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ON THE MEAN VALUES OF $L(1,\chi)$

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Abstract. Let $p > 2$ be a prime, and let $k \geq 1$ be an integer. Let χ be a Dirichlet character modulo p, and let $L(s,\chi)$ be the Dirichlet L-functions corresponding to χ. In this paper we consider the mean values of

$$\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(2^k)\,|L(1,\chi)|^2 .$$

1. Introduction

Let χ be a Dirichlet character modulo $q \geq 2$, and let $L(s,\chi)$ be the Dirichlet L-functions corresponding to χ. For $q = p$ a prime, H. Walum [8] showed that

$$\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} |L(1,\chi)|^2 = \frac{\pi^2 (p-1)^2 (p-2)}{12p^2}. \tag{1.1}$$

For a general integer q, S. Louboutin [5] proved that

$$\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} |L(1,\chi)|^2 = \frac{\pi^2}{12} \cdot \frac{\phi^2(q)}{q^2} \left(q \prod_{p|q} \left(1 + \frac{1}{p}\right) - 3 \right),$$

where $\phi(q)$ is the Euler function. Moreover, S. Louboutin [6] studied the mean value of $|L(1,\chi)|^2$ for odd primitive Dirichlet characters. M. Katsurada and K. Matsumoto [3] gave some asymptotic formulae for $\sum_{\substack{\chi \bmod q \\ \chi \neq \chi_0}} |L(1,\chi)|^2$, where χ_0 is the principal character modulo q.

Furthermore, S. Louboutin [7] proved the following:

Proposition 1.1. *Let $q > 2$, $k \geq 1$ and $l \geq 1$ denote integers. Set*

$$\phi_l(q) = \prod_{p|q} \left(1 - \frac{1}{p^l}\right) \qquad \textit{and} \qquad \phi(q) = q\phi_1(q).$$

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Then for any $k \geq 1$ there exists a polynomial $R_k(X) = \sum_{l=0}^{2k} r_{k,l} X^l$ of degree $2k$ with rational coefficients such that for all $q > 2$ we have

$$\frac{2}{\phi(q)} \sum_{\substack{\chi \bmod q \\ \chi(-1)=(-1)^k}} |L(k,\chi)|^2 = \frac{\pi^{2k}}{2\left((k-1)!\right)^2} \sum_{l=1}^{2k} r_{k,l} \phi_l(q) q^{l-2k}.$$

H. Liu and W. Zhang [4] determined the coefficients $r_{k,l}$.

Proposition 1.2. *Let $q \geq 2$, $m \geq 1$ and $n \geq 1$ be positive integers with $m \equiv n \bmod 2$. Set $\epsilon_{m,n} = 1$ if $m \equiv n \equiv 1 \bmod 2$ and $\epsilon_{m,n} = 0$ if $m \equiv n \equiv 0 \bmod 2$. Then,*

$$\frac{2}{\phi(q)} \sum_{\substack{\chi \bmod q \\ \chi(-1)=(-1)^m}} L(m,\chi) L(n,\overline{\chi})$$
$$= \frac{(-1)^{\frac{m-n}{2}} (2\pi)^{m+n}}{2m!n!} \left(\sum_{l=0}^{m+n} r_{m,n,l} \phi_l(q) q^{l-m-n} - \frac{\epsilon_{m,n}}{q} B_m B_n \phi_{m+n-1}(q) \right),$$

where

$$r_{m,n,l} = B_{m+n-l} \sum_{\substack{a=0 \\ a+b\geq m+n-l}}^{m} \sum_{b=0}^{n} B_{m-a} B_{n-b} \frac{\binom{m}{a}\binom{n}{b}\binom{a+b+1}{m+n-l}}{a+b+1},$$

B_m is the Bernoulli number, and $\binom{m}{a} = \frac{m!}{a!(m-a)!}$.

In this paper we consider the mean values of

$$S(p,k) := \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(2^k) |L(1,\chi)|^2,$$

and give some identities. The main results are the following.

Theorem 1.1. *Let $p > 2$ be a prime, $k \geq 1$ be an integer, and define*

$$g_p(c) := \left(\sum_{\substack{a=1 \\ 2\nmid a+R_p(ac)}}^{p-1} 1 \right) - \frac{p-1}{2},$$

where $R_p(n)$ denotes the unique $r \in \{0,1,\ldots,p-1\}$ with $r \equiv n \pmod{p}$. Then $S(p,0)$ is given by (1.1),

$$S(p,1) = \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(2) |L(1,\chi)|^2 = \frac{\pi^2 (p-1)^2 (p-5)}{24p^2},$$

and for $k \geq 2$, the mean values of $S(p,k)$ can be determined by the recursion formula

$$S(p,k) = \frac{5}{2}S(p,k-1) - S(p,k-2) + \frac{\pi^2(p-1)g_p(2^{k-1})}{4p}, \quad k \geq 2.$$

By the recursion formula we immediately get the following corollary.

Corollary 1.1. *Let $p > 2$ be a prime, and let $k \geq 1$ be an integer. Then we have*

$$S(p,k) = \frac{\pi^2(p-1)\left(p^2 - b_k(p)p + 4^k + 1\right)}{12 \cdot 2^k \cdot p^2},$$

where $b_0(p) = 3$, $b_1(p) = 6$, and $b_k(p) = 5b_{k-1}(p) - 4b_{k-2}(p) - 3 \cdot 2^k \cdot g_p(2^{k-1})$, $k \geq 2$.

From Corollary 1.1 and Lemma 2.2 (in Section 2) we get the following identities.

Corollary 1.2. *Let $p > 2$ be a prime. Then we have*

$$\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(4)\left|L(1,\chi)\right|^2 = \begin{cases} \dfrac{\pi^2(p-1)^2(p-17)}{48p^2}, & \text{if } p \equiv 1 \pmod 4, \\ \dfrac{\pi^2(p-1)(p^2-6p+17)}{48p^2}, & \text{if } p \equiv -1 \pmod 4 \end{cases}$$

and

$$\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(8)\left|L(1,\chi)\right|^2 = \begin{cases} \dfrac{\pi^2(p-1)^2(p-65)}{96p^2}, & \text{if } p \equiv 1 \pmod 8, \\ \dfrac{\pi^2(p-1)(p^2+18p+65)}{96p^2}, & \text{if } p \equiv -1 \pmod 8, \\ \dfrac{\pi^2(p-1)(p^2-30p+65)}{96p^2}, & \text{if } p \equiv 3 \pmod 8, \\ \dfrac{\pi^2(p-1)(p^2-18p+65)}{96p^2}, & \text{if } p \equiv -3 \pmod 8. \end{cases}$$

Furthermore, from Corollary 1.1 and Lemma 2.2 we can give the following conjecture.

Conjecture 1.1. *Let $p > 2$ be a prime, and let $k \geq 1$ be an integer with $2^k < p$. Then we have*

$$S(p,k) = \frac{\pi^2(p-1)\left(p^2 - b_k(p)p + 4^k + 1\right)}{12 \cdot 2^k \cdot p^2},$$

where $b_0(p) = 3$, $b_1(p) = 6$, and

$$b_k(p) = 5b_{k-1}(p) - 4b_{k-2}(p) + 3 \cdot 2^{k-1} \cdot \left(L_{2^k}(p) - 1\right), \quad k \geq 2,$$

and $L_{2^k}(p)$ denotes the unique $r \in \{1-2^{k-1}, \dots, -1, 0, 1, \dots, 2^{k-1}-1\}$ with $r \equiv p \pmod{2^k}$.

2. Some lemmas

Lemma 2.1. *Let $p > 2$ be a prime, and define*

$$f_p(c) := \sum_{\substack{a=1 \\ 2 \nmid a+R_p(ac)}}^{p-1} 1,$$

where $R_p(n)$ denotes the unique $r \in \{0, 1, \ldots, p-1\}$ with $r \equiv n \pmod p$. Then we have

$$f_p(c) = \frac{p-1}{2} - \frac{2p}{\pi^2(p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(c)\left(5 - 2\chi(2) - 2\overline{\chi}(2)\right)|L(1,\chi)|^2.$$

Proof. Let $\overline{b}$ be the inverse of b modulo p with $1 \leq \overline{b} \leq p-1$ and $b\overline{b} \equiv 1 \pmod p$. By using the properties of character sums we get

$$\begin{aligned} f_p(c) &= \frac{1}{2} \sum_{\substack{a=1 \\ abc\equiv 1(\bmod p)}}^{p-1} \sum_{b=1}^{p-1} \left(1 - (-1)^{a+\overline{b}}\right) \\ &= \frac{1}{2} \sum_{\substack{a=1 \\ abc\equiv 1(\bmod p)}}^{p-1} \sum_{b=1}^{p-1} 1 - \frac{1}{2} \sum_{\substack{a=1 \\ abc\equiv 1(\bmod p)}}^{p-1} \sum_{b=1}^{p-1} (-1)^{a+\overline{b}} \\ &= \frac{p-1}{2} - \frac{1}{2(p-1)} \sum_{\chi \bmod p} \chi(c) \sum_{a=1}^{p-1} (-1)^a \chi(a) \sum_{b=1}^{p-1} (-1)^{\overline{b}} \chi(b) \\ &= \frac{p-1}{2} - \frac{1}{2(p-1)} \sum_{\chi \bmod p} \chi(c) \sum_{a=1}^{p-1} (-1)^a \chi(a) \sum_{b=1}^{p-1} (-1)^{b} \overline{\chi}(b) \\ &= \frac{p-1}{2} - \frac{1}{2(p-1)} \sum_{\chi \bmod p} \chi(c) \left|\sum_{a=1}^{p-1} (-1)^a \chi(a)\right|^2. \end{aligned}$$

Now if $\chi(-1) = 1$, then

$$\sum_{a=1}^{p-1} (-1)^a \chi(a) = 0,$$

while if $\chi(-1) = -1$, then

$$\sum_{a=1}^{p-1} (-1)^a \chi(a) = 2\chi(2) \sum_{b=1}^{(p-1)/2} \chi(b).$$

By [2] we know that for any odd character χ mod p, we have

$$(1-2\chi(2))\sum_{c=1}^{p-1} c\chi(c) = \chi(2)p\sum_{c=1}^{(p-1)/2}\chi(c).$$

On the other hand, from Theorem 12.11 and Theorem 12.20 of [1] we also know that if χ is an odd character modulo p, then

$$\frac{1}{p}\sum_{b=1}^{p-1} b\chi(b) = \frac{i}{\pi}\tau(\chi)L(1,\overline{\chi}),$$

where

$$\tau(\chi) = \sum_{a=1}^{p-1}\chi(a)\mathrm{e}\left(\frac{a}{p}\right)$$

is the Gauss sum. Therefore

$$\sum_{a=1}^{p-1}(-1)^a\chi(a) = \begin{cases} 0, & \text{if } \chi(-1)=1,\\ \dfrac{2i}{\pi}\left(1-2\chi(2)\right)\tau(\chi)L(1,\overline{\chi}), & \text{if } \chi(-1)=-1.\end{cases}$$

Then we have

$$f_p(c) = \frac{1}{2}(p-1) - \frac{2p}{\pi^2(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\chi(c)\left|1-2\chi(2)\right|^2\left|L(1,\chi)\right|^2.$$

□

Since $f_p(p+c) = f_p(c)$, we only need to study $f_p(c)$ for $c<p$.

Lemma 2.2. *Let $p>2$ be a prime. Then we get*

$$f_p(1) = \sum_{\substack{a=1\\ 2\nmid a+a}}^{p-1} 1 = 0.$$

Furthermore, for $2^k < p$ we have

$$f_p(2^k) = \begin{cases} \dfrac{p-1}{2}, & \textit{if } p\equiv 1 \pmod{2^{k+1}},\\ \dfrac{p-(2^k-1)}{2}, & \textit{if } p\equiv 2^k-1 \pmod{2^{k+1}},\\ \dfrac{p+1}{2}, & \textit{if } p\equiv -1 \pmod{2^{k+1}},\\ \dfrac{p-(1-2^k)}{2}, & \textit{if } p\equiv 1-2^k \pmod{2^{k+1}}.\end{cases}$$

Proof. For $k\geq 1$ and $2^k<p$, we get

$$f_p(2^k) = \sum_{j=1}^{2^k}\sum_{\substack{a=1\\ (j-1)p<2^k a<jp\\ 2\nmid a+R_p(2^k a)}}^{p-1} 1 = \sum_{j=1}^{2^k}\sum_{\substack{a=1\\ (j-1)p<2^k a<jp\\ 2\nmid a+2^k a-(j-1)p}}^{p-1} 1$$

$$= \sum_{j=1}^{2^k} \sum_{\substack{a=1\\ (j-1)p<2^k a<jp\\ 2|a+j}}^{p-1} 1 = \sum_{j=1}^{2^k} \sum_{\substack{a=1\\ \frac{(j-1)p}{2^k}<a<\frac{jp}{2^k}\\ 2|a+j}}^{p-1} 1$$

$$= \left(\sum_{j=1}^{2^k-1} \sum_{\substack{\frac{(j-1)p-R_{2^k}((j-1)p)}{2^k}+1\le a\le \frac{jp-R_{2^k}(jp)}{2^k}\\ 2|a+j}} 1 \right) + \sum_{\substack{\frac{(2^k-1)p-R_{2^k}((2^k-1)p)}{2^k}+1\le a\le p-1\\ 2|a}} 1$$

$$= \left(\sum_{j=1}^{2^k-1} \sum_{\substack{1\le b\le \frac{p+R_{2^k}((j-1)p)-R_{2^k}(jp)}{2^k}\\ 2|b+\frac{(j-1)p-R_{2^k}((j-1)p)}{2^k}+j}} 1 \right) + \sum_{\substack{1\le b\le \frac{p+R_{2^k}(-p)}{2^k}-1\\ 2\nmid b-\frac{p+R_{2^k}(-p)}{2^k}}} 1.$$

If $R_{2^k}(p)=1$, we have

$$f_p(2^k) = \left(\sum_{j=1}^{2^k-1} \sum_{\substack{1\le b\le \frac{p-1}{2^k}\\ 2|b+\frac{(j-1)(p-1)}{2^k}+j}} 1 \right) + \sum_{\substack{1\le b\le \frac{p-1}{2^k}\\ 2\nmid b-\frac{p+2^k-1}{2^k}}} 1 = \sum_{j=0}^{2^k-1} \sum_{\substack{1\le b\le \frac{p-1}{2^k}\\ 2|b+\frac{(j-1)(p-1)}{2^k}+j}} 1$$

$$= \begin{cases} \dfrac{p-1}{2}, & \text{if } p\equiv 1 \pmod{2^{k+1}},\\ \dfrac{p-(1-2^k)}{2}, & \text{if } p\equiv 1-2^k \pmod{2^{k+1}}. \end{cases}$$

While if $R_{2^k}(p)=2^k-1$, then

$$f_p(2^k) = \left(\sum_{\substack{1\le b\le \frac{p+1}{2^k}-1\\ 2|b+1}} 1 \right) + \left(\sum_{j=2}^{2^k-1} \sum_{\substack{1\le b\le \frac{p+1}{2^k}\\ 2|b+\frac{(j-1)(p+1)}{2^k}+j-1}} 1 \right) + \sum_{\substack{1\le b\le \frac{p+1}{2^k}-1\\ 2\nmid b-\frac{p+1}{2^k}}} 1$$

$$= \begin{cases} \dfrac{p+1}{2}, & \text{if } p\equiv -1 \pmod{2^{k+1}},\\ \dfrac{p-(2^k-1)}{2}, & \text{if } p\equiv 2^k-1 \pmod{2^{k+1}}. \end{cases}$$

This proves Lemma 2.2. □

Corollary 2.1. *Let $p > 2$ be a prime. Then we have*

$$f_p(2) = \begin{cases} \dfrac{p-1}{2}, & \text{if } p \equiv 1 \pmod 4, \\ \dfrac{p+1}{2}, & \text{if } p \equiv -1 \pmod 4. \end{cases}$$

$$f_p(4) = \begin{cases} \dfrac{p-1}{2}, & \text{if } p \equiv 1 \pmod 8, \\ \dfrac{p-3}{2}, & \text{if } p \equiv 3 \pmod 8, \\ \dfrac{p+1}{2}, & \text{if } p \equiv -1 \pmod 8, \\ \dfrac{p+3}{2}, & \text{if } p \equiv -3 \pmod 8. \end{cases}$$

It is natural to give the following conjecture.

Conjecture 2.1. *Let $p > 2$ be a prime, and $k \geq 1$ be an integer. For $2^k < p$ we have*

$$f_p(2^k) = \frac{p - L_{2^{k+1}}(p)}{2},$$

where $L_{2^{k+1}}(p)$ denotes the unique $r \in \{1-2^k, \ldots, -1, 0, 1, \ldots, 2^k - 1\}$ with $r \equiv p \pmod{2^{k+1}}$.

3. Proof of Theorem 1.1

By Lemma 2.1 we have

$$f_p(1) = \frac{p-1}{2} - \frac{2p}{\pi^2(p-1)} \left(5S(p,0) - 4S(p,1)\right),$$

which, together with (1.1) and Lemma 2.2, show the formula for $S(p,1)$. We also have for $k \geq 2$

$$f_p(2^{k-1}) = \frac{p-1}{2} - \frac{2p}{\pi^2(p-1)} \left(5S(p,k-1) - 2S(p,k) - 2S(p,k-2)\right),$$

which, together with (1.1) and Lemma 2.2, proves the recursion formula.

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