < \infty$ be fixed and $A : W_0^{1,p}(\Omega) \to W^{1,p'}(\Omega)$ a nonlinear operator defined by*

$$A(u) = -\operatorname{div} a(x, u, Du),$$

where $a(x, s, \xi)$ is a Carathéodory function $a : \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ such that

$$\begin{aligned}
&|a(x, s, \xi)| \le \beta[|s|^{p-1} + |\xi|^{p-1} + k(x)],\\
&[a(x, s, \xi) - a(x, s, \eta)](\xi - \eta) > 0, \quad \xi \ne \eta,\\
&a(x, s, \xi)\xi \ge \alpha|\xi|^p,
\end{aligned}$$

where $k(x) \in L^{p'}(\Omega)$, $k \ge 0$, $\beta > 0$ and $\alpha > 0$.

Let $g(x, s, \xi)$ be a Carathéodory function such that

$$g(x, s, \xi)s \ge 0, \quad |g(x, s, \xi)| \le b(|s|)(|\xi|^p + c(x)),$$

where b is a continuous and increasing function with (finite) values on $\mathbb{R}^+$ and $c \in L^1(\Omega)$, $c \ge 0$. Then, for $h \in W^{-1,p'}(\Omega)$, the problem

$$Au + g(x, u, \nabla u) = h,$$

has at least one solution $u \in W_0^{1,p}(\Omega)$.

Lemma 2.3 ([16]). *If $u \in W_0^{1,p}(\Omega)$ is a solution to the equation*

$$-\Delta_p u + F(x, u) = h,$$

where $h \in W^{-1,r}$, $r > \frac{d}{p-1}$ and F satisfies $\xi F(x, \xi) \ge 0$ in $\Omega \times \mathbb{R}$, then $u \in L^\infty(\Omega)$.

Now, we state our main results as follows:

Theorem 2.4. *Under assumptions* (H1), (H2), (H3) (*or* (H3)$'$) *and* (H4), *there exists a unique solution u of* (1) *such that*

$$\begin{cases} u \in L^p(0, T; W_0^{1,p}(\Omega)) \cap L^\infty(0, T; L^\infty(\Omega)) & \text{if} \quad p \ge 2,\\ u \in L^2(0, T; L^2(\Omega)) & \text{if} \quad d^* \le p < 2. \end{cases}$$

3. Existence of scheme

For the problem (1), we consider the discrete scheme (DS) for $i = 1, 2, \ldots, N$,

$$\text{(DS)} \qquad \begin{cases} \frac{\beta(u_i) - \beta(u_{i-1})}{\tau} - \Delta_p u_i + f(x, i\tau, u_i) = 0 & \text{in } \Omega,\\ u_i = 0 & \text{in } \partial\Omega,\\ u_0 = u^0 & \text{in } \Omega, \end{cases}$$

where $N\tau = T$ and T is a fixed positive real. We shall show that (DS) has a solution u_i for $i = 1, 2, \ldots, N$.

Theorem 3.1. *Let* (H1)–(H3) *hold. Then for $i = 1, 2, \ldots, N$, there exists a unique solution $u_i \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ of* (DS) *for sufficiently small τ.*

Proof. First of all, we rewrite (DS) as

$$-\tau\Delta_p u_i + F(x,u_i) = \varphi_{i-1},$$

where $F(x,u_i) = \beta(u_i) + \tau f(x,i\tau,u_i) + \tau C_1 \mathrm{sign}(u_i)$ and $\varphi_{i-1} = \beta(u_{i-1}) + \tau C_1 \mathrm{sign}(u_i)$.

Now, we consider the equation

$$-\tau\Delta_p u + F(x,u) = \varphi_0 = \beta(u_0) + \tau C_1 \mathrm{sign}(u), \tag{3}$$

where $F(x,u) = \beta(u)+\tau f(x,\tau,u)+\tau C_1\mathrm{sign}(u)$ for fixed $\tau = T/N$. It is obvious that $a(x,u,Du) := |\nabla u|^{p-2}\nabla u$ satisfies all the three conditions of a in Lemma 2.2 (cf [11]). In particular, we used the inequality $|a|^{p-2}a(a-b) \geq \frac{1}{p}|a|^p - \frac{1}{p}|b|^p$ for the second condition. Since β is continuous, $\varphi_0 \in L^\infty(\Omega)$. And, by (H1) and (H2), $g(x,u,\nabla u) := F(x,u)$ is a Carathéodory function with $u\,F(x,u) \geq 0$. Also, by (H2), $|F(x,u)| \leq \beta(|u|) + 2\tau C_1$. Thus, all the conditions of g in Lemma 2.2 are satisfied. Therefore, there exists a solution $u \in W_0^{1,p}(\Omega)$ of (3). Moreover, by Lemma 2.3, $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. We put $u_1 := u$ and consider the equation $-\tau\Delta_p u + F(x,u) = \varphi_1 = \beta(u_1) + \tau C_1\mathrm{sign}(u)$ where $F(x,u) = \beta(u) + \tau f(x,2\tau,u) + \tau C_1\mathrm{sign}(u)$. Continuing this process, we have a solution u_i of (DS) for $i = 1,2,\ldots,N$ such that $u_i \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ $(i = 1,2,\ldots,N)$.

Next, we show the uniqueness of u_i $(i = 1,2,\ldots,N)$. Let u_i and u_i^* be two solutions of (DS) for $i = 1,2,\ldots,N$. Using the result which we will establish below (see Theorem 3.3), we have

$$||u_i||_{1,p} + ||u_i^*||_{1,p} \leq M,$$

where M is a suitable positive constant independent of τ. And, from [11] we have

$$\langle -\Delta_p u + \Delta_p v, u - v\rangle \geq \begin{cases} C_p||u-v||_{1,p}^p & \text{if} \quad p \geq 2, \\ C_p \frac{\|u-v\|_{1,p}^2}{(\|u\|_{1,p}+\|v\|_{1,p})^{2-p}} & \text{if} \quad 1 < p < 2, \end{cases} \tag{4}$$

for all $u, v \in W_0^{1,p}(\Omega)$.

Since u_i and u_i^* are solutions of (DS), we have

$$-\tau\Delta_p u_i + \tau\Delta_p u_i^* + \beta(u_i) - \beta(u_i^*) + \tau f(x,i\tau,u_i) - \tau f(x,i\tau,u_i^*) = 0. \tag{5}$$

Multiplying (5) by $u_i - u_i^*$ and integrating over Ω, we get

$$\langle -\tau\Delta_p u_i + \tau\Delta_p u_i^*, u_i - u_i^*\rangle + \tau\int_\Omega (f(x,i\tau,u_i) - f(x,i\tau,u_i^*))(u_i - u_i^*)dx \tag{6}$$
$$+ \int_\Omega (\beta(u_i) - \beta(u_i^*))(u_i - u_i^*)dx = 0.$$

(a) Suppose $p \geq 2$. By (4), the equation (6) becomes

$$\tau C_p\|u_i - u_i^*\|_{1,p}^p + \int_\Omega (\beta(u_i) - \beta(u_i^*))(u_i - u_i^*)dx$$

$$\le \int_\Omega \left[\frac{1}{C_M^{1/p'}}(f(x,i\tau,u_i)-f(x,i\tau,u_i^*))\right]\left(-C_M^{1/p'}\tau(u_i-u_i^*)\right)dx.$$

By Young's inequality and (H3),

$$\begin{aligned}&\tau C_p||u_i-u_i^*||_{1,p}^p+\int_\Omega(\beta(u_i)-\beta(u_i^*))(u_i-u_i^*)dx\\ \le\ &\frac{1}{p'}\int_\Omega(\beta(u_i)-\beta(u_i^*))(u_i-u_i^*)dx+\frac{1}{p}\tau^p\lambda_1 C_M^{\frac{p}{p'}}||u_i-u_i^*||_{1,p}^p.\end{aligned}$$

Since β is increasing, $0\le(\tau^p\lambda_1 C_M^{p/p'}-p\tau C_p)||u_i-u_i^*||_{1,p}^p$. It implies that for sufficiently small τ, i.e., $\tau<(\frac{pC_p}{\lambda_1 C_M^{p/p'}})^{1/(p-1)}$, $u_i=u_i^*$ holds.

(b) Suppose $d^*\le p<2$. Since $||u_i||_{1,p}+||u_i^*||_{1,p}\le M$, by (H3) the equation (6) becomes

$$\begin{aligned}&\tau C_p\frac{||u_i-u_i^*||_{1,p}^2}{(||u_i||_{1,p}+||u_i^*||_{1,p})^{2-p}}+\int_\Omega(\beta(u_i)-\beta(u_i^*))(u_i-u_i^*)dx\\ \le\ &-\tau\int_\Omega(f(x,i\tau,u_i)-f(x,i\tau,u_i^*))(u_i-u_i^*)dx\\ \le\ &\int_\Omega\frac{1}{2C_M}|f(x,i\tau,u_i)-f(x,i\tau,u_i^*)|^2dx+\int_\Omega\frac{\tau^2C_M}{2}|u_i-u_i^*|^2dx\\ \le\ &\frac{1}{2}\int_\Omega(\beta(u_i)-\beta(u_i^*))(u_i-u_i^*)dx+\int_\Omega\frac{\tau^2C_M}{2}|u_i-u_i^*|^2dx.\end{aligned}$$

Therefore,

$$\begin{aligned}&\tau C_p\frac{1}{M^{2-p}}||u_i-u_i^*||_{1,p}^2+\int_\Omega(\beta(u_i)-\beta(u_i^*))(u_i-u_i^*)dx\\ \le\ &\frac{1}{2}\int_\Omega(\beta(u_i)-\beta(u_i^*))(u_i-u_i^*)dx+\frac{\tau^2C_M\lambda_2}{2}\|u_i-u_i^*\|_{1,p}^2.\end{aligned}$$

In other words, since β is increasing

$$0\le\frac{1}{2}\int_\Omega(\beta(u_i)-\beta(u_i^*))(u_i-u_i^*)dx\le\left(\frac{C_M\tau^2\lambda_2}{2}-\frac{\tau C_p}{M^{2-p}}\right)||u_i-u_i^*||_{1,p}^2.$$

Hence,

$$0\le\left(\frac{C_M\tau^2\lambda_2}{2}-\frac{\tau C_p}{M^{2-p}}\right)||u_i-u_i^*||_{1,p}^2.$$

It means that for sufficiently small τ, i.e., $\tau<\frac{2C_p}{M^{2-p}C_M\lambda_2}$, $u_i=u_i^*$ holds. □

Proposition 3.2. *We assume* (H3)$'$ *holds instead of* (H3) *in Theorem 3.1. Then the same results hold provided that* $\tau<1/C_2$.

Proof. In the proof of Theorem 3.1, we have only used (H1)–(H2) in showing the existence of solution $u_i \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ $(i = 1, 2, \dots, N)$ for (DS). Hence, we are going to show the uniqueness using (H3)$'$. Let u_i and u_i^* be two solutions of (DS). Then, from (H3)$'$, we have

$$\int_\Omega (f(x, i\tau, u_i) - f(x, i\tau, u_i^*))(u_i - u_i^*)dx \geq -C_2 \int_\Omega (\beta(u_i) - \beta(u_i^*))(u_i - u_i^*)dx.$$

Applying the above inequality to (6), by the monotonicity of $-\Delta_p$ (the p-Laplacian operator),

$$(1 - \tau C_2) \int_\Omega (\beta(u_i) - \beta(u_i^*))(u_i - u_i^*)dx \leq 0.$$

Then by (H1), if $\tau < 1/C_2$, we get $u_i = u_i^*$. □

Now, we consider the bounds of $\{u_i\}$ $(i = 1, 2, \dots, N)$, which is constructed in Theorem 3.1 and Proposition 3.2 as solutions of (DS).

Theorem 3.3. *We assume* (H1)–(H2). *Then there exist C_3, C_4, C_5, which are positive constants and independent of τ, such that for all $i = 1, 2, \dots, N$,*

$$\text{(a) } ||u_i||_\infty \leq C_3,$$

$$\text{(b) } \tau \sum_{i=1}^{m} ||u_i||_{1,p}^p \leq C_4,$$

$$\text{(c) } ||\beta(u_m)||_2^2 + \sum_{i=1}^{m} ||\beta(u_i) - \beta(u_{i-1})||_2^2 \leq C_5,$$

where $m = 1, 2, \dots, N$.

Proof. (a) Multiplying (DS) by $|\beta(u_i)|^k \beta(u_i)$ and integrating over Ω, we have by (H2) and Hölder's inequality,

$$\int_\Omega |\beta(u_i)|^k \beta(u_i)\beta(u_i)dx - \int_\Omega \tau \Delta_p u_i |\beta(u_i)|^k \beta(u_i)dx$$
$$\leq ||\beta(u_i)||_{k+2}^{k+1} ||\beta(u_{i-1})||_{k+2} + m(\Omega)^{1/(k+2)} \tau C_1 ||\beta(u_i)||_{k+2}^{k+1}.$$

Then

$$||\beta(u_i)||_{k+2} \leq m(\Omega)^{1/k+2} \tau C_1 + ||\beta(u_{i-1})||_{k+2}.$$

By induction, we have $||\beta(u_i)||_{k+2} \leq m(\Omega)^{1/(k+2)} C_1 T + ||\beta(u_0)||_{k+2}$. Letting $k \to \infty$, $||u_i||_\infty \leq C(C_1, T, u_0) =: C_3$ by (H1).

(b) Let $z \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ be fixed. Multiplying the equation (DS) by $u_i - z$ and integrating over Ω, we have, by (H2),

$$\langle \beta(u_i) - \beta(u_{i-1}), u_i \rangle - \langle \beta(u_i) - \beta(u_{i-1}), z \rangle + \tau ||u_i||_{1,p}^p \tag{7}$$
$$\leq \tau \int_\Omega |\nabla u_i|^{p-1} |\nabla z| dx + \tau C_1 \int_\Omega |u_i - z| dx.$$

Now, we apply Young's inequality to (7) to get

$$\begin{aligned}(8)\qquad &\langle\beta(u_i)-\beta(u_{i-1}),u_i\rangle-\langle\beta(u_i)-\beta(u_{i-1}),z\rangle+\frac{\tau}{2}||u_i||_{1,p}^p\\ &\le\left\{\tau(\frac{1}{2}\frac{p}{p-1})^{-(p-1)}+2^p\lambda_1\tau^2\right\}\frac{1}{p}||z||_{1,p}^p+\tau C_1||u_i||_\infty m(\Omega)\\ &\quad+\tau C_1||z||_\infty m(\Omega)+\frac{\tau^2}{p'}C_1^{p'}m(\Omega)+2^p\lambda_1\frac{\tau}{p}\sum_{j=1}^{i}\tau||u_j||_{1,p}^p\end{aligned}$$

for $i=1,2,\ldots,m$ and for arbitrary $m=1,2,\ldots,N$. By the property of the Legendre transform ψ^* of $\psi=\int_0^t\beta(s)ds$,

$$\int_\Omega(\psi^*(\beta(u_i))-\psi^*(\beta(u_{i-1})))\,dx\le\int_\Omega(\beta(u_i)-\beta(u_{i-1}))u_i dx$$

and by (8),

$$\begin{aligned}(9)\qquad &\int_\Omega(\psi^*(\beta(u_i))-\psi^*(\beta(u_{i-1})))\,dx-\langle\beta(u_i)-\beta(u_{i-1}),z\rangle+\frac{\tau}{2}||u_i||_{1,p}^p\\ &\le\left\{\tau(\frac{1}{2}\frac{p}{p-1})^{-(p-1)}+2^p\lambda_1\tau^2\right\}\frac{1}{p}||z||_{1,p}^p+\tau C_1||u_i||_\infty m(\Omega)\\ &\quad+\tau C_1||z||_\infty m(\Omega)+\frac{\tau^2}{p'}C_1^{p'}m(\Omega)+2^p\lambda_1\frac{\tau}{p}\sum_{j=1}^{i}\tau||u_j||_{1,p}^p\end{aligned}$$

for $i=1,2,\ldots,m$. By summing (9) with respect to $i=1,2,\ldots,m$ and by (a),

$$\begin{aligned}(10)\qquad &\int_\Omega\psi^*(\beta(u_m))-\psi^*(\beta(u_0))dx-\langle\beta(u_m)-\beta(u_0),z\rangle+\frac{\tau}{2}\sum_{i=1}^{m}||u_i||_{1,p}^p\\ &\le C_6+C_7\tau\sum_{i=1}^{m}\sum_{j=1}^{i}\tau||u_j||_{1,p}^p,\end{aligned}$$

where $C_6:=C(T,p,\lambda_1,C_1,C(C_1,T,u_0),m(\Omega),||z||_\infty,||z||_{1,p})$ and $C_7:=C(\lambda_1,$ $p)$ and for $m=1,2,\ldots,N$. By (10) and for arbitrary $\tau<\bar{\tau}=1/(4C_7)$,

$$\begin{aligned}(11)\qquad &\int_\Omega\psi^*(\beta(u_m))dx-\langle\beta(u_m),z\rangle+\frac{\tau}{4}\sum_{i=1}^{m}||u_i||_{1,p}^p\\ &\le\int_\Omega\psi^*(\beta(u_0))dx-\langle\beta(u_0),z\rangle+C_6+C_7\tau\sum_{i=1}^{m-1}\sum_{j=1}^{i}\tau||u_j||_{1,p}^p.\end{aligned}$$

Applying the discrete Gronwall's lemma to (11),

$$\begin{aligned}(12)\qquad &\int_\Omega\psi^*(\beta(u_m))dx-\langle\beta(u_m),z\rangle+\frac{\tau}{4}\sum_{i=1}^{m}||u_i||_{1,p}^p\\ &\le C(\beta(u_0),T,p,\lambda_1,C_1,C(C_1,T,u_0),m(\Omega),||z||_\infty,||z||_{1,p}).\end{aligned}$$

Hence by $\int_\Omega \psi^*(\beta(u_m))dx - \langle \beta(u_m), z\rangle > -\infty$ and (12), $\tau\sum_{i=1}^m ||u_i||_{1,p}^p \le C_4$.

(c) Multiplying (DS) by $\beta(u_i)$ and integrating over Ω, we get by (H2),

$$\int_\Omega (\beta(u_i) - \beta(u_{i-1}))\beta(u_i)dx \le C_1\tau \int_\Omega |\beta(u_i)|dx.$$

Using $a(a-b) = \frac{1}{2}a^2 - \frac{1}{2}b^2 + \frac{1}{2}(a-b)^2$,

$$||\beta(u_i)||_2^2 - ||\beta(u_{i-1})||_2^2 + ||\beta(u_i) - \beta(u_{i-1})||_2^2 \le 2C_1\tau||\beta(u_i)||_1.$$

Summing the above inequality with respect to $i = 1, 2, \dots, m$, we have by (a),

$$\begin{aligned} ||\beta(u_m)||_2^2 + \sum_{i=1}^m ||\beta(u_i) - \beta(u_{i-1})||_2^2 &\le 2C_1\tau\sum_{i=1}^m ||\beta(u_i)||_1 + ||\beta(u_0)||_2^2 \\ &\le 2C_1 Tm(\Omega)C(C_1, T, u_0) + ||\beta(u_0)||_2^2 \end{aligned}$$

for $m = 1, 2, \dots, N$. Thus $||\beta(u_m)||_2^2 + \sum_{i=1}^m ||\beta(u_i) - \beta(u_{i-1})||_2^2 \le C_5$. □

In the forthcoming discussion, the following notations will be used extensively. For vectors u_i $(i = 0, 1, \dots, N)$ in Theorem 3.1, we define two functions u_τ and $\bar{u}_\tau$ on $[0, T]$ by

$$\begin{aligned} &u_\tau(0) := u_0, \quad u_\tau(t) := u_i + \frac{u_i - u_{i-1}}{\tau}(t - i\tau), \\ &\bar{u}_\tau(0) := u_0, \quad \bar{u}_\tau(t) := u_i, \end{aligned}$$

for $t \in ((i-1)\tau, i\tau]$ $(i = 1, 2, \dots, N)$ and $\tau = T/N$. Similarly, we define

$$\begin{aligned} &v_\tau(0) := v_0, \quad v_\tau(t) := v_i + \frac{v_i - v_{i-1}}{\tau}(t - i\tau), \\ &\bar{v}_\tau(0) := v_0, \quad \bar{v}_\tau(t) := v_i, \end{aligned}$$

where $v_i := \beta(u_i)$ for $i = 0, 1, \dots, N$. Also, we let $\bar{f}_\tau(t) := f_i = f(\cdot, i\tau, u_i)$ for $t \in ((i-1)\tau, i\tau]$ $(i = 1, 2, \dots, N)$.

Hence we can rewrite (DS) in a more compact form as

$$\begin{aligned} &v_\tau' - \Delta_p \bar{u}_\tau + \bar{f}_\tau = 0, \\ &\bar{v}_\tau = \beta(\bar{u}_\tau), \end{aligned} \quad \text{a.e.} \quad \text{in} \quad [0, T]. \tag{13}$$

4. Estimates and limits

In this section, we assume the hypotheses (H1)–(H4).

4.1. Estimates

First of all, by the consequence of Theorem 3.3, the followings are very easily proven.

$$\bar{u}_\tau \text{ is bounded in } L^p(0, T; W_0^{1,p}(\Omega)) \cap L^\infty(0, T; L^\infty(\Omega)), \tag{14}$$

$$\bar{v}_\tau \text{ is bounded in } L^\infty(0, T; L^2(\Omega)). \tag{15}$$

Moreover, by (14) and boundedness of p-Laplacian operator,

$$-\Delta_p \bar{u}_\tau \text{ is bounded in } L^{p'}(0,T;W^{-1,p'}(\Omega)). \tag{16}$$

We emphasis that all the above boundedness are independent of τ.

First of all, we have $u \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(0,T;L^\infty(\Omega))$ which is a weak limit of $\bar{u}_\tau$ as $\tau \to 0$ by (14), i.e., $\bar{u}_\tau$ converges weakly to u as $\tau \to 0$ in $L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(0,T;L^\infty(\Omega))$. Moreover, by Theorem 3.3(c) we may use the term $w(x) := \sup_{t\in[0,T]} \frac{\partial\beta(u(x,t))}{\partial t}$ which is bounded in $L^2(\Omega)$ a.e.

Now, we consider the boundedness of v'_τ.

(a) We suppose $p \geq 2$. By (H3) and Theorem 3.3(c),

$$\begin{aligned}
&\sum_{i=1}^N \int_{(i-1)\tau}^{i\tau} ||f(x,t,u) - f(x,t,\bar{u}_\tau)||_{-1,p'}^{p'} dt \\
&\leq \sum_{i=1}^N \int_{(i-1)\tau}^{i\tau} \left\{ \sup_{||v||_{1,p}\leq 1} (\int_\Omega |v(x)|^p dx)^{\frac{1}{p}} (\int_\Omega |f(x,t,u) - f(x,t,\bar{u}_\tau)|^{p'} dx)^{\frac{1}{p'}} \right\}^{p'} dt \\
&\leq {\lambda_1}^{p'} C_M \tau \sum_{i=1}^N \int_{(i-1)\tau}^{i\tau} (||u||_2 + ||\bar{u}_\tau||_2) \|w\|_2 dt \\
&\leq {\lambda_1}^{p'} C_M \tau (||u||_{L^2(0,T;L^2(\Omega))} + ||\bar{u}_\tau||_{L^2(0,T;L^2(\Omega))}) ||w||_{L^2(0,T;L^2(\Omega))}.
\end{aligned}$$

And, by (H4),

$$\begin{aligned}
&\sum_{i=1}^N \int_{(i-1)\tau}^{i\tau} ||f(x,t,\bar{u}_\tau(x,t)) - f(x,i\tau,\bar{u}_\tau(x,t))||_{-1,p'}^{p'} dt \\
&\leq \widetilde{C}_M^{p'} \tau \sum_{i=1}^N \int_{(i-1)\tau}^{i\tau} [\sup_{||v||_{1,p}\leq 1} | \int_\Omega |v(x)| dx]^{1/p'} dt \\
&\leq \widetilde{C}_M^{p'} \tau \sum_{i=1}^N \int_{(i-1)\tau}^{i\tau} \lambda_4^{p'} dt = \widetilde{C}_M^{p'} \tau \lambda_4^{p'} T.
\end{aligned}$$

Hence

$$\begin{aligned}
&||\bar{f}_\tau(x,t,\bar{u}_\tau) - f(x,t,u)||_{L^{p'}(0,T;W^{-1,p'}(\Omega))}^{p'} \\
&\leq 2^{p'} ({\lambda_1}^{p'} C_M \tau (||u||_{L^2(0,T;L^2(\Omega))} + ||\bar{u}_\tau||_{L^2(0,T;L^2(\Omega))}) ||w||_{L^2(0,T;L^2(\Omega))} \\
&\quad + \widetilde{C}_M^{p'} \tau \lambda_4^{p'} T).
\end{aligned} \tag{17}$$

(b) Now, we suppose $d^* \leq p < 2$. In a very similar way, we have

$$\begin{aligned}
&||\bar{f}_\tau(x,t,\bar{u}_\tau) - f(x,t,u)||_{L^2(0,T;L^2(\Omega))}^2 \\
&\leq 2(\tau C_M (||u||_{L^2(0,T;L^2(\Omega))} + ||\bar{u}_\tau||_{L^2(0,T;L^2(\Omega))}) ||w||_{L^2(0,T;L^2(\Omega))} \\
&\quad + T\widetilde{C}_M^2 \tau m(\Omega)).
\end{aligned} \tag{18}$$

Since $v'_\tau = \Delta_p \bar{u}_\tau - \bar{f}_\tau$ in (13), by (16)–(18), we conclude that

$$v'_\tau \text{ is bounded in } \begin{cases} L^{p'}(0,T;W_0^{-1,p'}(\Omega)) & \text{if} \quad p \geq 2, \\ L^2(0,T;L^2(\Omega)) & \text{if} \quad d^* \leq p < 2. \end{cases} \tag{19}$$

4.2. Limits

(a) We suppose $p \geq 2$. As we mentioned in (14) and (15), we have u and v such that

$$\bar{u}_\tau \to u \quad \text{weakly in} \quad L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(0,T;L^\infty(\Omega)), \tag{20}$$

$$v_\tau \to v \quad \text{weakly star in} \quad W^{1,p'}(0,T;W^{-1,p'}(\Omega)), \tag{21}$$
$$v_\tau \to v \quad \text{weakly in} \quad C(0,T;L^2(\Omega)),$$

$$\bar{v}_\tau \to v \quad \text{weakly star in} \quad L^{p'}(0,T;W^{-1,p'}(\Omega)), \tag{22}$$
$$\bar{v}_\tau \to v \quad \text{strongly in} \quad L^2(0,T;L^2(\Omega)).$$

We note that the above sequences with τ are for some not relabeled subsequence. Also we note that

$$\begin{aligned} \lim_{\tau\to 0} \int_0^T \langle \phi, \bar{v}_\tau - v_\tau \rangle dt &\leq \lim_{\tau \to 0} \tau \sum_{i=1}^N \int_{(i-1)\tau}^{i\tau} \int_\Omega \phi(x,t) (\frac{v_i - v_{i-1}}{\tau}) dx dt \\ &= \lim_{\tau\to 0} \tau \sum_{i=1}^N \int_{(i-1)\tau}^{i\tau} \int_\Omega \phi(x,t) v'_\tau(t) dx dt \\ &= \lim_{\tau \to 0} \tau \int_0^T \langle \phi, v'_\tau \rangle dt = 0 \end{aligned}$$

for all $\phi \in L^p(0,T;W_0^{1,p}(\Omega))$ by (19). Hence, v_τ and $\bar{v}_\tau$ have the same limit v in (21) and (22).

(b) Now, we suppose $d^* \leq p < 2$. In a very similar way, we have

$$\bar{u}_\tau \to u \quad \text{strongly in} \quad L^2(0,T;L^2(\Omega)), \tag{23}$$

$$v_\tau \to v \quad \text{strongly in} \quad W^{1,2}(0,T;L^2(\Omega)), \tag{24}$$
$$v_\tau \to v \quad \text{weakly in} \quad C(0,T;L^2(\Omega)),$$

$$\bar{v}_\tau \to v \quad \text{strongly in} \quad L^2(0,T;L^2(\Omega)). \tag{25}$$

Proof of Theorem 2.4. First of all, by (17) and (18), we have f such that

$$\bar{f}_\tau \to f \quad \text{strongly in} \quad \begin{cases} L^{p'}(0,T;W_0^{-1,p'}(\Omega)) & \text{if } p \geq 2, \\ L^2(0,T;L^2(\Omega)) & \text{if } d^* \leq p < 2. \end{cases} \tag{26}$$

In addition, by (20), (22) [(23), (25), respectively] and (H1), we have $v = \beta(u)$. Therefore,

(27)
$$-\Delta_p \bar{u}_\tau \to -\frac{\partial \beta(u)}{\partial t} - f \text{ weakly in } \begin{cases} L^{p'}(0,T;W_0^{-1,p'}(\Omega)) & \text{if } p \geq 2, \\ L^2(0,T;L^2(\Omega)) & \text{if } d^* \leq p < 2, \end{cases}$$

from (13) since $\beta(\bar{u}_\tau(t)) = \bar{v}_\tau(t)$ for $t \in [0,T]$. And, by the property

$$\int_\Omega \psi^*(\beta(u_i)) - \psi^*(\beta(u_{i-1}))dx \leq \int_\Omega (\beta(u_i) - \beta(u_{i-1}))u_i dx$$

of the Legendre transform ψ^* of $\psi = \int_0^t \beta(s)ds$,

$$\begin{aligned} &\limsup_{\tau\to 0} \int_0^T \langle -\Delta_p \bar{u}_\tau, \bar{u}_\tau \rangle dt \\ &\leq \limsup_{\tau\to 0} \sum_{i=1}^N \int_{(i-1)\tau}^{i\tau} \int_\Omega \frac{-\psi^*(\beta(u_i)) + \psi^*(\beta(u_{i-1}))}{\tau} dxdt \\ &\quad + \limsup_{\tau\to 0} \int_0^T \int_\Omega -\bar{f}_\tau \bar{u}_\tau dxdt. \end{aligned}$$

Moreover, since $\psi^*(\beta(u(i\tau))) - \psi^*(\beta(u((i-1)\tau))) = \frac{\partial \psi^*}{\partial s}(\beta(u(\sigma)))(i\tau - (i-1)\tau)$ for $\sigma \in ((i-1)\tau, i\tau]$,

$$\begin{aligned} &\limsup_{\tau\to 0} \int_0^T \langle -\Delta_p \bar{u}_\tau, \bar{u}_\tau \rangle dt \\ &\leq \limsup_{\tau\to 0} \sum_{i=1}^N \int_{(i-1)\tau}^{i\tau} \frac{1}{\tau} \int_\Omega -\tau \frac{\partial \psi^*(\beta(u(s)))}{\partial s} dxdt + \int_0^T \langle -f, u \rangle dt \\ &= \int_0^T \int_\Omega -\frac{\partial \psi^*(\beta(u(s)))}{\partial s} dxdt + \int_0^T \langle -f, u \rangle dt \end{aligned}$$

by (20) and (26) [(23), (26), respectively]. Also, since $\psi(t) = \int_0^t \beta(s)ds$, $\psi'(t) = \beta(t)$ and $(\psi^*)' = (\psi')^{-1}$,

(28)
$$\begin{aligned} &\limsup_{\tau\to 0} \int_0^T \langle -\Delta_p \bar{u}_\tau, \bar{u}_\tau \rangle dt \\ &\leq \int_0^T \int_\Omega -\frac{\partial \beta(u(s))}{\partial s} u(s) dxdt + \int_0^T \langle -f, u \rangle dt \\ &= \int_0^T \langle -\frac{\partial \beta}{\partial t} - f, u \rangle dt. \end{aligned}$$

Therefore, by Lemma 2.1 (Minty's Theorem), (27) and (28), there exists a unique solution u of (1) such that

$$\begin{cases} u \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(0,T;L^\infty(\Omega)) & \text{if } \ p \geq 2, \\ u \in L^2(0,T;L^2(\Omega)) & \text{if } \ d^* \leq p < 2. \end{cases}$$ □

Remark 4.1. We may consider more generalized equation of the form

$$\begin{cases} \frac{\partial\beta(u)}{\partial t} - \operatorname{div} a(x,u,Du) + f(x,t,u) = 0 & \text{in} \quad \Omega\times[0,T], \\ u = 0 & \text{on} \quad \partial\Omega\times[0,T], \\ u(\cdot,0) = u^0 & \text{in} \quad \Omega, \end{cases}$$

where the nonlinear operator $-\operatorname{div} a(x,u,Du)$ is same as the operator in Lemma 2.2. As we mentioned in the proof of Theorem 3.1, since the p-Laplacian operator $\Delta_p u=\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ $(1<p<\infty)$ is a special case of $-\operatorname{div} a(x,u,Du)$, the above equation has the same results provided we assume the conditions (H1)–(H4) and (H3)$'$.

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