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GLOBAL STABILITY ANALYSIS FOR A CLASS OF COHEN-GROSSBERG NEURAL NETWORK MODELS

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Abstract. By constructing suitable Lyapunov functionals and combining with matrix inequality technique, a new simple sufficient condition is presented for the global asymptotic stability of the Cohen-Grossberg neural network models. The condition contains and improves some of the previous results in the earlier references.

1. Introduction

In this paper, we are concerned with the model of continuous-time neural networks described by the following systems of the form:

(1)

$$x_i'(t) = c_i(x_i(t))\Big[-d_i(x_i(t)) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t-\tau_j(t))) + J_i\Big],$$
$$i = 1, 2, \ldots, n,$$

or equivalently

$$x'(t) = C(x(t))[-D(x(t)) + Af(x(t)) + Bf(x(t-\tau(t))) + J], \tag{2}$$

where n denotes the number of the neurons in the network, $x_i(t)$ is the state of the ith neuron at time t, $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T \in \mathbb{R}^n$, $f(x(t)) = (f_1(x_1(t)), f_2(x_2(t)), \ldots, f_n(x_n(t)))^T \in \mathbb{R}^n$ denote the activation functions of the jth neuron at time t, $C(x(t)) = \operatorname{diag}(c_1(x_1(t)), c_2(x_2(t)), \ldots, c_n(x_n(t))) > 0$, $D(x(t)) = \operatorname{diag}(d_1(x_1(t)), d_2(x_2(t)), \ldots, d_n(x_n(t))) > 0, A = (a_{ij})_{n\times n}, B = (b_{ij})_{n\times n}$ are the feedback matrix and the delayed feedback matrix, respectively, $J = (J_1, J_2, \ldots, J_n)^T \in \mathbb{R}^n$ be a constant external input vector, the time delay $\tau(t)$ is any nonnegative continuous function with $0 \le \tau_j(t) \le \tau$, and $0 < \tau_j'(t) \le \delta < 1$, where τ, δ are constants.

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System (1) is one of the most popular and generic neural network models. The Cohen-Grossberg neural network models were firstly proposed and studied by Cohen and Grossberg [6], which have been widely applied in various engineering and scientific fields such as neural-biology, population biology, and computing technology. In such applications, it is important to know the convergence properties of the designed neural networks. Usually, this kind of neural networks can be described by the system (1).

In hardware implementation, however, time delays occur due to finite switching speed of the amplifier and communication time [5]. And time delays may lead to oscillation, divergence, or instability, which may be harmful to the systems [2, 13]. On the other hand, it has also been shown that the process of moving images required the introduction of delay in signal transmission through the networks [15]. The stability of dynamical neural networks with time delay which have been used in many applications such as optimization, control and image processing, has received much attention recently (see, for example [1, 3, 4, 7, 9, 10, 11]).

During the past several years, considerable attention has been paid to the delay-dependent stability and control problems of linear neutral systems. Many efforts have been made to obtain less conservative delay-dependent conditions. One important index of measuring the conservatism of the conditions obtained is the maximum allowable upper bound on the delay. Delay dependent conditions via Lyapunov functionals are often based on a fixed model transformation technique that rewrites the delayed term via integration. By transforming the original system into a distributed-delay system, a Lyapunov-Krasovskii functional is constructed for the distributed-delay system As mentioned previously, it is often the case in practice that the network parameters may contain uncertainties due to modeling errors, and the neural network is disturbed by environmental noises that affect the stability of the equilibrium.

In this paper, we will consider the global asymptotic stability of the Cohen-Grossberg neural networks with distributed delays described by (1). The organization of this paper is as follows. In Section 2, problem formulation and preliminaries are given. In Section 3, some new results are given to the Cohen-Grossberg neural networks with distributed delays described by (1) based on Lyapunov method. Section 4 gives an example to illustrate the effectiveness of our results.

2. Preliminaries and lemmas

In our analysis, we assume that the following conditions are satisfied

(H1) There exist constant scalers $l_i > 0$ such that

$$0 \leq \frac{f_i(\eta_1) - f_i(\eta_2)}{\eta_1 - \eta_2} \leq l_i, \qquad \forall\ \eta_1, \eta_2 \in \mathbb{R}, \eta_1 \neq \eta_2;$$

(H2) $c_i(x_i(t)) > 0, c_i$ are bounded, $i = 1, 2, \ldots, n$;

(H3) For all $\eta_1, \eta_2 \in \mathbb{R}, \eta_1 \neq \eta_2$, there exist constant scalers $\mu_i > 0$ such that

$$\frac{d_i(\eta_1) - d_i(\eta_2)}{\eta_1 - \eta_2} \geq \mu_i > 0.$$

The initial conditions associated with system (1) are of the form

$$x_i(s) = \phi_i(s), \qquad s \in [-\tau, 0], \qquad i = 1, 2, \dots, n,$$

in which $\phi_i(s)$ are bounded and continuous for $s \in [-\tau, 0]$.

In the following, we will use the notation $A > 0$ (or $A < 0$) to denote the matrix A is a symmetric and positive definite (or negative definite) matrix. The notation A^T and A^{-1} means the transpose of and the inverse of a square matrix A.

In order to obtain our result, we need establishing the following lemma.

Lemma 1. *For any vectors $a, b \in \mathbb{R}^n$, the inequality*

$$2a^T b \leq \varepsilon a^T a + \frac{1}{\varepsilon} b^T b$$

holds for $\forall\ \varepsilon > 0$.

Assume $x^* = (x_1^*, x_2^*, \dots, x_n^*)^T$ is an equilibrium of Eq.(1), one can derive from (1) that the transformation $y_i(t) = x_i(t) - x_i^*$ transforms system (1) or (2) into the following system:

$$(3) \quad y_i'(t) = \alpha_i(y_i(t))\Big[-\beta_i(y_i(t)) + \sum_{j=1}^{n} a_{ij} g_j(y_j(t)) + \sum_{j=1}^{n} b_{ij} g_j(y_j(t - \tau_j(t)))\Big]$$

for $i = 1, 2, \dots, n$. Where

$$\begin{aligned} \alpha_i(y_i(t)) &= c_i(y_i(t) + x_i^*), \\ \beta_i(y_i(t)) &= d_i(y_i(t) + x_i^*) - d_i(x_i^*), \\ g_j(y_j(t)) &= f_j(y_j(t) + x_j^*) - f_j(x_j^*). \end{aligned}$$

Note that since each function $f_j(\cdot)$ satisfies the hypothesis (H1), hence, each $g_j(\cdot)$ satisfies

$$0 \leq \frac{g_j(y_j)}{y_j} \leq l_j, \forall\ y_j \in \mathbb{R},\ y_j \neq 0, \text{ and } g_j(0) = 0, j = 1, 2, \dots, n$$

and since each function $d_j(\cdot)$ satisfies the hypothesis (H3), hence, each $\beta_j(\cdot)$ satisfies

$$\frac{\beta_j(y_j)}{y_j} \geq \mu_j > 0, \forall\ y_j \in \mathbb{R},\ y_j \neq 0, \text{ and } \beta_j(0) = 0, j = 1, 2, \dots, n.$$

To prove the stability of x^* of Eq.(1), it is sufficient to prove the stability of the trivial solution of Eq.(3).

3. Stability analysis

In the section, we present and prove our main results.

Theorem 1. *Assume that* (H1)-(H3) *are satisfied and there exists a positive diagonal matrix M such that*

$$-2l^{-1}\mu M + MA + A^T M + \frac{1}{1-\delta} MBB^T M + E < 0,$$

where E denotes the identity matrix of size n, $M = \mathrm{diag}(m_i)_{n\times n}$, $l = \mathrm{diag}(l_i)_{n\times n}$, $\mu = \mathrm{diag}(\mu_i)_{n\times n}$. Then the equilibrium point of system (1) *is globally asymptotically stable.*

Proof. Consider the following positive definite Lyapunov function defined by:

$$V(y_t) = 2\sum_{i=1}^n m_i \int_0^{y_i(t)} \frac{g_i(s)}{\alpha_i(s)} ds + \sum_{i=1}^n \int_{t-\tau_i(t)}^t g_i^2(y_i(s))ds.$$

We calculate and estimate the time derivative of $V(y_t)$ along the trajectories of system (3) as follows:

$$\begin{aligned}
V'(y_t) &= 2\sum_{i=1}^n m_i \frac{g_i(y_i(t))}{\alpha(y_i(t))} y_i'(t) + \sum_{i=1}^n [g_i^2(y_i(s)) - g_i^2(y_i(t-\tau_i(t)))(1-\tau_i'(t))] \\
&\le 2\sum_{i=1}^n m_i g_i(y_i(t)) \Big[-\beta_i(y_i(t)) + \sum_{j=1}^n a_{ij} g_j(y_j(t)) \\
&\quad + \sum_{j=1}^n b_{ij} g_j(y_j(t-\tau_j(t)))\Big] + \sum_{i=1}^n [g_i^2(y_i(s)) - g_i^2(y_i(t-\tau_i(t)))(1-\tau_i'(t))] \\
&\le -2\sum_{i=1}^n m_i g_i(y_i(t))\beta_i(y_i(t)) + 2\sum_{i=1}^n\sum_{j=1}^n a_{ij} m_i g_i(y_i(t)) g_j(y_j(t)) \\
&\quad + 2\sum_{i=1}^n\sum_{j=1}^n b_{ij} m_i g_i(y_i(t)) g_j(y_j(t-\tau_j(t))) \\
&\quad + \sum_{i=1}^n [g_i^2(y_i(s)) - g_i^2(y_i(t-\tau_i(t)))(1-\tau_i'(t))] \\
&\le -2\sum_{i=1}^n m_i g_i(y_i(t))\beta_i(y_i(t)) + g^T(y(t))(MA + A^T M)g(y(t)) \\
&\quad + 2g^T(y(t))MBg(y(t-\tau)) + g^T(y(t))g(y(t)) \\
&\quad - (1-\delta)g^T(y(t-\tau))g(y(t-\tau)).
\end{aligned}$$

From Lemma 1, we have

$$2g^T(y(t))MBg(y(t-\tau)) \le \frac{1}{1-\delta}g^T(y(t))MBB^TMg(y(t)) + (1-\delta)g^T(y(t-\tau))g(y(t-\tau)).$$

Then we have

$$\begin{aligned} V'(y_t) &\le -2l^{-1}\mu g^T(y(t))Mg(y(t)) + g^T(y(t))(MA + A^TM)g(y(t)) \\ &\quad + \frac{1}{1-\delta}g^T(y(t))MBB^TMg(y(t)) + g^T(y(t))g(y(t)) \\ &\le g^T(y(t))\Big(-2l^{-1}\mu M + MA + A^TM + \frac{1}{1-\delta}MBB^TM + E\Big)g(y(t)) \\ &< 0. \end{aligned}$$

The proof is complete. □

When the delayed feedback matrix $B = 0$ in Theorem 1, we can easily obtain the following corollary.

Corollary 1. *The equilibrium point of Eq.*(1) *is globally asymptotically stable if there exists a positive diagonal matrix M such that*

$$-2l^{-1}\mu M + MA + A^TM + E < 0.$$

Remark. In Theorem 1 and Corollary 1, we do not need the assumptions of boundedness, monotonicity, and differentiability for the activation functions, moreover, the model discussed is with time-varying delays. Clearly, the proposed results are different from those in [2, 5, 6, 8, 12, 13, 14, 15] and the references cited therein. Therefore, the results of this paper are new and they complement previously known results.

4. An example

In this section, an example is used to demonstrate that the method presented in this paper is effective.

Example. Consider the following two state neural networks:

$$A = (a_{ij})_{2\times 2} = \begin{pmatrix} -2 & 0.5 \\ 0.5 & -2 \end{pmatrix},$$

$$B = (b_{ij})_{2\times 2} = \begin{pmatrix} 0.5 & 0.5 \\ -0.5 & -0.5 \end{pmatrix},$$

and we take $l = \mu = E,\ \delta = 0.5$. Let $M = E$. Then we have

$$-2l^{-1}\mu M + MA + A^TM + \frac{1}{1-\delta}MBB^TM^T + E = \begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix} < 0.$$

Therefore, by Theorem 1, the equilibrium point of Eq.(1) is globally asymptotically stable.

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