Bull. Korean Math. Soc. **49** (2012), No. 6, pp. 1213–1222
http://dx.doi.org/10.4134/BKMS.2012.49.6.1213

EXISTENCE OF THREE SOLUTIONS FOR A MIXED BOUNDARY VALUE PROBLEM WITH THE STURM-LIOUVILLE EQUATION

Diego Averna, Nicoló Giovannelli, and Elisabetta Tornatore

Abstract. The aim of this paper is to establish the existence of three solutions for a Sturm-Liouville mixed boundary value problem. The approach is based on multiple critical points theorems.

1. Introduction

The aim of this paper is to establish, under a suitable set of assumptions, the existence of at least three solutions for the following Sturm-Liouville problem with mixed boundary conditions

$$(RS_\lambda) \qquad \begin{cases} -(pu')' + qu = \lambda f(t,u) \ \text{ in } \ I =]a,b[\\ u(a) = u'(b) = 0, \end{cases}$$

where λ is a positive parameter and p, q, f are regular functions. To be precise, if $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is a L^2-Carathéodory function and $p, q \in L^\infty([a,b])$ such that

$$p_0 := \operatorname*{ess\,inf}_{t\in[a,b]} p(t) > 0, \ \ q_0 := \operatorname*{ess\,inf}_{t\in[a,b]} q(t) \geq 0,$$

then we prove the existence of three weak solutions for problem (RS_λ) (see Theorems 3.1 and 3.2). Clearly, when $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $p \in C^1([a,b])$ and $q \in C^0([a,b])$, the solutions of (RS_λ) are actually classical (see for instance Corollaries 3.1 and 3.2).

The problem (RS_λ) with $p = q = 1$ has been studied in [5] (see also [1]) but it is worth noticing that our results assure a more precise conclusion. In fact, in [5] precise values of parameters λ were not established, and in [1] an asymptotic condition at infinity was assumed (see Remark 4.2).

In our main results a precise interval of real parameters λ for which the problem (RS_λ) admits at least three solutions is established and, in addition, in Theorem 3.2 no asymptotic condition at infinity is assumed. Further, since p and q are variable functions, our results can be applied when the equation is in

Received July 9, 2010; Revised April 19, 2012.
2010 *Mathematics Subject Classification.* 34B15.
Key words and phrases. boundary value problem, mixed conditions.

a complete form (see Section 4). Here, as an example, we present the following result which is a particular case of Theorem 4.3 (see Remark 4.1).

Theorem 1.1. *Let $g : \mathbb{R} \to \mathbb{R}$ be a non-negative continuous and non-zero function such that*

$$\lim_{x\to 0^+} \frac{g(x)}{x} = \lim_{x\to +\infty} \frac{g(x)}{x} = 0.$$

Then, the problem

$$(P) \qquad \begin{cases} -u'' + u' + u = \lambda g(u) \text{ in } I =]0,1[\\ u(0) = u'(1) = 0 \end{cases}$$

for each $\lambda > \bar{\lambda}$, where $\bar{\lambda} = \frac{4e}{3(\sqrt{e}-1)} \inf\{\frac{d^2}{\int_0^d g(\xi)d\xi} : d > 0, \int_0^d g(\xi)d\xi > 0\}$, admits at least two non-zero classical solutions.

Our main tools are three critical points theorems that here recall in a convenient form. Theorem 1.2 has been obtained in [3], it is a more precise version of Theorem 3.2 of [2] and the coercivity of the functional $\Phi - \lambda\Psi$ is required, Theorem 1.3 has been established in [2] and a suitable sign hypothesis is assumed.

Theorem 1.2 ([3, Theorem 3.6])**.** *Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable, coercive and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Assume that*

$$\Phi(0) = \Psi(0) = 0,$$

and that there exist $r \in \mathbb{R}$ and $\bar{u} \in X$, with $0 < r < \Phi(\bar{u})$, such that

- (a_1) $\frac{\sup_{\Phi(u)\le r} \Psi(u)}{r} < \frac{\Psi(\bar{u})}{\Phi(\bar{u})}$;
- (a_2) *for each $\lambda \in \Lambda_r :=]\frac{\Phi(\bar{u})}{\Psi(\bar{u})}, \frac{r}{\sup_{\Phi(u)\le r} \Psi(u)}[$ the functional $\Phi - \lambda\Psi$ is coercive.*

Then, for each $\lambda \in \Lambda_r$, the functional $\Phi - \lambda\Psi$ has at least three distinct critical points in X.

Theorem 1.3 ([2, Corollary 3.1])**.** *Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable, coercive and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact, such that*

$$\Phi(0) = \Psi(0) = 0.$$

Assume that there exist two constants r_1 and r_2 and a function $\bar{u} \in X$ with $2r_1 < \Phi(\bar{u}) < \frac{r_2}{2}$, such that

- (b_1) $\frac{\sup_{\Phi(u)\le r_1} \Psi(u)}{r_1} < \frac{2}{3}\frac{\Psi(\bar{u})}{\Phi(\bar{u})}$;

(b_2) $\frac{\sup_{\Phi(u)\le r_2}\Psi(u)}{r_2} < \frac{1}{3}\frac{\Psi(\bar{u})}{\Phi(\bar{u})}$;

(b_3) *for each* $\lambda \in \Lambda_{r_1,r_2} :=]\frac{3}{2}\frac{\Phi(\bar{u})}{\Psi(\bar{u})}, \min\{\frac{r_1}{\sup_{\Phi(u)\le r_1}\Psi(u)}, \frac{\frac{r_2}{2}}{\sup_{\Phi(u)\le r_2}\Psi(u)}\}[$ *and for every* u_1, $u_2 \in X$, *which are local minima for the functional* $\Phi - \lambda\Psi$, *and such that* $\Psi(u_1) \ge 0$ *and* $\Psi(u_2) \ge 0$ *one has* $\inf_{s\in[0,1]}\Psi(su_1 + (1-s)u_2) \ge 0$.

Then, for each $\lambda \in \Lambda_{r_1,r_2}$ *the functional* $\Phi - \lambda\Psi$ *admits at least three critical points which lie in* $\Phi^{-1}(]-\infty, r_2[)$.

2. Mixed boundary value problem

Consider problem (RS_λ), assume that $p, q \in L^\infty([a,b])$ such that

$$p_0 := \operatorname*{ess\,inf}_{t\in[a,b]} p(t) > 0, \quad q_0 := \operatorname*{ess\,inf}_{t\in[a,b]} q(t) \ge 0.$$

We recall that a function $f : [a,b]\times\mathbb{R} \to \mathbb{R}$ is said a L^1-Carathéodory function if

- $t \to f(t,x)$ is measurable for every $x \in \mathbb{R}$;
- $x \to f(t,x)$ is continuous for almost every $t \in [a,b]$;
- for every $\rho > 0$ one has $\sup_{|x|\le\rho}|f(t,x)| \in L^1([a,b])$ for almost every $t \in [a,b]$.

Put

$$F(t,x) := \int_0^x f(t,\xi)d\xi$$

for all $(t,x) \in [a,b]\times\mathbb{R}$.
Denote by

$$X := \{u \in W^{1,2}([a,b]) : u(a) = 0\}$$

the space endowed with the following norm

$$||u||_X := \left(\int_a^b u^2(t)dt + \int_a^b (u'(t))^2 dt\right)^{\frac{1}{2}}.$$

For every $u, v \in X$, we define

$$(u,v) := \int_a^b p(t)u'(t)v'(t)dt + \int_a^b q(t)u(t)v(t)dt. \tag{1}$$

We observe that (1) defines an inner product on X whose corresponding norm is

$$||u|| := \left(\int_a^b p(t)(u'(t))^2 dt + \int_a^b q(t)(u(t))^2 dt\right)^{\frac{1}{2}}.$$

A simple computation shows that the norm $||\cdot||$ is equivalent to the usual one.

A function $u \in X$ is said a weak solution of problem (RS_λ) if

$$\int_a^b p(t)u'(t)v'(t)dt + \int_a^b q(t)u(t)v(t)dt = \lambda\int_a^b f(t,u(t))v(t)dt \qquad \forall v \in X.$$

Clearly, if f is continuous, $p \in C^1([a,b])$ and $q \in C^0([a,b])$, then the weak solutions of (RS_λ) are classical solutions.

It is well known that $(X, ||\cdot||)$ is compactly embedded in $(C^0([a,b]), ||\cdot||_\infty)$ and one has

$$||u||_\infty \leq \sqrt{\frac{b-a}{p_0}}||u|| \qquad \forall u \in X. \tag{2}$$

In order to study problem (RS_λ), we will use the functionals Φ, $\Psi : X \to \mathbb{R}$ defined by putting

$$\Phi(u) := \frac{1}{2}||u||^2, \quad \Psi(u) := \int_a^b F(t,u(t))dt \qquad \forall u \in X, \tag{3}$$

Φ is continuous and convex, hence it is weakly sequentially lower semicontinuous. Moreover Φ is continuously Gâteaux differentiable and its Gâteaux derivative admits a continuous inverse. On the other hand, Ψ is Gâteaux differentiable with compact derivative and one has

$$\Phi'(u)(v) = \int_a^b p(t)u'(t)v'(t)dt + \int_a^b q(t)u(t)v(t)dt,$$
$$\Psi'(u)(v) = \int_a^b f(t,u(t))v(t)dt \quad \forall v \in X,$$

moreover

$$\Phi(0) = \Psi(0) = 0.$$

A critical point for the functional $\Phi - \lambda\Psi$ is any $u \in X$ such that

$$\Phi'(u)(v) - \lambda\Psi'(u)(v) = 0 \qquad \forall v \in X.$$

We can observe that each critical point for functional $\Phi - \lambda\Psi$ is a weak solution for problem (RS_λ).

Now, put

$$k := \frac{3p_0}{6||p||_\infty + 2(b-a)^2||q||_\infty}, \tag{4}$$

where

$$||p||_\infty := \operatorname*{ess\,sup}_{t\in[a,b]} p(t), \qquad ||q||_\infty := \operatorname*{ess\,sup}_{t\in[a,b]} q(t).$$

3. Main results

Our main results are the following theorems.

Theorem 3.1. *Assume that there exist three positive constants c, d and s with $c < d$, $s < 2$ and a function $\mu \in L^1([a,b])$ such that*

(i) $\int_a^{\frac{a+b}{2}} F(t,\xi)dt > 0 \quad \forall \xi \in [0,d]$;

(ii) $\frac{\int_a^b \max_{|\xi|\leq c} F(t,\xi)dt}{c^2} < k\frac{\int_{\frac{a+b}{2}}^b F(t,d)dt}{d^2}$ *where k is given by* (4);

(iii) $F(t,\xi) \le \mu(t)(1+|\xi|^s)\, \forall t \in [a,b] \quad \forall \xi \in \mathbb{R}$.

Then, for each $\lambda \in \left] \frac{p_0 d^2}{2(b-a)k\int_{\frac{a+b}{2}}^b F(t,d)dt}, \frac{p_0 c^2}{2(b-a)\int_a^b \max_{|\xi|\le c} F(t,\xi)dt} \right[$ *the problem* (RS_λ) *has at least three weak solutions.*

Proof. Our goal is to apply Theorem 1.2. Consider the Sobolev space X and the operators defined in (3).

Now, we claim that (ii) ensures (a_1) of Theorem 1.2. In fact, set $r = \frac{p_0 c^2}{2(b-a)}$ and consider the function $\bar{u} \in X$ defined by putting

$$\bar{u}(t) := \begin{cases} \frac{2d}{b-a}(t-a) \text{ if } t \in [a, \frac{a+b}{2}[\\ d \text{ if } t \in [\frac{a+b}{2}, b]. \end{cases} \tag{5}$$

We observe

$$\begin{aligned}\Phi(\bar{u}) &:= \frac{1}{2}||\bar{u}||^2 \\ &= \frac{1}{2}\left(\frac{4d^2}{(b-a)^2}\int_a^{\frac{a+b}{2}} p(t)dt + \frac{4d^2}{(b-a)^2}\int_a^{\frac{a+b}{2}} (t-a)^2 q(t)dt + d^2\int_{\frac{a+b}{2}}^b q(t)dt\right)\end{aligned}$$

from $0 < c < d$ by using the previous relation and (4) we have

$$0 < r < \Phi(\bar{u}) < \frac{p_0 d^2}{2(b-a)k}.$$

In virtue of (i) we have

$$\Psi(\bar{u}) \ge \int_{\frac{a+b}{2}}^b F(t,d)dt.$$

Therefore, one has

$$\frac{\Psi(\bar{u})}{\Phi(\bar{u})} \ge \frac{2(b-a)k}{p_0 d^2}\int_{\frac{a+b}{2}}^b F(t,d)dt. \tag{6}$$

From (2) if $\Phi(u) \le r$, we have $\max_{t\in[a,b]} |u(t)| \le c$ therefore

$$\sup_{\Phi(u)\le r} \Psi(u) \le \int_a^b \max_{|\xi|\le c} F(t,\xi)dt. \tag{7}$$

Hence, owing to (6), (7) and (ii) condition (a_1) of Theorem 1.2 is verified.

We prove that the operator $\Phi - \lambda\Psi$ is coercive, in fact, for each $u \in X$, by using (iii) one has

$$\begin{aligned}\Phi(u) - \lambda\Psi(u) &= \frac{1}{2}||u||^2 - \lambda\int_a^b F(t,u(t))dt \\ &\ge \frac{1}{2}||u||^2 - \lambda\int_a^b \mu(t)(1+|u(t)|^s)dt \\ &\ge \frac{1}{2}||u||^2 - \lambda\int_a^b \mu(t)dt - \lambda\int_a^b \mu(t)|u(t)|^s dt.\end{aligned}$$

By using (2), we obtain

$$\int_a^b |\mu(t)||u(t)|^s dt \leq ||u||_\infty^s \int_a^b |\mu(t)| dt \leq \left(\frac{b-a}{p_0}\right)^{\frac{s}{2}} ||u||^s ||\mu||_1, \tag{8}$$

substituting (8) into previous relation, we have

$$\Phi(u) - \lambda\Psi(u) \geq \frac{1}{2}||u||^2 - \lambda||\mu||_1 - \lambda\left(\frac{b-a}{p_0}\right)^{\frac{s}{2}} ||u||^s ||\mu||_1$$

hence condition (a_2) of Theorem 1.2 is verified. All assumptions of Theorem 1.2 are satisfied and the proof is complete. □

Now, we point out the following consequence of Theorem 3.1.

Corollary 3.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a nonnegative continuous function and $p \in C^1([a,b])$, $q \in C^0([a,b])$. Assume that there exist positive constants μ, c, d and s, with $c < d$ and $s < 2$, such that*

(i') $\frac{F(c)}{c^2} < \frac{k}{2}\frac{F(d)}{d^2}$ *where k is given by* (4);
(ii') $F(\xi) \leq \mu(1 + |\xi|^s)$ $\forall \xi \in \mathbb{R}$.

Then, for each $\lambda \in \left]\frac{p_0 d^2}{(b-a)^2 kF(d)}, \frac{p_0 c^2}{2(b-a)^2 F(c)}\right[$ the problem (RS_λ) has at least three classical solutions.

Other main result is the following theorem.

Theorem 3.2. *Assume that there exist three positive constants c_1, c_2 and d such that $c_1 < d < \sqrt{\frac{k}{2}}c_2$ and*

(j)

$$\frac{\int_a^b \max_{|\xi|\leq c_1} F(t,\xi)dt}{c_1^2} < \frac{2}{3}k\frac{\int_{\frac{a+b}{2}}^b F(t,d)dt}{d^2},$$

$$\frac{\int_a^b \max_{|\xi|\leq c_2} F(t,\xi)dt}{c_2^2} < \frac{1}{3}k\frac{\int_{\frac{a+b}{2}}^b F(t,d)dt}{d^2},$$

where k is given by (4).

Then, for each $\lambda \in \left]\frac{p_0 d^2}{2(b-a)k\int_{\frac{a+b}{2}}^b F(t,d)dt}, \frac{p_0}{2(b-a)}\min\left\{\frac{c_1^2}{\int_a^b \max_{|\xi|\leq c_1} F(t,\xi)dt}, \frac{\frac{c_2^2}{2}}{\int_a^b \max_{|\xi|\leq c_2} F(t,\xi)dt}\right\}\right[$ the problem (RS_λ) has at least three weak solutions u_i $(i = 1,2,3)$ such that

$$||u_i||_\infty < c_2 \qquad i = 1,2,3.$$

Proof. Our goal is to apply Theorem 1.3. Consider the Sobolev space X and the operators defined in (3). Taking into account that the regularity assumptions on Φ and Ψ are satisfied and that, owing to the Maximum Principle (see [4]), (b_3) holds, our aim is to verify (b_1) and (b_2). To this end, put $\bar{u}$ as in (5), $r_1 = \frac{p_0 c_1^2}{2(b-a)}$, $r_2 = \frac{p_0 c_2^2}{2(b-a)}$ one has $2r_1 < \Phi(\bar{u}) < \frac{r_2}{2}$ and, by using (j)

$$\frac{\int_a^b \max_{|\xi|\leq c_1} F(t,\xi)dt}{c_1^2} < \frac{2}{3}\frac{\Psi(\bar{u})}{\Phi(\bar{u})}, \quad \frac{\int_a^b \max_{|\xi|\leq c_2} F(t,\xi)dt}{c_2^2} < \frac{1}{3}\frac{\Psi(\bar{u})}{\Phi(\bar{u})}.$$

All assumptions of Theorem 1.3 are satisfied and the proof is complete. □

If $f : \mathbb{R} \to \mathbb{R}$ is a non negative continuous function, $p \in C^1([a,b])$ and $q \in C^0([a,b])$ are positive functions, the assumptions of Theorem 3.2 take a simpler form:

Corollary 3.2. *Assume that there exist three positive constants* c_1, c_2 *and* d *with* $c_1 < d < \sqrt{\frac{k}{2}}c_2$

$$\frac{F(c_1)}{c_1^2} < \frac{1}{3}k\frac{F(d)}{d^2},$$
$$\frac{F(c_2)}{c_2^2} < \frac{1}{6}k\frac{F(d)}{d^2},$$

where k *is given by* (4).

Then, for each $\lambda \in \left]\frac{p_0 d^2}{(b-a)^2 kF(d)}, \frac{p_0}{2(b-a)^2}\min\{\frac{c_1^2}{F(c_1)}, \frac{\frac{c_2^2}{2}}{F(c_2)}\}\right[$ *the problem* (RS_λ) *has at least three classical solutions* u_i $(i = 1,2,3)$ *such that*

$$||u_i||_\infty < c_2, \quad i = 1,2,3.$$

4. Consequences and examples

Now, consider the following problem

$$\begin{cases} -(\bar{p}u')' + \bar{r}u' + \bar{q}u = \lambda g(t,u) \text{ in } I =]a,b[\\ u(a) = u'(b) = 0, \end{cases} \tag{9}$$

where $g : [a,b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $\bar{p} \in C^1([a,b])$, $\bar{q}$, $\bar{r} \in C^0([a,b])$ and λ is a positive parameter. Moreover $\bar{p}$, $\bar{q}$ are positive functions and R is a primitive of $\frac{\bar{r}}{\bar{p}}$.

Put $G(t,x) = \int_0^x g(t,\xi)d\xi$ for all $(t,x) \in [a,b] \times \mathbb{R}$

$$\bar{k} := \frac{3\min_{t\in[a,b]}(e^{-R}\bar{p})}{6||e^{-R}\bar{p}||_\infty + 2(b-a)^2||e^{-R}\bar{q}||_\infty}. \tag{10}$$

Observe that the solutions of the problem

$$\begin{cases} -(e^{-R}\bar{p}u')' + e^{-R}\bar{q}u = \lambda e^{-R}g(t,u) \text{ in } I =]a,b[\\ u(a) = u'(b) = 0 \end{cases} \tag{11}$$

are solutions of the problem (9). Hence, in virtue of Theorems 3.1 and 3.2 we obtain the following results.

Theorem 4.1. *Assume that there exist three positive constants* c, d *and* s *with* $c < d$, $s < 2$ *and a function* $\mu \in L^1([a,b])$*such that*

(i$_1$) $G(t,\xi) > 0 \quad \forall t \in [a,b] \quad \forall \xi \in [0,d]$;

(ii$_2$) $\frac{\int_a^b e^{-R(t)}\max_{|\xi|\le c} G(t,\xi)dt}{c^2} < \bar{k}\frac{\int_{\frac{a+b}{2}}^b e^{-R(t)}G(t,d)dt}{d^2}$ *where* $\bar{k}$ *is given by* (10);

(iii$_3$) $G(t,\xi) \le \mu(t)(1+|\xi|^s)$, $\forall t \in [a,b]$, $\quad \forall \xi \in \mathbb{R}$.

Then, for each $\lambda \in \left]\frac{\min_{t\in[a,b]}(e^{-R(t)}\bar{p}(t))d^2}{2(b-a)\bar{k}\int_{\frac{a+b}{2}}^b e^{-R(t)}G(t,d)dt}, \frac{\min_{t\in[a,b]}(e^{-R(t)}\bar{p}(t))c^2}{2(b-a)\int_a^b \max_{|\xi|\le c} e^{-R(t)}G(t,\xi)dt}\right[$ *the problem* (9) *has at least three classical solutions.*

Theorem 4.2. *Let* g *be a positive continuous function and assume that there exist three positive constants* c_1, c_2 *and* d *with* $c_1 < d < \sqrt{\frac{\bar{k}}{2}}c_2$ *such that*

(j')

$$\frac{\int_a^b e^{-R(t)}\max_{|\xi|\le c_1} G(t,\xi)dt}{c_1^2} < \frac{2}{3}\bar{k}\frac{\int_{\frac{a+b}{2}}^b e^{-R(t)}G(t,d)dt}{d^2},$$

$$\frac{\int_a^b e^{-R(t)}\max_{|\xi|\le c_2} G(t,\xi)dt}{c_2^2} < \frac{1}{3}\bar{k}\frac{\int_{\frac{a+b}{2}}^b e^{-R(t)}G(t,d)dt}{d^2},$$

where $\bar{k}$ *is given by* (10).

Then, for each

$\lambda \in \left]\frac{\min_{t\in[a,b]}(e^{-R(t)}\bar{p}(t))d^2}{2(b-a)\bar{k}\int_{\frac{a+b}{2}}^b e^{-R(t)}G(t,d)dt}, \frac{\min_{t\in[a,b]}(e^{-R(t)}\bar{p}(t))}{2(b-a)}\min\{\frac{c_1^2}{\int_a^b \max_{|\xi|\le c_1} e^{-R(t)}G(t,\xi)dt}, \frac{\frac{c_2^2}{2}}{\int_a^b \max_{|\xi|\le c_2} e^{-R(t)}G(t,\xi)dt}\}\right[$

the problem (9) *has at least three classical solutions* u_i $(i = 1,2,3)$ *such that*

$$||u_i||_\infty < c_2, \quad i = 1,2,3.$$

Now we point out the following application of Theorem 4.2 to the autonomous case.

Theorem 4.3. *Let* $g : \mathbb{R} \to \mathbb{R}$ *be a non-negative continuous and non-zero function such that*

$$\lim_{x\to 0^+}\frac{g(x)}{x} = \lim_{x\to+\infty}\frac{g(x)}{x} = 0. \tag{12}$$

Then, for each $\lambda > \lambda^*$, *where*

$$\lambda^* = \inf\{\frac{\min_{t\in[a,b]}(e^{-R(t)}\bar{p}(t))d^2}{2(b-a)\bar{k}\int_{\frac{a+b}{2}}^b e^{-R(t)}dt\int_0^d g(\xi)d\xi} : d > 0, \int_0^d g(\xi)d\xi > 0\}$$

the problem

$$\begin{cases} -(\bar{p}u')' + \bar{r}u' + \bar{q}u = \lambda g(u) \text{ in } I =]a,b[\\ u(a) = u'(b) = 0 \end{cases}$$

has at least two non-zero classical solutions.

Proof. Fix $\lambda > \lambda^*$ and put $G(x) = \int_0^x g(\xi)d\xi$ for all $x \in \mathbb{R}$, there exists $d > 0$ such that $\lambda > \frac{\min_{t\in[a,b]}(e^{-R(t)}\bar{p}(t))d^2}{2(b-a)\bar{k}\int_{\frac{a+b}{2}}^b e^{-R(t)}dtG(d)}$.

From (12) there is $c_1 > 0$ such that $c_1 < d$ and $\frac{G(c_1)}{c_1^2} < \frac{\min_{t\in[a,b]}(e^{-R(t)}\bar{p}(t))}{3\lambda(b-a)\int_a^b e^{-R(t)}dt}$, and there is $c_2 > 0$ such that $d < \sqrt{\frac{\bar{k}}{2}}c_2$ and $\frac{G(c_2)}{c_2^2} < \frac{\min_{t\in[a,b]}(e^{-R(t)}\bar{p}(t))}{6\lambda(b-a)\int_a^b e^{-R(t)}dt}$, where $\bar{k}$ is given by (10). The conclusion follows from Theorem 4.2. □

Remark 4.1. Theorem 1.1 in the introduction is a consequence of Theorem 4.3 taking (10) into account.

Example 4.1. The problem

$$\begin{cases} -u'' + u' + e^{2t}u = 2\lambda te^t u^{10}(11-u) \text{ in } I =]0,1[\\ u(0) = u'(1) = 0 \end{cases}$$

admits at least three classical solutions for each $\lambda \in]\frac{3+e}{2^6 \cdot 15}, \frac{6}{11e}[$.

In fact, if we choose, for example, $c = 1$ and $d = 2$, hypotheses of Theorem 4.1 are satisfied.

Remark 4.2. In ([1]), it has been studied a mixed boundary problem of type

$$\begin{cases} -\left(|u'|^{s-2}u'\right)' + |u|^{s-2}u = \lambda f(t,u) \text{ in } I =]a,b[\\ u(a) = u'(b) = 0. \end{cases}$$

We observe that the case studied, when $s = 2$, gives back our case with $p = q = 1$. It is important noticing that, differently from the previous cited paper the coefficients p and q of our equation can depend on variable t, then the results of ([1]) can not applied to Example 4.1.

Example 4.2. The problem

$$\begin{cases} -u'' + u = \lambda e^{-u}u^2(3-u) \text{ in } I =]0,1[\\ u(0) = u'(1) = 0 \end{cases}$$

admits at least three classical solutions for each $\lambda \in]\frac{8e}{3}, \frac{50}{7}e^{-\frac{7}{100}}[\subset [0,8]$.

In fact, if we choose, for example, $c = \frac{7}{100}$ and $d = 1$, hypotheses of Theorem 3.1 are satisfied.

Remark 4.3. The Example 4.2 has been studied in ([5]) obtaining the existence of at least three solutions for each $\lambda \in \Lambda \subseteq [0,8]$, but the open interval Λ was not established while we obtain precise values of parameter λ.

Example 4.3. Consider the following problem

$$\begin{cases} -u'' + u = \lambda t \cdot h(u) \text{ in } I =]0,1[\\ u(0) = u'(1) = 0 \end{cases}$$

where $h : \mathbb{R} \to \mathbb{R}$

$$h(x) := \begin{cases} 1, \text{ if } t \in]-\infty, 1]; \\ x^{10}, \text{ if } t \in]1,2]; \\ 2^{10}, \text{ if } t \in]2,800]; \\ x^2, \text{ if } t \in]800,+\infty[; \end{cases}$$

admits at least three classical solutions u_i $(i = 1,2,3)$ for each

$$\lambda \in]\frac{2^7 \cdot 11}{3^2(5+2^{10})}, \frac{2^8 \cdot 5^3}{\frac{1-2^{11}}{11} + 2^{14} \cdot 5}[$$

such that $|u_i(t)| < 800$ for all $t \in [0,1]$.

In fact, if we choose, for example, $c_1 = 1$, $c_2 = 800$ and $d = 2$, hypotheses of Theorem 3.2 are satisfied.

Remark 4.4. The Theorem 3.1 can not applied to Example 4.3 because the function is positive but it is not sub-linear at infinity.

Acknowledgments. The authors wish to thank Professor Gabriele Bonanno of University of Messina for his useful comments during the writing of this paper.

References

[1] D. Averna and R. Salvati, *Three solutions for a mixed boundary value problem involving the one-dimensional p-Laplacian*, J. Math. Anal. Appl. **298** (2004), no. 1, 245–260.

[2] G. Bonanno and P. Candito, *Non-differentiable functionals and applications to elliptic problems with discontinuous nonlinearities*, J. Differential Equations **244** (2008), no. 12, 3031–3059.

[3] G. Bonanno and S. A. Marano, *On the structure of the critical set of non-differentiable functions with a weak compactness condition*, Appl. Anal. **89** (2010), no. 1, 1–10.

[4] H. Brézis, *Analyse fonctionelle - théorie et applications*, Masson, Paris, 1983.

[5] R. Salvati, *Multiple Solution for a mixed boundary value problem*, Math. Sci. Res. J. **7** (2003), no. 7, 275–283.

[6] E. Zeidler, *Nonlinear functional analysis and its applications, vol III*, Springer, Berlin, 1990.

Diego Averna
Dipartimento di Matematica ed Informatica
Facoltà di Scienze
Università degli Studi di Palermo
Via Archirafi 34, 90123 - Palermo (Italy)
E-mail address: averna@math.unipa.it

Nicoló Giovannelli
Dipartimento di Metodi e Modelli Matematici
Facoltà di Ingegneria
Università degli Studi di Palermo Viale delle Scienze
90128 - Palermo (Italy)
E-mail address: giovanic@unipa.it

Elisabetta Tornatore
Dipartimento DIEETCAM
Facoltà di Ingegneria
Università degli Studi di Palermo
Viale delle Scienze, 90128 - Palermo (Italy)
E-mail address: elisa.tornatore@unipa.it