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A NOTE ON QUASI-PERIODIC PERTURBATIONS OF ELLIPTIC EQUILIBRIUM POINTS

Houyu Zhao

Abstract. The system

$$\dot{x} = (A + \varepsilon Q(t, \varepsilon))x + \varepsilon g(t, \varepsilon) + h(x, t, \varepsilon),$$

where A is elliptic whose eigenvalues are not necessarily simple and h is $\mathcal{O}(x^2)$. It is proved that, under suitable hypothesis of analyticity, for most values of the frequencies, the system is reducible.

1. Introduction and main result

Before stating our questions, we first give some definitions and notations.

Definition 1.1. A function $f : \mathbb{R} \to \mathbb{R}$ is called real analytic quasi-periodic with the frequencies $\omega = (\omega_1, \omega_2, \ldots, \omega_r)$ if it can be represented as a Fourier series of the type

$$f(t) = \sum_k f_k e^{\mathrm{i}\langle k,\omega\rangle t}, \tag{1.1}$$

where $k = (k_1, \ldots, k_r)$, $\langle k, \omega\rangle = \sum k_j \omega_j \neq 0$ if $k \neq 0$, the coefficients f_k decay exponentially with $|k| = |k_1| + \cdots + |k_r|$.

We denote by $Q(\omega)$ the set of real analytic quasi-periodic functions with the frequencies ω.

From the above definition, we see that the function $F : \vartheta = (\vartheta_1, \ldots, \vartheta_r) \in \mathbb{R}^r \to \mathbb{R}$ defined by

$$F(\vartheta) = \sum_k f_k e^{\mathrm{i}\langle k,\vartheta\rangle}$$

is 2π-periodic in each variable and bounded in a complex neighborhood of $\mathbb{R}^r : |\mathrm{Im}\vartheta_j| \le \rho$ for some $\rho > 0$. This function is called a shell function of f.

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Definition 1.2. Let $v(t)$ be a quasi-periodic function with rationally independent frequencies $\omega_1,\dots,\omega_r$ and shell function V, satisfying $v(t) = V(\omega_1 t,\dots,\omega_r t)$. Then the limit

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T v(t)dt$$

is called the time average of v, coincides with the space average

$$v_0 = \frac{1}{(2\pi)^r}\int_{\mathbb{T}^r} V(\vartheta)d\vartheta$$

and we denote the common value by $\overline{v}$ as well.

Definition 1.3. Let $Q_\rho(\omega)\subset Q(\omega)$ be the set of real analytic functions f such that the corresponding shell functions F are bounded on the subset $\Theta(\rho) = \{(\vartheta_1,\dots,\vartheta_r)\in\mathbb{C}^r : |\mathrm{Im}\vartheta_j|\le\rho\}$, with the sup-norm

$$\|f\|_\rho = \sup_{\vartheta\in\Theta(\rho)}\left|\sum_k f_k e^{\mathrm{i}\langle k,\vartheta\rangle}\right| = \sup_{\vartheta\in\Theta(\rho)}|F(\vartheta)|.$$

By Cauchy's formula, it follows that

$$|f_k|\le\|f\|_\rho e^{-|k|\rho}\quad\text{and}\quad |f_\vartheta|_{\rho'}\le(\rho-\rho')^{-1}\|f\|_\rho \tag{1.2}$$

for $f\in Q_\rho(\omega)$ and $0<\rho'<\rho$.

Since N. N. Bogoljubov et al. [1] in 1960's proved reducibility of non-autonomous finite-dimensional linear systems to constant coefficient equations by KAM-technique, establishing the reducibility of finite-dimensional systems by means of the KAM tools has become an active field of research. Such results are also included in [11]. In this directions, we refer [2]-[16] and references therein for a detailed description. In particular, A. Jorba and C. Simó [9] investigated the reducibility of the quasi-periodic ordinary differential equation

$$\dot{x} = (A+\varepsilon Q(t,\varepsilon))x + \varepsilon g(t,\varepsilon) + h(x,t,\varepsilon), \tag{1.3}$$

where A is an elliptic constant square matrix with order d (that is, all the eigenvalues are purely imaginary and nonzero), Q is a square matrix with order d, h is a vector function of second order in x, and Q,g,h are quasi-periodic in time t with frequency vector $\omega = (\omega_1,\omega_2\dots,\omega_r)$. Such equation (1.3) is an autonomous differential equation under quasi-periodic time-dependent perturbations near an elliptic equilibrium point. More precisely, under suitable conditions of analyticity, nonresonance and nondegeneracy with respect to ε, A. Jorba and C. Simó [9] proved that the system (1.3) is reducible for ε in some Cantorian set $\mathcal{E}\subset(0,\varepsilon_0)$ with ε_0 sufficiently small, provided that

(1) the eigenvalues $\lambda_1,\dots,\lambda_d$ of A are different;

(2) the eigenvalues $\lambda_1^0(\varepsilon),\dots,\lambda_d^0(\varepsilon)$ of $\underline{A} = A+\varepsilon\overline{Q}(\varepsilon)+\overline{D_x h(\underline{x}(t,\varepsilon),t,\varepsilon)}$ satisfy

$$0<2\delta|\varepsilon_1-\varepsilon_2|<|\lambda_i^0(\varepsilon_1)-\lambda_j^0(\varepsilon_1)-\lambda_i^0(\varepsilon_2)+\lambda_j^0(\varepsilon_2)|<\frac{\overline{\delta}}{2}|\varepsilon_1-\varepsilon_2|, \tag{1.4}$$

$$(1.5) \qquad 0 < 2\delta|\varepsilon_1 - \varepsilon_2| < |\lambda_k^0(\varepsilon_1) - \lambda_k^0(\varepsilon_2)| < \frac{\overline{\delta}}{2}|\varepsilon_1 - \varepsilon_2|$$

for all i, j, and k satisfying $1 \leq i < j \leq d$ and $1 \leq k \leq d$, and provided that $|\varepsilon_1|$ and $|\varepsilon_2|$ are less than some small value ε_0. Here $\underline{x}(t)$ is the unique analytical quasi-periodic solution of $\dot{x} = Ax + \varepsilon g(t, \varepsilon)$.

Naturally, we should ask that whether or not (1.3) is reducible when conditions (1) or (2) is not satisfied. In this paper, we will give an answer to this question.

Throughout this paper, we always assume that the following hypothesis is satisfied:

(H) $Q(t), g(t)$, and $h(t)$ are in $Q_\rho(\omega)$ and $\omega \in D_\gamma$:

$$D_\gamma : \;=\; \left\{\omega \in \mathbb{R}^r : \left||\sqrt{-1}(k,\omega)E_d - A|_e\right| > \frac{\gamma}{|k|^\tau}, \right.$$
$$\left. \left||\sqrt{-1}(k,\omega)E_{d^2} - E_d \otimes A + A^\top \otimes E_d|_e\right| > \frac{\gamma}{|k|^\tau}\right\},$$

where $k \in \mathbb{Z}^r \setminus \{0\}$, $1 \leq i, j \leq d$, $\gamma > 0$, $\tau \geq r - 1$ and $|\cdot|_e$ denotes the determinant of a matrix.

If $x \in \mathbb{R}^n$, we denote by $\|x\|$ the sup norm of x. If A is a matrix, $\|A\|$ denotes the corresponding sup-norm, and for a matrix-valued function $Q(t)$, define

$$||Q||^U = \sup_{t \in U} ||Q(t)||,$$

where $||\cdot||$ is the sup-norm of the matrix.

In the present paper, we will follow the techniques developed in [9] and [16] to prove the following result.

Theorem 1.1. *Assume that* (H) *is satisfied. Let $\Omega_0 \subset \mathbb{R}^r$ be a compact set with positive Lebesgue measure. Consider the system* (1.3) *and assume that*

(A1) $\det A \neq 0$, *and $Q(t, \varepsilon)$, $g(t, \varepsilon)$ and $h(x, t, \varepsilon)$ are in $Q_\rho(\omega)$ with $\omega \in \Omega_0$.*

(A2) *$h(x, t, \varepsilon)$ is analytic with respect to x on the ball $B_\kappa(0)$ centered in the origin with radius κ, $h(0, t, \varepsilon) = 0$ and $\|D_{xx}h(x, t, \varepsilon)\| \leq K$ with $\|x\| \leq \kappa$ and K positive constant. Then for a sufficiently small positive constant γ, there exist a subset $\Omega \subset \Omega_0$ with $Meas(\Omega_0 \setminus \Omega) = Meas(\Omega_0)(1 - O(\gamma^{\frac{1}{d^2}}))$ and a sufficiently small constant $\epsilon(\rho, \gamma) > 0$ such that for any $\varepsilon \in (0, \epsilon)$, there is an analytic quasi-periodic transformation $y = \Psi(t)x$ with the basic frequencies ω such that the system* (1.3) *is transformed into*

$$\dot{y} = A_\infty(\varepsilon)y + h_\infty(y, t, \varepsilon),$$

where A_∞ is a constant matrix and $h_\infty(y, t, \varepsilon)$ is of second order in y.

Corollary 1. *Under the hypothesis of Theorem 1.1, for any $\varepsilon \in (0, \epsilon)$,* (1.3) *has a quasi-periodic solution $x_\varepsilon(t)$ with basic frequencies ω such that*

$$\lim_{\substack{\varepsilon \to 0 \\ \varepsilon \in E}} ||x_\varepsilon|| = 0.$$

2. Preliminary lemmas

Lemma 2.1. *Let us consider the equation*

$$\dot{x} = Ax + \varepsilon g(t), \tag{2.1}$$

where A is a $d \times d$ matrix with $\det A \neq 0$ and $g(t) \in Q_{\rho_1}(\omega)$. If $0 < \rho_2 < \rho_1$ and $\omega \in D_\gamma$, then the equation (2.1) has a unique quasi-periodic solution $x(t) \in Q_{\rho_2}(\omega)$ satisfying

$$\|x\|_{\rho_2} \leq \varepsilon L_1 \|g\|_{\rho_1},$$

where $L_1 = \|A^{-1}\| + \frac{C}{\gamma}\left(\frac{\tau}{e}\right)^\tau \frac{(1+e)^r}{(\frac{\rho_1-\rho_2}{2})^{\tau+r}}$.

Proof. We write

$$x(t) = \sum_{k \in \mathbb{Z}^r} x_k e^{\sqrt{-1}(k,\omega)t}, \quad g(t) = \sum_{k \in \mathbb{Z}^r} g_k e^{\sqrt{-1}(k,\omega)t},$$

where x_k and g_k are vectors. From $\dot{x} = Ax + \varepsilon g(t)$ we get

$$\left(\sqrt{-1}(k,\omega)E_d - A\right)x_k = \varepsilon g_k, \tag{2.2}$$

where E_d is a d dimension unit matrix.

When $k = 0$, because $\det A \neq 0$, we get

$$|x_0| \leq \varepsilon ||A^{-1}||_{\rho_2} |g_0| \leq \varepsilon ||A^{-1}||_{\rho_1} ||g||_{\rho_1}.$$

When $k \neq 0$, by (H) and (1.2), the equation (2.2) is solvable for $\omega \in D_\gamma$, and

$$|x_k| \leq \varepsilon \left\| \left(\sqrt{-1}(k,\omega)E_d - A\right)^{-1} \right\|_{\rho_2} |g_k| \leq \varepsilon C \frac{|k|^\tau}{\gamma} ||g||_{\rho_1} e^{-\rho_1 |k|}. \tag{2.3}$$

Further, we have

$$\begin{aligned} ||x||_{\rho_2} &\leq \varepsilon ||A^{-1}||_{\rho_1} ||g||_{\rho_1} + \sum_{k \neq 0} \varepsilon C \frac{|k|^\tau}{\gamma} ||g||_{\rho_1} e^{-(\rho_1-\rho_2)|k|} \\ &\leq \varepsilon \Big[||A^{-1}|| + \frac{C}{\gamma}\Big(\frac{\tau}{e}\Big)^\tau \frac{(1+e)^r}{(\frac{\rho_1-\rho_2}{2})^{\tau+r}} \Big] ||g||_{\rho_1} \\ &= \varepsilon L_1 ||g||_{\rho_1}, \end{aligned} \tag{2.4}$$

where the second inequality follows from Lemma 6.1. □

Lemma 2.2. *Consider the equation*

$$\dot{P} = AP - PA + Q^*, \tag{2.5}$$

where A is a $d \times d$ matrix and $Q^*(t) = \sum_{0 < |k| \leq M} Q_k e^{\sqrt{-1}(k,\omega)t} \in Q_{\rho_1}(\omega)$. Let $0 < \rho_2 < \rho_1$ and $\omega \in D_\gamma$. Then the equation (2.5) has a unique quasi-periodic solution $P(t) \in Q_{\rho_2}(\omega)$ satisfying

$$||P||_{\rho_2} \leq L_2 ||Q||_{\rho_1},$$

where $Q = \sum_{k \in \mathbb{Z}^r} Q_k e^{\sqrt{-1}(k,\omega)t}$, $L_2 = \frac{C}{\gamma}\left(\frac{\tau}{e}\right)^\tau \frac{(1+e)^r}{(\frac{\rho_1-\rho_2}{2})^{\tau+r}}$.

Proof. Let

$$P(t) = \sum_{0<|k|\le M} P_k e^{\sqrt{-1}(k,\omega)t}.$$

Then it is easy to see that

$$\Big(\sqrt{-1}(k,\omega)E_d - A\Big)P_k + P_k A = Q_k. \tag{2.6}$$

By (H), (1.2) and Lemma 6.1, we have

$$\begin{aligned} |P_k| &\le \Big\|\Big(\sqrt{-1}(k,\omega)E_{d^2} - E_d\otimes A + A^\top \otimes E_d\Big)^{-1}\Big\|_{\rho_2}|Q_k| \\ &\le C\frac{|k|^\tau}{\gamma}||Q||_{\rho_1}e^{-\rho_1|k|} \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} ||P||_{\rho_2} &\le \sum_{0<|k|\le M} C\frac{|k|^\tau}{\gamma}||Q||_{\rho_1}e^{-\rho_1|k|}e^{\rho_2|k|} \\ &\le \sum_{k\in\mathbb{Z}^r} C\frac{|k|^\tau}{\gamma}||Q||_{\rho_1}e^{-(\rho_1-\rho_2)|k|} \\ &\le \frac{C}{\gamma}\Big(\frac{\tau}{e}\Big)^\tau\frac{(1+e)^r}{(\frac{\rho_1-\rho_2}{2})^{\tau+r}}||Q||_{\rho_1}. \end{aligned} \tag{2.8}$$

□

Lemma 2.3. *Let us consider*

$$\dot{x} = (A+\varepsilon Q(t))x + \varepsilon g(t) + h(x,t), \tag{2.9}$$

where $Q(t), g(t), h(x,t) \in Q_{\rho_1}(\omega)$ and $0<\rho_2<\rho_1$. Also, we assume that $h(x,t)$ is analytic with respect to x on the ball $B_\kappa(0)$ and satisfies $||D_{x,x}h(x,t)||_{\rho_1} \le K, \forall x \in B_\kappa(0)$. Then for a solution $\underline{x}(t) \in Q_{\rho_2}(\omega)$ of the equation (2.1)*, the change of variables $x = y + \underline{x}(t)$ transforms the initial equation* (2.9) *into*

$$\dot{y} = (\overline{A} + \varepsilon\widetilde{Q}_1(t))y + \varepsilon_1 g_1(t) + h_1(y,t),$$

where $\varepsilon_1 = \varepsilon^{1+\iota}$ and $\widetilde{Q}_1$ has zero average and the following bounds hold for $\omega \in D_\gamma$:

1. $||\widetilde{Q}_1||_{\rho_2} \le 2||Q||_{\rho_1} + 2KL_1||g||_{\rho_1}$*, where L_1 was defined in Lemma 2.1.*
2. $||g_1||_{\rho_2} \le \Big(\frac{KL_1^2(||g||_{\rho_1})^2}{2} + L_1||Q||_{\rho_1}||g||_{\rho_1}\Big)\varepsilon^{1-\iota}$.
3. $||\overline{A}|| \le ||A|| + \varepsilon(||Q||_{\rho_1} + KL_1||g||_{\rho_1})$.
4. $||D_{yy}h_1||_{\rho_2} \le K$.
5. $||\underline{x}||_{\rho_2} \le \varepsilon L_1||g||_{\rho_1}$.

Here $y \in B_{\kappa_1}(0)$, $\kappa_1 = \kappa - ||\underline{x}||_{\rho_2}$, and ε is small enough.

Proof. Let $\underline{x}$ be such that $\dot{\underline{x}} = A\underline{x} + \varepsilon g$. In Lemma 2.1, we have

$$||\underline{x}||_{\rho_2} \le \varepsilon L_1||g||_{\rho_1}.$$

By the change of variables $x = y + \underline{x}(t)$, we can obtain

$$\dot{y} = (A + \varepsilon Q_1(t))y + \varepsilon_1 g_1(t) + h_1(y,t),$$

where $Q_1 = Q + \frac{1}{\varepsilon} D_x h(\underline{x}(t), t)$, $g_1 = \frac{1}{\varepsilon^{1+\iota}} h(\underline{x}(t), t) + \frac{1}{\varepsilon^{\iota}} Q\underline{x}(t)$ $(\varepsilon \neq 0)$, and $h_1(y,t) = h(\underline{x}(t) + y, t) - h(\underline{x}(t), t) - D_x h(\underline{x}(t), t)y$. Like [9], the terms of this equation must be bounded. First, by Lemma 6.4, we have

$$||Q_1||_{\rho_2} \leq ||Q||_{\rho_2} + \frac{1}{\varepsilon} K ||\underline{x}||_{\rho_2} \leq ||Q||_{\rho_1} + KL_1 ||g||_{\rho_1}.$$

Then we bound $||g_1||_{\rho_2}$, also from Lemma 6.4, we get

$$\begin{aligned} ||g_1||_{\rho_2} &\leq \frac{K}{2} \frac{1}{\varepsilon^{1+\iota}} (||\underline{x}||_{\rho_2})^2 + \frac{1}{\varepsilon^{\iota}} ||Q||_{\rho_1} ||\underline{x}||_{\rho_2} \\ &\leq \Big(\frac{KL_1^2 (||g||_{\rho_1})^2}{2} + L_1 ||Q||_{\rho_1} ||g||_{\rho_1} \Big) \varepsilon^{1-\iota}. \end{aligned}$$

The third one is $D_{yy} h_1(y,t)$,

$$||D_{yy} h_1||_{\rho_2} = ||D_{xx} h(\underline{x}(t) + y, t)|| \leq K.$$

So we must require that $y \in B_{\kappa_1}(0)$, where $\kappa_1 = \kappa - ||\underline{x}||_{\rho_2}$. We define $Q_1(t) = \overline{Q}_1 + \widetilde{Q}_1(t)$, $\overline{A} = A + \varepsilon \overline{Q}_1$, we obtain

$$\dot{y} = (\overline{A} + \varepsilon \widetilde{Q}_1(t))y + \varepsilon_1 g_1(t) + h_1(y,t).$$

At last,

$$\begin{aligned} ||\overline{A}|| &\leq ||A|| + \varepsilon ||\overline{Q}_1||_{\rho_2} \\ &\leq ||A|| + \varepsilon ||Q_1||_{\rho_2} \\ &\leq ||A|| + \varepsilon (||Q||_{\rho_1} + KL_1 ||g||_{\rho_1}), \end{aligned}$$

and

$$||\widetilde{Q}_1||_{\rho_2} \leq 2||Q_1||_{\rho_2} \leq 2(||Q||_{\rho_1} + KL_1 ||g||_{\rho_1}). \qquad \square$$

Lemma 2.4. *Let us consider*

$$\dot{x} = (A + \varepsilon Q(t))x + \varepsilon_1 g(t) + h(x,t), \tag{2.10}$$

where $Q(t), g(t), h(x,t) \in \mathcal{Q}_{\rho_1}(\omega), 0 < \rho_2 < \rho_1$ *and* Q *has zero average. Also, we assume that* $h(x,t)$ *is analytic with respect to* x *on the ball* $B_\kappa(0)$ *and satisfies* $||D_{xx} h(x,t)||_{\rho_1}^{\mathcal{O}_1} \leq K, \forall x \in B_\kappa(0)$. *Then the change of variables* $x = (E + \varepsilon P(t))y$ *with* $P(t) \in \mathcal{Q}_{\rho_2}(\omega)$ *transforms the initial equation* (2.10) *into*

$$\dot{y} = (\overline{A} + \varepsilon_1 \widetilde{Q}_1(t))y + \varepsilon_1 g_1(t) + h_1(y,t),$$

where $\widetilde{Q}_1$ *has zero average,* E *is the identity* $d \times d$ *matrix and the following bounds hold for* $\omega \in D_\gamma$:

1. $||\widetilde{Q}_1||_{\rho_2} \leq 2\varepsilon^{1-\iota} \frac{(L_2' + ||P||_{\rho_2})}{1 - \varepsilon ||P||_{\rho_2}} ||Q||_{\rho_1}$, *where* $||P||_{\rho_2} \leq L_2 ||Q||_{\rho_1}$ *and* L_2 *was defined in Lemma 2.2.*
2. $||g_1||_{\rho_2} \leq \frac{1}{1 - \varepsilon ||P||_{\rho_2}} ||g||_{\rho_1}$.
3. $||\overline{A}|| \leq ||A|| + \varepsilon^2 \frac{(L_2' + ||P||_{\rho_2})}{1 - \varepsilon ||P||_{\rho_2}} ||Q||_{\rho_1}$.

4. $||D_{yy}h_1||_{\rho_2} \le K\frac{(1+\varepsilon||P||_{\rho_2})^2}{1-\varepsilon||P||_{\rho_2}}$.
Here $y \in B_{\kappa_2}(0)$, *where* $\kappa_2 = \frac{\kappa}{1+\varepsilon||P||_{\rho_2}}$, *and* ε *is small enough.*

Proof. By the change of variables $x = (E + \varepsilon P)y$, we can obtain

$$\dot{y} = \Big(E + \varepsilon P\Big)^{-1}\Big(A + \varepsilon(Q^* + AP - \dot{P}) + (\varepsilon Q^{**} + \varepsilon^2 QP)\Big)y + \varepsilon_1 g_1(t) + h_1(y,t),$$

where

$$\begin{aligned} Q &= Q^* + Q^{**}, \\ Q^*(t) &= \sum_{0<|k|\le M} Q_k e^{\sqrt{-1}(k,\omega)t}, \\ Q^{**}(t) &= \sum_{|k|>M} Q_k e^{\sqrt{-1}(k,\omega)t}, \\ g_1 &= (E + \varepsilon P)^{-1} g, \text{ and} \\ h_1(y,t) &= (E+\varepsilon P)^{-1} h((E+\varepsilon P)y, t). \end{aligned}$$

We would like to have

$$(E + \varepsilon P)^{-1}(A + \varepsilon(Q^* + AP - \dot{P})) = A,$$

this implies that

$$\dot{P} = AP - PA + Q^*.$$

From Lemma 2.2 we have

$$||P||_{\rho_2} < L_2||Q||_{\rho_1}.$$

We obtain the equation

$$\dot{y} = (A + \varepsilon_1 Q_1(t))y + \varepsilon_1 g_1(t) + h_1(y,t),$$

where $\varepsilon_1 Q_1 = (E + \varepsilon P)^{-1}(\varepsilon Q^{**} + \varepsilon^2 QP)$.

Now we bounded the terms of this equation. In fact

$$\begin{aligned} ||Q^{**}||_{\rho_2} &\le \sum_{|k|>M} ||Q_k||_{\rho_2} e^{\rho_2|k|} \\ &\le ||Q||_{\rho_1} \sum_{|k|>M} e^{-(\rho_1-\rho_2)|k|} \\ &= ||Q||_{\rho_1} \sum_{|k|>0} e^{-(\rho_1-\rho_2)(|k|+M)} \\ &\le \varepsilon||Q||_{\rho_1} \sum_{|k|>0} e^{-(\rho_1-\rho_2)|k|} \\ &\le \varepsilon L_2'||Q||_{\rho_1}, \end{aligned} \tag{2.11}$$

then

$$||\varepsilon_1 Q_1||_{\rho_2} \le \frac{1}{1 - \varepsilon||P||_{\rho_2}}(\varepsilon||Q^{**}||_{\rho_2} + \varepsilon^2||Q||_{\rho_2}||P||_{\rho_2})$$

$$\leq \varepsilon^2 \frac{(L_2' + ||P||_{\rho_2})}{1 - \varepsilon||P||_{\rho_2}} ||Q||_{\rho_1}, \tag{2.12}$$

and

$$||Q_1||_{\rho_2} \leq \varepsilon^{1-\iota} \frac{(L_2' + ||P||_{\rho_2})}{1 - \varepsilon||P||_{\rho_2}} ||Q||_{\rho_1},$$

$$||g_1||_{\rho_2} \leq \frac{1}{1 - \varepsilon||P||_{\rho_2}} ||g||_{\rho_1},$$

$$||D_{yy}h_1||_{\rho_2} \leq \frac{K}{1 - \varepsilon||P||_{\rho_2}} (||E + \varepsilon P||_{\rho_2})^2 \leq K \frac{(1 + \varepsilon||P||_{\rho_2})^2}{1 - \varepsilon||P||_{\rho_2}},$$

in this we require $y \in B_{\kappa_2}(0)$, where $\kappa_2 = \frac{\kappa}{1+\varepsilon||P||_{\rho_2}}$, and ε is small enough. Let $Q_1(t) = \overline{Q}_1 + \widetilde{Q}_1(t), \overline{A} = A + \varepsilon_1 \overline{Q}_1$. Then we obtain

$$\dot{y} = (\overline{A} + \varepsilon_1 \widetilde{Q}_1(t))y + \varepsilon_1 g_1(t) + h_1(y, t).$$

At last,

$$||\overline{A}|| \leq ||A|| + \varepsilon_1 ||Q_1||_{\rho_2} \leq ||A|| + \varepsilon^2 \frac{(L_2' + ||P||_{\rho_2})}{1 - \varepsilon||P||_{\rho_2}} ||Q||_{\rho_1},$$

$$||\widetilde{Q}_1||_{\rho_2} \leq 2||Q_1||_{\rho_2} \leq 2\varepsilon^{1-\iota} \frac{(L_2' + ||P||_{\rho_2})}{1 - \varepsilon||P||_{\rho_2}} ||Q||_{\rho_1}.$$

□

3. Iterative lemma

We will invoke the KAM iterative technique to prove Theorem 1.1. To that end, for given $\varepsilon > 0$, $\rho > 0$ and $r > 0$, we first introduce some iterative sequences.

(i) $\varepsilon_0 = \varepsilon, \quad \varepsilon_n = \varepsilon^{(1+\iota)^{n-1}}, \quad \iota = \frac{1}{4}, \quad n = 1, 2, \ldots;$

(ii) $\rho_0 = \rho, \quad \rho_n = \rho_0 - \frac{\rho_0}{2} \frac{(1+2^{-2}+\cdots+n^{-2})}{\sum_{j=1}^{\infty} j^{-2}}, \quad n = 0, 1, 2, \ldots;$

(iii) $q_n = \varepsilon_{n+1}^{\frac{1}{4d^2}}, \quad n = 0, 1, 2, \ldots,$ where d is the dimensional number of system (1.3);

(iv) $\Theta_n = \Theta(\rho_n) = \{\vartheta = (\vartheta_1, \vartheta_2, \ldots, \vartheta_r) \in \mathbb{C}^r / 2\pi\mathbb{Z}^r : |\mathrm{Im}\vartheta| < \rho_n\}, \quad n = 0, 1, 2, \ldots;$

(v) $\sigma_n = \rho_0 - \frac{\rho_0}{2} \frac{(1+2^{-2}+\cdots+n^{-2}+\frac{(n+1)^{-2}}{2})}{\sum_{j=1}^{\infty} j^{-2}}, \quad \widetilde{q}_n = \frac{q_n + q_{n+1}}{2}.$

Let $\Pi_0 = \Omega_0$ and define the sets

$$\mathcal{R}_k^*(n) = \Big\{\omega \in \Pi_{n-1} : \Big| |\sqrt{-1}(k, \omega) E_{d^2} - E_d \otimes A_n + A_n^\top \otimes E_d|_e \Big| < \frac{\gamma_n}{|k|^{\tau_1}} \Big\},$$

$$\mathcal{R}_k^{**}(n) = \Big\{\omega \in \Pi_{n-1} : \Big| |\sqrt{-1}(k, \omega) E_d - A_n|_e \Big| < \frac{\gamma_n}{|k|^{\tau_1}} \Big\},$$

$$\mathcal{R}_k(n) = \mathcal{R}_k^*(n) \bigcup \mathcal{R}_k^{**}(n), \tag{3.1}$$

where $\gamma_n = \frac{\gamma}{n^{2d^2}}$ and $\tau_1 = (r+1)d^2$ and

$$\Pi_n = \Pi_{n-1} \setminus \bigcup_{0<|k|<M_n} \mathcal{R}_k(n), \tag{3.2}$$

where $M_n = \frac{|\ln \varepsilon_n|}{\rho_{n-1}-\rho_n}$. Then Π_n is the sequence of compact closed subsets of $\mathbb{R}^r_+$ with

$$\Omega_0 = \Pi_0 \supset \Pi_1 \cdots \supset \Pi_n \supset \cdots$$

and from Lemma 5.2, we have

$$Meas\mathcal{R}_k(n) \le \frac{2}{n^2}\frac{R\gamma^{\frac{1}{d^2}}}{|k|^{r+1}},$$

and

$$\begin{aligned} Meas \bigcup_{0<|k|<M_n} \mathcal{R}_k(n) &\le \sum_{0\neq k\in\mathbb{Z}^r} Meas\mathcal{R}_k(n) \\ &\le \frac{2}{n^2}R\gamma^{\frac{1}{d^2}} \sum_{0\neq k\in\mathbb{Z}^r} \frac{1}{|k|^{r+1}} \\ &\le \frac{4r}{n^2}R\gamma^{\frac{1}{d^2}} \sum_{m=1}^{\infty} m^{r-1}\frac{1}{m^{r+1}} = \frac{2\pi^2 r}{3n^2}R\gamma^{\frac{1}{d^2}}, \end{aligned}$$

where we use that $\#\{k\in\mathbb{Z}^r/|k|=m\} \le 2rm^{r-1}$.

Then, we can prove that

$$\begin{aligned} Meas\Pi_n &= Meas\left(\Pi_0 - \bigcup_{i=1}^{n}\bigcup_{0<|k|<M_i} \mathcal{R}_k(i)\right) \\ &> \left(1 - \frac{2\pi^2}{3}rR\left(\sum_{i=1}^{n}\frac{1}{i^2}\right)\gamma^{\frac{1}{d^2}}\right) Meas\Pi_0, \end{aligned}$$

and

$$Meas\Pi_\infty = (1 - O(\gamma^{\frac{1}{d^2}}))Meas\Pi_0.$$

We let that $\mathcal{O}_n$ and U_n are the complex q_n-neighborhood and $\widetilde{q}_n$-neighborhood of Π_n respectively, and construct iteratively a series of equations of the form

$$(E)_n \qquad \dot{x}_n = (A_n(\omega) + \varepsilon_n Q_n(t,\omega))x_n + \varepsilon_n g_n(t,\omega) + h_n(x_n,t,\omega),$$

and where the following conditions are satisfied:

$(\mathrm{H1})_n$ $Q_n(t,\omega)$ and $g_n(t,\omega)$ are analytic in $\Theta_n \times \mathcal{O}_n$, and

$$\max\{||Q_n(t,\omega)||^{\Theta_n\times\mathcal{O}_n}, ||g_n||^{\Theta_n\times\mathcal{O}_n}\} \le 1. \tag{3.3}$$

$(\mathrm{H2})_n$ $h_n(x_n,t,\varepsilon)$ is analytic with respect to x_n on the ball $B_{\kappa_n}(0)$, where κ_n is a sequence with $\kappa_0 = \kappa$ and $\lim_{n\to\infty}\kappa_n > 0$.

Lemma 3.1 (Iterative lemma)**.** *Assume that* $(\mathrm{H1})_n$ *and* $(\mathrm{H2})_n$ *are satisfied. Then for* ε *small sufficiently there is a transformation*

$$\Phi_n: \quad x_n = (E + \varepsilon_n P_n(t,\omega))x_{n+1} + \underline{x_n}, \tag{3.4}$$

where $\underline{x_n}$ *satisfies* $\dot{\underline{x_n}}(t) = A_n\underline{x_n}(t) + \varepsilon_n g_n(t)$ *and* $P_n(t,\omega)$ *is analytic in* $\Theta_n \times \mathcal{O}_n$ *and quasi-periodic with frequency* ω*, such that* $(E)_n$ *is changed into* $(E)_{n+1}$ *and* $(\mathrm{H1})_{n+1}$ *and* $(\mathrm{H2})_{n+1}$ *are fulfilled.*

Proof. Now we apply the change of variables $x_n(t) = y_n(t) + \underline{x_n}(t)$, where $\underline{x_n}$ satisfies $\dot{\underline{x_n}}(t) = A_n\underline{x_n}(t) + \varepsilon_n g_n(t)$, and Lemma 2.3 to $(E)_n$. Then we get

$$\dot{y}_n = (\widehat{A}_n(\omega) + \varepsilon_n \widehat{Q}_n(t,\omega))x_n + \varepsilon_{n+1}\widehat{g}_n(t,\omega) + \widehat{h}_n(x_n,t,\omega), \tag{3.5}$$

where the analyticity strip reduce to σ_n, and

$$\begin{aligned}
\widehat{A}_n &= A_n + \varepsilon_n\overline{Q}_n(t) + \overline{D_x h_n(\underline{x_n}(t),t)},\\
\widehat{Q}_n(t) &= Q_n(t) - \overline{Q}_n(t) + \frac{1}{\varepsilon_n}\Big(D_x h_n(\underline{x_n}(t),t) - \overline{D_x h_n(\underline{x_n}(t),t)}\Big),\\
\widehat{g}_n(t) &= \varepsilon_n^{-(1+\iota)} h_n(\underline{x_n}(t),t) + \varepsilon_n^{-\iota} Q_n(t)\underline{x_n}(t),\\
\widehat{h}_n(t) &= h_n(\underline{x_n}(t) + y_n, t) - h_n(\underline{x_n}(t),t) - D_x h_n(\underline{x_n}(t),t)y_n,
\end{aligned}$$

where we omit the dependence on ω to simplify the notation.

Then we apply the change of variables $y_n(t) = (E + \varepsilon_n P_n(t))x_{n+1}(t)$ and Lemma 2.4 to (3.5), we have

$$\dot{x}_{n+1} = (A_{n+1}(\omega) + \varepsilon_{n+1}Q_{n+1}(t,\omega))x_{n+1} + \varepsilon_{n+1}g_{n+1}(t,\omega) + h_{n+1}(x_{n+1},t,\omega). \tag{3.6}$$

Now the analyticity strip has been reduced to ρ_{n+1}, and

$$\begin{aligned}
A_{n+1} &= \widehat{A}_n + \overline{(E + \varepsilon_n P_n(t))^{-1}}\Big(\varepsilon_n\overline{\widehat{Q}_n^{**}}(t) + \varepsilon_n^2\overline{\widehat{Q}}_n(t)\overline{P}_n(t)\Big)\\
&= A_n + \varepsilon_n\overline{Q}_n(t) + \overline{D_x h_n(\underline{x_n}(t),t)},
\end{aligned}$$

$$\begin{aligned}
Q_{n+1}(t) &= \varepsilon_{n+1}^{-1}(E + \varepsilon_n P_n(t))^{-1}\Big(\varepsilon_n\widehat{Q}_n^{**}(t) + \varepsilon_n^2\widehat{Q}_n(t)P_n(t)\Big)\\
&\quad - \varepsilon_{n+1}^{-1}\overline{(E + \varepsilon_n P_n(t))^{-1}}\Big(\varepsilon_n\overline{\widehat{Q}_n^{**}}(t) + \varepsilon_n^2\overline{\widehat{Q}}_n(t)\overline{P}_n(t)\Big)\\
&= \varepsilon_{n+1}^{-1}(E + \varepsilon_n P_n(t))^{-1}\Big(\varepsilon_n\widehat{Q}_n^{**}(t) + \varepsilon_n^2\widehat{Q}_n(t)P_n(t)\Big)\\
&= \varepsilon_n^{-(1+\iota)}(E + \varepsilon_n P_n(t))^{-1}\Big(\varepsilon_n Q_n^{**}(t) + D_x h_n^{**}(\underline{x_n}(t),t)\Big)\\
&\quad + \varepsilon_n^{-\iota}(E + \varepsilon_n P_n(t))^{-1}\\
&\quad \left(\varepsilon_n Q_n(t) - \varepsilon_n\overline{Q}_n(t) + D_x h_n(\underline{x_n}(t),t) - \overline{D_x h_n(\underline{x_n}(t),t)}\right)P_n(t),
\end{aligned}$$

$$\begin{aligned}
g_{n+1}(t) &= (E + \varepsilon_n P_n(t))^{-1}\widehat{g}_n(t)\\
&= (E + \varepsilon_n P_n(t))^{-1}h_n(\underline{x_n}(t),t)\varepsilon_n^{-(1+\iota)} + (E + \varepsilon_n P_n(t))^{-1}Q_n(t)\underline{x_n}(t)\varepsilon_n^{-\iota},
\end{aligned}$$

$$\begin{aligned} h_{n+1}(t) &= (E+\varepsilon_n P_n(t))^{-1}\widehat{h}_n((E+\varepsilon_n P_n(t))x_{n+1}(t),t) \\ &= (E+\varepsilon_n P_n(t))^{-1}\Big(h_n(\underline{x_n}(t)+(E+\varepsilon_n P_n(t))x_{n+1},t) \\ &\quad - h_n(\underline{x_n}(t),t) - D_x h_n(\underline{x_n}(t),t)(E+\varepsilon_n P_n(t))x_{n+1}\Big), \end{aligned}$$

where $\overline{\widehat{Q}_n}(t) = \overline{\widehat{Q}^{**}_n}(t) = 0$. So, by the change of variables $x_n(t) = (E + \varepsilon_n P_n(t))x_{n+1}(t) + \underline{x_n}(t)$, Lemma 2.3 and Lemma 2.4, we can transform $(E)_n$ to $(E)_{n+1}$. □

In the following, we will prove the terms of $(E)_{n+1}$ is bounded by the terms of $(E)_n$, and we will use $L_{1,n}, L_{2,n}$ denote the values of L_1, L_2 which is introduced in Lemmas 2.1 and 2.2, also $L'_{2,n}$ in place of L'_2. Moreover, as in [9], we use the symbol K_n to be the bound of the second derivative of h_n and by the same method in [9], we can find $\{K_n\}$ is convergent, we omit the proof.

First we note that a fact $\varepsilon_n||P_n||^{\Theta_{n+1}\times\mathcal{O}_{n+1}} \le \frac{1}{2}$ which will be proved in the below, then for ε_n small enough, we will find

$$||Q_{n+1}||^{\Theta_{n+1}\times\mathcal{O}_{n+1}} \le 1, \quad ||g_{n+1}||^{\Theta_{n+1}\times\mathcal{O}_{n+1}} \le 1.$$

In fact, from Lemmas 2.3 and 2.4, we have

$$\begin{aligned} ||Q_{n+1}||^{\Theta_{n+1}\times\mathcal{O}_{n+1}} &\le 2\varepsilon_n^{1-\iota}\frac{(L'_{2,n}+||P_n||^{\Theta_{n+1}\times\mathcal{O}_{n+1}})}{1-\varepsilon_n||P_n||^{\Theta_{n+1}\times\mathcal{O}_{n+1}}}||\widehat{Q}_n||^{\Theta(\sigma_n)\times U_n} \\ &\le 4\varepsilon_n^{1-\iota}(L'_{2,n}+L_{2,n}||\widehat{Q}_n||^{\Theta(\sigma_n)\times U_n})||\widehat{Q}_n||^{\Theta(\sigma_n)\times U_n} \\ &\le 8L_{1,n}(1+K_nL_{1,n})(3+2K_nL_{1,n})\varepsilon_n^{1-\iota}, \end{aligned}$$

since $L_{1,n} > \max\{L_{2,n}, L'_{2,n}\}$ and

$$||\widehat{Q}_n||^{\Theta(\sigma_n)\times U_n} = 2(||Q_n||^{\Theta_n\times\mathcal{O}_n} + K_nL_{1,n}||g||^{\Theta_n\times\mathcal{O}_n}),$$

and $K_n \le (\frac{9}{2})^n K_0$ is the bound of the second derivative of h_n. We can assume $\gamma \le 1$, then $L_{1,n} > 1$, there is

$$||Q_{n+1}||^{\Theta_{n+1}\times\mathcal{O}_{n+1}} \le 24L_{1,n}^3(1+K_n)^2\varepsilon_n^{1-\iota} \le 1.$$

From Lemmas 2.3 and 2.4 now bound the norm of g_{n+1} as

$$\begin{aligned} ||g_{n+1}||^{\Theta_{n+1}\times\mathcal{O}_{n+1}} &\le 2||\widehat{g}_n||^{\Theta(\sigma_n)\times U_n} \\ &\le \Big(K_nL_{1,n}^2||g_n||^{\Theta_n\times\mathcal{O}_n} + 2L_{1,n}||Q_n||^{\Theta_n\times\mathcal{O}_n}||g_n||^{\Theta_n\times\mathcal{O}_n}\Big)\varepsilon_n^{1-\iota} \\ &\le (K_n+2)L_{1,n}^2\varepsilon_n^{1-\iota} \le 1. \end{aligned}$$

Thus, if ε is a sufficiently small constant, we have

$$\lim_{n\to\infty}\varepsilon_n||Q_n||^{\Theta_n\times\mathcal{O}_n} = \lim_{n\to\infty}\varepsilon_n||g_n||^{\Theta_n\times\mathcal{O}_n} = 0.$$

Now we bound $||P_n||^{\Theta_{n+1}\times\mathcal{O}_{n+1}}$. From Lemmas 2.3 and 2.4, we have

$$||P_n||^{\Theta_{n+1}\times\mathcal{O}_{n+1}} \le 2L_{2,n}\Big(||Q_n||^{\Theta_n\times\mathcal{O}_n} + K_nL_{1,n}||g_n||^{\Theta_n\times\mathcal{O}_n}\Big)$$

$$\leq 4(\frac{9}{2})^n L_{1,n} L_{2,n} K_0$$

and $\lim_{n\to\infty} \varepsilon_n ||P_n||^{\Theta_{n+1}\times\mathcal{O}_{n+1}} = 0$, for ε small enough, thus we do not mind to take $\varepsilon_n ||P_n||^{\Theta_{n+1}\times\mathcal{O}_{n+1}} < \frac{1}{2}$. The bound of $||\underline{x_n}||^{\Theta(\sigma_n)\times U_n}$ is

$$||\underline{x_n}||^{\Theta(\sigma_n)\times U_n} \leq \varepsilon_n L_{1,n} ||g_n||^{\Theta_n\times\mathcal{O}_n} < \varepsilon_n L_{1,n}.$$

Now we will show that if ε is small enough, the limit radius κ_n of the ball where h_n analytic with respect to x is positive. Then from Lemma 2.4 and Lemma 2.3 we have

$$\kappa_{n+1} = \frac{1}{1+\varepsilon_n ||P_n||^{\Theta_{n+1}\times\mathcal{O}_{n+1}}} \kappa_n - \frac{||x_n||^{\Theta(\sigma_n)\times U_n}}{1+\varepsilon_n ||P_n||^{\Theta_{n+1}\times\mathcal{O}_{n+1}}},$$

like [9] we define $a_n = \frac{1}{1+\varepsilon_n ||P_n||^{\Theta_{n+1}\times\mathcal{O}_{n+1}}}$, $b_n = \frac{||x_n||^{\Theta(\sigma_n)\times U_n}}{1+\varepsilon_n ||P_n||^{\Theta_{n+1}\times\mathcal{O}_{n+1}}}$, and when ε is small enough, by Lemma 6.5 it is easy to find $\prod_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are convergent, so $\kappa_\infty \geq a\kappa_0 - b > 0$.

Now let us bound $||A_n||$ as

$$\begin{aligned}
||A_{n+1}|| &\leq ||\widehat{A}_n|| + \varepsilon_n^2 \frac{(L'_{2,n} + ||P_n||^{\Theta_{n+1}\times\mathcal{O}_{n+1}})}{1-\varepsilon_n ||P_n||^{\Theta_{n+1}\times\mathcal{O}_{n+1}}} ||\widehat{Q}_n||^{\Theta(\sigma_n)\times U_n} \\
&\leq ||A_n|| + \varepsilon_n (||Q_n||^{\Theta_n\times\mathcal{O}_n} + K_n L_{1,n} ||g_n||^{\Theta_n\times\mathcal{O}_n}) \\
&\quad + \varepsilon_n^2 \frac{(L'_{2,n} + ||P_n||^{\Theta_{n+1}\times\mathcal{O}_{n+1}})}{1-\varepsilon_n ||P_n||^{\Theta_{n+1}\times\mathcal{O}_{n+1}}} ||\widehat{Q}_n||^{\Theta(\sigma_n)\times U_n} \\
&\leq ||A_n|| + \varepsilon_n \Big(1 + (\frac{9}{2})^n K_0 L_{1,n}\Big) \\
&\quad + 4\varepsilon_n^2 \Big(1 + (\frac{9}{2})^n K_0 L_{1,n}\Big)\Big(L'_{2,n} + 4(\frac{9}{2})^n L_{1,n} L_{2,n} K_0\Big) \\
&\leq ||A_n|| + \varsigma_n,
\end{aligned}$$

where $\varsigma_n \leq \varepsilon_n C'$ for a suitable C', when ε is small enough, we have $\sum_{n=1}^{\infty} \varsigma_n$ convergent, so we can ensure that the matrices $A_n \to B$, when $n \to \infty$.

4. Proof of Theorem 1.1

We remark that the system (1.3) satisfies (E_ν), $(H1)_\nu$ and $(H2)_\nu$ with $\nu = 0$, the iterative procedure in Lemma 3.1 can run repeatedly. So, there exists a sequence of transformation $x_n = (E + \varepsilon_n P_n(t,\omega)) x_{n+1} + \underline{x_n}, n = 0, 1, 2, \ldots$, such that $P_n(t)$ are analytic in the domains $\Theta_{n+1} \times \mathcal{O}_{n+1}$. Let

$$\Theta_\infty \times \mathcal{O}_\infty = \bigcap_{n=1}^{\infty} \Theta_n \times \mathcal{O}_n.$$

Then, all the $P_n, n = 1, 2, \ldots$, are well defined in the domain $\Theta_\infty \times \mathcal{O}_\infty$, and we have

$$\begin{aligned}
x_{n+1} &= \Phi_n^{-1}(t) x_n = \Phi_n^{-1}(t) \circ \Phi_{n-1}^{-1}(t) x_{n-1} \\
&= \cdots
\end{aligned}$$

$$= \Phi_n^{-1}(t) \circ \Phi_{n-1}^{-1}(t) \circ \cdots \circ \Phi_0^{-1}(t)x_0,$$

i.e.,

$$\begin{aligned} x_{n+1} &= (E+\varepsilon_n P_n)^{-1}x_n - (E+\varepsilon_n P_n)^{-1}\underline{x_n} \\ &= (E+\varepsilon_n P_n)^{-1} \circ \Big((E+\varepsilon_{n-1}P_{n-1})^{-1}x_{n-1} \\ &\quad - (E+\varepsilon_{n-1}P_{n-1})^{-1}\underline{x_{n-1}}\Big) - (E+\varepsilon_n P_n)^{-1}\underline{x_n} \\ &= (E+\varepsilon_n P_n)^{-1} \circ (E+\varepsilon_{n-1}P_{n-1})^{-1}x_{n-1} \\ &\quad - (E+\varepsilon_n P_n)^{-1} \circ (E+\varepsilon_{n-1}P_{n-1})^{-1}\underline{x_{n-1}} \\ &\quad - (E+\varepsilon_n P_n)^{-1}\underline{x_n} \\ &= \cdots \\ &= (E+\varepsilon_n P_n)^{-1} \circ (E+\varepsilon_{n-1}P_{n-1})^{-1} \circ \cdots \circ (E+\varepsilon_0 P_0)^{-1}x_0 \\ &\quad - \sum_{i=0}^{n}(E+\varepsilon_n P_n)^{-1} \circ \cdots \circ (E+\varepsilon_i P_i)^{-1}\underline{x_i}, \end{aligned}$$

where $\underline{x_i}$ is a solution of the equation as in Lemma 2.1.

We set

$$\Psi_n(t) = \Phi_n^{-1}(t) \circ \Phi_{n-1}^{-1}(t) \circ \cdots \circ \Phi_0^{-1}(t).$$

Note that

$$||\underline{x_n}||^{\Theta_\infty \times U_\infty} \leq ||\underline{x_n}||^{\Theta(\sigma_n)\times U_n} < \varepsilon_n L_{1,n}, \quad \lim_{n\to\infty} \varepsilon_n ||P_n||^{\Theta_{n+1}\times \mathcal{O}_{n+1}} = 0,$$

then

$$\begin{aligned} &||\Psi_{n+1}(t) - \Psi_n(t)||^{\Theta_\infty \times U_\infty} \\ &= \Big|\Big|\Phi_{n+1}^{-1}(t) \circ \Psi_n(t) - \Psi_n(t)\Big|\Big|^{\Theta_\infty \times U_\infty} \\ &\leq \Big|\Big|\Phi_{n+1}^{-1}(t) - E\Big|\Big|^{\Theta_\infty \times U_\infty} ||\Psi_n(t)||^{\Theta_\infty \times U_\infty} \\ &\leq \Big(||(E+\varepsilon_{n+1}P_{n+1})^{-1} - E||^{\Theta_\infty \times U_\infty} \\ &\quad + \Big|\Big|(E+\varepsilon_{n+1}P_{n+1})^{-1}\underline{x_{n+1}}\Big|\Big|^{\Theta_\infty \times U_\infty}\Big)||\Psi_n(t)||^{\Theta_\infty \times U_\infty} \\ &\leq \Big(\frac{\varepsilon_{n+1}||P_{n+1}||^{\Theta_{n+2}\times U_{n+2}}}{1-\varepsilon_{n+1}||P_{n+1}||^{\Theta_{n+2}\times U_{n+2}}} + \frac{\varepsilon_{n+1}L_{1,n+1}}{1-\varepsilon_{n+1}||P_{n+1}||^{\Theta_{n+2}\times U_{n+2}}}\Big)||\Psi_n(t)||^{\Theta_\infty \times U_\infty} \\ &\leq (2\varepsilon_{n+1}||P_{n+1}||^{\Theta_{n+2}\times U_{n+2}} + 2\varepsilon_{n+1}L_{1,n+1})||\Psi_n(t)||^{\Theta_\infty \times U_\infty}, \end{aligned}$$

it is easy to find that

$$||\Psi_n(t)||^{\Theta_\infty \times U_\infty} < +\infty,$$

so $\{\Psi_n(t)\}$ is convergent on $\Theta_\infty \times U_\infty$, where we use $\varepsilon_{n+1}||P_{n+1}||^{\Theta_{n+2}\times \mathcal{O}_{n+2}} < \frac{1}{2}$ as in Lemma 3.1. We see that $\Psi(t) = \lim_{n\to\infty}\Psi_n(t)$ is well defined, let

$$y = \Psi(t)x.$$

Then, the transformation $y = \Psi(t)x$ change (1.3) into

$$\dot{y} = A_\infty(\varepsilon)y + h_\infty(y, t, \varepsilon).$$

This completes the proof of Theorem 1.1.

5. Measure lemma of the allowed frequencies set

In this section, we bound the measure of the resonances and in the proving, C is positive constants.

Lemma 5.1. *Let* $\tau = \tau_1 + d^2 - 1$ *and*

$$G^*(\omega) = \sqrt{-1}(k, \omega)E_{d^2} - E_d \otimes A_n + A_n^\top \otimes E_d, \tag{5.1}$$

$$G^{**}(\omega) = \sqrt{-1}(k, \omega)E_d - A_n(\omega). \tag{5.2}$$

Then, for $\omega \in \mathcal{O}_n$ *and* $0 < |k| \leq M_n$*, the inverse of* $G^*(\omega)$ *and* $G^{**}(\omega)$ *exist, moreover, they are analytic in the domain* $\mathcal{O}_n$ *with*

$$(G^*)^{-1}(\omega) \leq C\frac{|k|^\tau}{\gamma_n}, \quad (G^{**})^{-1}(\omega) \leq C\frac{|k|^\tau}{\gamma_n}, \tag{5.3}$$

respectively.

Proof. This proof like the Lemma 3.1 in [16], we omit it. □

Lemma 5.2. *Let* $R = d^2(d+1)^2(diam\Pi_0)^{r-1}$*. Then the Lebesgue measure of* $\mathcal{R}_k(n)$ *satisfies*

$$Meas\mathcal{R}_k(n) \leq \frac{2}{n^2}\frac{R\gamma^{\frac{1}{d^2}}}{|k|^{r+1}}. \tag{5.4}$$

Proof. From Lemma 2.4 and Lemma 2.3 we have

$$A_{n+1}(\omega) \quad = \quad A_n(\omega) + \varepsilon_n\overline{Q}_n(\omega) + \overline{D_x h(\underline{x}_n(t, \omega), t)}. \tag{5.5}$$

First, we know that $q_s = \varepsilon_{s+1}^{\frac{1}{4d^2}}$, let $q_{1,s} = \frac{5}{6}q_s + \frac{1}{6}q_{s+1}, q_{2,s} = \frac{11}{12}q_s + \frac{1}{12}q_{s+1}$, and $\mathcal{O}_{i,s}$ are the $q_{i,s}$-neighborhood of Π_s respectively for $i = 1, 2$, moreover, $\rho_{1,s} = \frac{5}{6}\rho_s + \frac{1}{6}\rho_{s+1}, \rho_{2,s} = \frac{11}{12}\rho_s + \frac{1}{12}\rho_{s+1}$ are the size of the analyticity domain in the angular variables. Obviously, $\mathcal{O}_{s+1} \subset \mathcal{O}_{1,s} \subset \mathcal{O}_{2,s} \subset \mathcal{O}_s$ and $\text{dist}(\partial\mathcal{O}_{1,s}, \partial\mathcal{O}_s) > \text{dist}(\partial\mathcal{O}_{2,s}, \partial\mathcal{O}_s) = \text{dist}(\partial\mathcal{O}_{1,s}, \partial\mathcal{O}_{2,s}) > \frac{1}{24}q_s$, and $\rho_{s+1} < \rho_{1,s} < \rho_{2,s} < \rho_s$. Recall that $||Q_s||^{\Theta_s\times\mathcal{O}_s} \leq 1$ and $||g_s||^{\Theta_s\times\mathcal{O}_s} \leq 1$, then for $1 \leq l \leq d^2$ and $0 \leq s \leq n$,

(5.6)

$$\varepsilon_s||\partial_\omega^l\overline{Q}_s(\omega)||^{\Theta_{1,s}\times\mathcal{O}_{1,s}} \leq \varepsilon_s\frac{24^l}{q_s^l}||\overline{Q}_s(\omega)||^{\Theta_s\times\mathcal{O}_s} \leq \varepsilon_s\frac{24^l}{q_s^l}||Q_s(\omega)||^{\Theta_s\times\mathcal{O}_s} \leq \varepsilon_s^{\frac{1}{2}},$$

$$||\partial_\omega^l\overline{D_x h(\underline{x}_s(t, \omega), t)}||^{\Theta_{1,s}\times\mathcal{O}_{1,s}} \leq \frac{24^l}{q_s^l}||\overline{D_x h(\underline{x}_s(t, \omega), t)}||^{\Theta_{2,s}\times\mathcal{O}_{2,s}}$$

$$\begin{aligned}
&\leq \frac{24^l}{q_s^l}||D_x h(\underline{x}_s(t,\omega),t)||^{\Theta_{2,s}\times\mathcal{O}_{2,s}}\\
&\leq \frac{24^l}{q_s^l}K_s||\underline{x}_s(t,\omega)||^{\Theta_{2,s}\times\mathcal{O}_{2,s}}\\
&\leq \varepsilon_s\frac{24^l}{q_s^l}K_s\widetilde{L}_{1,s}||g_s(\omega)||^{\Theta_s\times\mathcal{O}_s}\leq \varepsilon_s^{\frac{1}{2}}.
\end{aligned} \tag{5.7}$$

Now from (5.6) and (5.7), the norm boundedness of every terms in (5.5) and recursiveness, for a sufficiently small ε, we have

$$||\partial_\omega^l A_{n+1}(\omega)||^{\Theta_{1,n}\times\mathcal{O}_{1,n}}\leq \varepsilon^{\frac{1}{2}}.$$

Let $B(\omega)=-E_d\otimes A_n(\omega)+A_n^\top(\omega)\otimes E_d=(b_{ij})$. Then

$$||\partial_\omega^l B(\omega)||^{\Theta_{1,n-1}\times\mathcal{O}_{1,n-1}}\leq \varepsilon^{\frac{1}{2}},$$

obviously, for $1\leq l\leq d^2$ and $\omega\in\mathcal{O}_{1,n-1}$,

$$|\partial_{\omega_1}^l b_{ij}|\leq ||\partial_\omega^l B(\omega)||^{\Theta_{1,n-1}\times\mathcal{O}_{1,n-1}}\leq \varepsilon^{\frac{1}{2}}. \tag{5.8}$$

Set $\mathcal{T}(\omega)=|\sqrt{-1}(k,\omega)E_{d^2}+B(\omega)|_e$. Then

$$\mathcal{T}(\omega)=\sqrt{-1}^{d^2}(k,\omega)^{d^2}+\sum_{1\leq\nu\leq d^2-1}\chi_\nu(\omega)(k,\omega)^{d^2-\nu}, \tag{5.9}$$

where

$$\chi_\nu(\omega)=\sum_{1\leq j_i\leq d^2}\sigma_{j_1\cdots j_\nu}b_{1j_1}\cdots b_{\nu j_\nu} \text{ and } \sigma_{j_1\cdots j_\nu}\in\{-1,+1,-\sqrt{-1},+\sqrt{-1}\}.$$

For $\omega\in\mathcal{O}_{1,n-1}, 1\leq l\leq d^2$, by (5.8),

$$\begin{aligned}
\Big|\frac{d^l}{d\omega_1^l}(b_{1j_1}\cdots b_{lj_\nu})\Big| &\leq \Bigg|\sum_{l_1+\cdots+l_\nu=l}\Big(\frac{d^{l_1}}{d\omega_1^{l_1}}b_{1j_1}\Big)\cdots\Big(\frac{d^{l_\nu}}{d\omega_1^{l_\nu}}b_{1j_\nu}\Big)\Bigg|\\
&\leq \varepsilon^{\frac{l}{2}}\sum_{l_1+\cdots+l_\nu=l}1\\
&\leq 2^l\varepsilon^{\frac{l}{2}},
\end{aligned} \tag{5.10}$$

and

$$\left|\frac{d^l}{d\omega_1^l}\chi_\nu(\omega)\right|\leq \binom{d^2}{\nu}2^l\varepsilon^{\frac{l}{2}}. \tag{5.11}$$

Now, without loss of generality, we can assume that $|k|=|k_1|+\cdots+|k_r|\leq r|k_1|$. Then, for every $\omega\in\mathcal{O}_{1,n-1}$,

$$\left|\frac{d^{d^2}}{d\omega_1^{d^2}}\sum_{1\leq\nu\leq d^2-1}\chi_\nu(\omega)(k,\omega)^{d^2-\nu}\right|$$

$$
\begin{aligned}
&\leq \sum_{1\leq\nu\leq d^2-1}\left|\frac{d^{d^2}}{d\omega_1^{d^2}}\Big(\chi_\nu(\omega)(k,\omega)^{d^2-\nu}\Big)\right| \\
&\leq \sum_{1\leq\nu\leq d^2-1}\sum_{1\leq l\leq d^2}\binom{d^2}{l}\left|\frac{d^l}{d\omega_1^l}\chi_\nu(\omega)\right|\left|\frac{d^{d^2-l}}{d\omega_1^{d^2-l}}(k,\omega)^{d^2-\nu}\right| \\
&\leq \sum_{1\leq\nu\leq d^2-1}\sum_{1\leq l\leq d^2}\binom{d^2}{\nu}\binom{d^2}{l}2^l\varepsilon^{\frac{l}{2}}|k_1|^{d^2-l}|(k,\omega)|^{l-\nu}d^2! \\
&\leq \varepsilon^{\frac{1}{2}}C_1|k_1|^{d^2-1}d^2!, \qquad (5.12)
\end{aligned}
$$

where C_1 is some constant which depends only on d, r and on the maximum of $|\omega|$ in Π_0. Obviously,

$$\frac{d^{d^2}}{d\omega_1^{d^2}}(k,\omega)^{d^2} = d^2!|k_1|^{d^2}, \tag{5.13}$$

moreover, in $\mathcal{O}^1_{n-1}$, for ε small enough so that $C_1\varepsilon^{\frac{1}{2}} < \frac{1}{2}$, by (5.12) and (5.13) we have

$$\left|\frac{d^{d^2}}{d\omega_1^{d^2}}\mathcal{T}(\omega)\right| \geq d^2!|k_1|^{d^2}\Big(1-\varepsilon^{\frac{1}{2}}C_1|k_1|^{-1}\Big) \geq \frac{1}{2}d^2!|k_1|^{d^2}. \tag{5.14}$$

Thus, from (5.14) and Lemma 6.3, we get

$$Meas\mathcal{R}^*_k(n) \leq d^2(d+1)^2\Big(\frac{\gamma_n}{|k|^{\tau_1}}\Big)^{\frac{1}{d^2}}(\text{diam}\Pi_0)^{r-1} \leq \frac{1}{n^2}\frac{R\gamma^{\frac{1}{d^2}}}{|k|^{r+1}}.$$

For the $Meas\mathcal{R}^{**}_k(n)$, we also have

$$Meas\mathcal{R}^{**}_k(n) \leq \frac{1}{n^2}\frac{R\gamma^{\frac{1}{d^2}}}{|k|^{r+1}}.$$

It means

$$Meas\mathcal{R}_k(n) \leq Meas\mathcal{R}^*_k(n) + Meas\mathcal{R}^{**}_k(n) \leq \frac{2}{n^2}\frac{R\gamma^{\frac{1}{d^2}}}{|k|^{r+1}}.$$

□

We can find the nest sequence of closed sets

$$\Omega_0 = \Pi_0 \supset \Pi_1 \supset \cdots \supset \Pi_n \supset \cdots$$

is defined inductively. Moreover, we have the following lemma.

Lemma 5.3. *Let* $\Pi_\infty = \bigcap_{n=0}^\infty \Pi_m$. *Then the Lebesgue measure of* Π_∞ *satisfies*

$$Meas\Pi_\infty = (Meas\Pi_0)(1 - O(\gamma^{\frac{1}{d^2}})).$$

6. Appendix

Lemma 6.1. *For $\delta > 0$ and $\nu > 0$, the following inequality holds true*:

$$\sum_{k\in\mathbb{Z}^N} |k|^\nu e^{-2|k|\delta} \le \left(\frac{\nu}{e}\right)^\nu \frac{(1+e)^N}{\delta^{\nu+N}}.$$

This lemma can be found in [1].

The following lemma can be found in many books on matrix theory; for example in [10].

Lemma 6.2. *Let A, B and C be $l\times l$, $m\times m$ and $l\times m$ matrixes, respectively*; *and let X be an $l\times m$ unknown matrix. Then the matrix equation*

$$AX + XB = C$$

is solvable if and only if the vector equation

$$(E_m \otimes A + B^\top \otimes E_l)X' = C'$$

is solvable, where $X' = (X_1^\top, \dots, X_m^\top)^\top, C' = (C_1^\top, \dots, C_m^\top)^\top$ if we write $X = (X_1, \dots, X_m)$ and $C = (C_1, \dots, C_m)$. Moreover,

$$||X|| \le \left|\left|(E_m \otimes A + B^\top \otimes E_l)^{-1}\right|\right| ||C||$$

if the inverse exists.

Lemma 6.3 ([15], Lemma 4.1)**.** *Let Δ be an interval in $\mathbb{R}^1$ and $\overline{\Delta}$ its closure. Suppose that $\psi : \overline{\Delta} \to \mathbb{C}$ is k times continuously differentiable. Let $\Delta_j = \{x \in \overline{\Delta} : |\psi(x)| \le j\}, j > 0$. If, for some constant $\nu > 0$, $|\frac{d^k\psi(x)}{dx^k}| \ge \nu$ for any $x \in \Delta$, then $MeasI_j \le cj^{\frac{1}{k}}$ where $c = 2(2 + 3 + \cdots + k + \nu^{-1})$.*

Lemma 6.4 ([9], Lemma 2)**.** *Let $h : U \subset \mathbb{R}^d \to \mathbb{R}^d$ be a function of class $\mathcal{C}^2$ on a ball $B_\kappa(0)$ that satisfies $h(0) = 0$, $D_xh(0) = 0$, and $||D_{xx}h(x)|| \le K$, where $x \in B_\kappa(0)$. Then $||h(x)|| \le \frac{K}{2}||x||^2$ and $||D_xh(x)|| \le K||x||$.*

Lemma 6.5 ([9], Lemma 10)**.** *Let $\{a_n\}$ be a sequence of positive real numbers that satisfies $a_n \in (0, 1]$, $\prod_{n=0}^\infty a_n = a \in (0, 1]$. Let $\{b_n\}$ be another sequence of positive real numbers that satisfies $\sum_{n=0}^\infty b_n = b < +\infty$. We define a new sequence $\{\kappa_n\}$ is $\kappa_{n+1} = a_n\kappa_n - b_n$. Then $\kappa_n \to \kappa_\infty$, $n \to \infty$, and $\kappa_\infty \ge a\kappa_0 - b$.*

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School of Mathematics
Chongqing Normal University
Chongqing 401331, P. R. China
E-mail address: houyu19@gmail.com