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MULTIPLICITY OF NONTRIVIAL SOLUTIONS TO PERTURBED SCHRÖDINGER SYSTEM WITH MAGNETIC FIELDS

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Abstract. We are concerned with the multiplicity of semiclassical solutions of the following Schrödinger system involving critical nonlinearity and magnetic fields

$$\begin{cases} -(\varepsilon\nabla + iA(x))^2 u + V(x)u = H_u(u,v) + K(x)|u|^{2^*-2}u, \ x \in \mathbb{R}^N, \\ -(\varepsilon\nabla + iB(x))^2 v + V(x)v = H_v(u,v) + K(x)|v|^{2^*-2}v, \ x \in \mathbb{R}^N, \end{cases}$$

where $2^* = 2N/(N-2)$ is the Sobolev critical exponent and i is the imaginary unit. Under proper conditions, we prove the existence and multiplicity of the nontrivial solutions to the perturbed system.

1. Introduction

This paper is motivated by some works that have appeared in recent years concerning the nonlinear Schrödinger equation with critical nonlinearity and magnetic fields of the form

$$(1.1)\quad i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}(\nabla + iA(x))^2\psi + W(x)\psi - K(x)|\psi|^{2^*-2}\psi - h(x,|\psi|^2)\psi,$$

where $\hbar$ is Planck's constant, i is the imaginary unit, 2^* is the critical exponent, $2^* = \frac{2N}{N-2}$, for $N \geq 3$, $A(x) = (A_1(x), A_2(x), \ldots, A_N(x)) : \mathbb{R}^N \to \mathbb{R}^N$ is a real vector potential with magnetic field $B = \text{curl}A$ and $W(x)$ is a scalar electric potential. Knowledge of the solutions for the elliptic equation

$$(1.2)\quad -(\nabla + iA(x))^2 u(x) + \lambda(W(x) - E)u(x) = \lambda K(x)|u|^{2^*-2}u + \lambda h(x,|u|^2)u$$

has a great importance in the study of the standing-wave solutions of (1.2), i.e., the solutions of the type

$$\psi(x,t) = \exp(-\frac{iEt}{\hbar})u(x),$$

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where $\lambda^{-1} = \frac{\hbar^2}{2m}$.

The transition from quantum mechanics to classical mechanics can be conducted by making $\hbar \to 0$. Therefore, the existence and multiplicity of solutions for $\hbar$ small have important physical interest.

The equation in the case $A(x) \equiv 0$ has been explored by many authors including M. Del. Pino and P. Felmer [16, 17], A. Floer and A. Weinstein [14], Y. G. Oh [14] and F. Wang [20]. For much more results, we refer the reader to [1, 2, 8, 9, 11] and the reference therein.

As far as the equation (1.2) in the case of $A(x) \neq 0$ is concerned, we recall P. L. Lions [13], G. Arioli and A. Szulkin [3], S. Cingolani [7] and the works of [4, 6, 10, 18, 19, 21].

Among the works mentioned above, the corresponding authors have obtained many valuable results. However, many results have only been established in subcritical case by using various methods.

Motivated by the results just described, a natural question is whether the existence and multiplicity of results occur for the following perturbed Schrödinger system with critical nonlinearity and magnetic fields as follows

$$\begin{cases} -(\varepsilon\nabla + iA(x))^2 u + V(x)u = H_u(u,v) + K(x)|u|^{2^*-2}u, \ x \in \mathbb{R}^N, \\ -(\varepsilon\nabla + iB(x))^2 v + V(x)v = H_v(u,v) + K(x)|v|^{2^*-2}v, \ x \in \mathbb{R}^N. \end{cases} \tag{1.3}$$

To my knowledge, there seems some work on the existence of solutions to (1.2), but to the system (1.3), there is few work on the existence and multiplicity of solutions.

By using the similar ideas of [12, 22], we will establish the two main results to (1.3).

Firstly, we make the following assumptions throughout the paper.

(V_0) $V \in C(\mathbb{R}^N, \mathbb{R})$, $V(0) = \inf_{x\in\mathbb{R}^N} V(x) = 0$ and there is a constant $b > 0$ such that the set $\nu^b = \{x \in \mathbb{R}^N : V(x) < b\}$ has finite Lebesgue measure;

(A_0) $A(x), B(x) \in C(\mathbb{R}^N, \mathbb{R}^N), A(0) = B(0) = 0$;

(K_0) $K(x) \in C(\mathbb{R}^N)$, $0 < \inf K \leq \sup K < \infty$;

(H_1) $H \in C^1(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R})$, $H_s(s,t) = o(|s| + |t|)$ and $H_t(s,t) = o(|s| + |t|)$;

(H_2) $\lim_{|s|+|t|\to\infty} \frac{H_s(s,t)+H_t(s,t)}{|s|^\alpha+|t|^\beta} = 0$ for some $2 < \alpha, \beta < 2^\star - 1$;

(H_3) There exist $a_0 > 0$, $p, q > 2$, $\theta \in (2, 2^\star)$ such that $H(s,t) \geq a_0(|s|^p + |t|^q)$ and $0 < \theta H(s,t) \leq sH_s(s,t) + tH_t(s,t)$ for all $s > 0, t > 0$.

Next, we show the two main results.

Theorem 1. *Assume that (V_0), (A_0), (K_0), (H_1)-(H_3) hold. Then, for any $\sigma > 0$, there exists $\varepsilon_\sigma > 0$, such that $\varepsilon \leq \varepsilon_\sigma$, the perturbed Schrödinger system (1.3) has one least energy solution $(u_\varepsilon, v_\varepsilon)$ satisfying*

$$\frac{\theta - 2}{2\theta} \int_{\mathbb{R}^N} \varepsilon^2(|\nabla|u_\varepsilon||^2 + |\nabla|v_\varepsilon||^2) + V(x)(|u_\varepsilon|^2 + |v_\varepsilon|^2) \leq \sigma\varepsilon^N. \tag{1.4}$$

Theorem 2. *Let (V_0), (A_0), (K_0) and (H_1)-(H_3) be satisfied. Moreover, assume that $H(u,v)$ is even in (u,v), then, for any $m \in N$ and $\sigma > 0$, there is*

$\Lambda_{m\sigma} > 0$, *such that the system* (1.3) *has at least m pairs of solutions* $(u_\varepsilon, v_\varepsilon)$ *which satisfy the estimate* (1.4) *whenever* $\lambda \geq \Lambda_{m\sigma}$.

The two results above mentioned are complement for the ones in [12]. Observe that though the method used in this paper is similar to the one of [12], the procedure of the main results is not trivial. We must face our problem with complex-valued functions, at the same time, we need much more delicate estimates for the appearance of magnetic potentials $A(x)$ and $B(x)$.

This paper is organized as follows. In Section 2, we describe some preliminaries. Section 3 contains the behavior of (PS) sequences and technical Lemmas. Section 4 includes the proofs of the main results.

2. Preliminaries

Let $\lambda = \varepsilon^{-2}$, we think about the following equivalent system

(2.1)
$$\begin{cases} -(\nabla + i\sqrt{\lambda}A(x))^2 u + \lambda V(x)u = \lambda H_u(u,v) + \lambda K(x)|u|^{2^*-2}u, \ x \in \mathbb{R}^N, \\ -(\nabla + i\sqrt{\lambda}B(x))^2 v + \lambda V(x)v = \lambda H_v(u,v) + \lambda K(x)|v|^{2^*-2}v, \ x \in \mathbb{R}^N. \end{cases}$$

In order to prove Theorem 1, we need only prove the following result.

Theorem 3. *Assume that* (V_0), (A_0), (K_0), (H_1)-(H_3) *hold. Then, for* $\sigma > 0$, *there exists* $\Lambda_\sigma > 0$ *such that if* $\lambda \geq \Lambda_\sigma$, *the system* (2.1) *has at least one least energy solution* (u_λ, v_λ) *which satisfies*

$$\frac{\theta-2}{2\theta}\int_{\mathbb{R}^N}(|\nabla|u_\lambda||^2 + |\nabla|v_\lambda||^2 + \lambda V(x)(|u_\lambda|^2 + |v_\lambda|^2) \leq \sigma\lambda^{1-\frac{N}{2}}.$$

Similarly, if we can prove the multiplicity of nontrivial solutions to (2.1), Theorem 2 will be obtained.

For the convenience, we quote the necessary notations.

Let $\nabla_A u = (\nabla + i\sqrt{\lambda}A)u$, $\nabla_B v = (\nabla + i\sqrt{\lambda}B)v$, $E_{\lambda,A}(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) : \nabla_A u \in L^2(\mathbb{R}^N)\}$ and $E_{\lambda,B}(\mathbb{R}^N) = \{v \in L^2(\mathbb{R}^N) : \nabla_B v \in L^2(\mathbb{R}^N)\}$. It is obvious that $E_{\lambda,A}$ is the Hilbert subspace under the scalar product

$$(u,v)_{\lambda,A} = \mathrm{Re}\int_{\mathbb{R}^N}((\nabla_A u, \overline{\nabla_A} v) + \lambda V(x)u\overline{v}),$$

the norm induced by the product $(\cdot,\cdot)$ is

$$\|u\|^2_{\lambda,A} = \int_{\mathbb{R}^N}(|\nabla_A u|^2 + \lambda V(x)|u|^2).$$

It is easily known that $E_{\lambda,A}$ is the closure of $C_0^\infty(\mathbb{R}^N, \mathbb{C})$. For $E_{\lambda,B}$, there exists the similar results to $E_{\lambda,A}$.

Remark 2.1. We have the following diamagnetic inequality (see [13]):

$$|\nabla_A u(x)| \geq |\nabla|u(x)||, \ u \in E_{\lambda,A}(\mathbb{R}^N),$$
$$|\nabla_B v(x)| \geq |\nabla|v(x)||, \ v \in E_{\lambda,B}(\mathbb{R}^N).$$

Indeed, since A, B is real-valued, we have

$$|\nabla|u(x)|| = |\mathrm{Re}(\nabla u \tfrac{\bar{u}}{|u|})| = |\mathrm{Re}(\nabla u + i\sqrt{\lambda}Au)\tfrac{\bar{u}}{|u|}| \leq |\nabla u + i\sqrt{\lambda}Au| = |\nabla_A u(x)|$$

and

$$|\nabla|v(x)|| = |\mathrm{Re}(\nabla v \tfrac{\bar{v}}{|v|})| = |\mathrm{Re}(\nabla v + i\sqrt{\lambda}Bv)\tfrac{\bar{v}}{|v|}| \leq |\nabla v + i\sqrt{\lambda}Bv| = |\nabla_B v(x)|$$

(the bar denotes complex conjugation). These facts mean if $u \in E_{\lambda,A}(\mathbb{R}^N)$, $v \in E_{\lambda,B}(\mathbb{R}^N)$, then $|u|, |v| \in H^1(\mathbb{R}^N)$ and therefore $u, v \in L^p(\mathbb{R}^N)$ for any $p \in [2, 2^*)$, i.e., if $u_n \rightharpoonup u$ in $E_{\lambda,A}$ ($v_n \rightharpoonup v$ in $E_{\lambda,B}$), then $u_n \to u$ in $L^p_{loc}(\mathbb{R}^N)$ for any $p \in [2, 2^*)$ ($v_n \to v$ in $L^p_{loc}(\mathbb{R}^N)$) and $u_n \to u$ a.e. in $\mathbb{R}^N (v_n \to v$ a.e. in $\mathbb{R}^N)$.

Remark 2.2. In general, $E_{\lambda,A}(\mathbb{R}^N) \not\subseteq H^1(\mathbb{R}^N)$ and $H^1(\mathbb{R}^N) \not\subseteq E_{\lambda,A}(\mathbb{R}^N)$. However, it is proved by Szulkin that if Ω is a bounded domain with regular boundary, then, the spaces $E_{\lambda,A}(\Omega)$ and $H^1(\Omega)$ are equivalent, where $E_{\lambda,A}(\Omega) = \{u \in L^2(\Omega) : \nabla_A u \in L^2(\Omega)\}$ with the norm

$$\|u\|^2_{E_{\lambda,A}(\Omega)} = \int_\Omega (|\nabla_A u|^2 + |u|^2).$$

From Remark 2.1, for each $s \in [2, 2^*)$, there is $c_s > 0$ (independent of λ) such that, if $\lambda > 1$, we have

$$\left(\int_{\mathbb{R}^N} |u|^s\right)^{\frac{1}{s}} \leq c_s \left(\int_{\mathbb{R}^N} |\nabla|u||^2\right)^{\frac{1}{2}} \leq c_s \left(\int_{\mathbb{R}^N} |\nabla_A u|^2\right)^{\frac{1}{2}} \leq c_s \|u\|_{E_{\lambda,A}}.$$

Set $E_\lambda = E_{\lambda,A} \times E_{\lambda,B}$ and $\|(u,v)\|^2_\lambda = \|u\|^2_{\lambda,A} + \|v\|^2_{\lambda,B}$ for $(u,v) \in E_\lambda$. The energy functional associated with (2.1) is defined by

$$\begin{aligned} J_\lambda(u,v) &= \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u + i\sqrt{\lambda}Au|^2 + |\nabla v + i\sqrt{\lambda}Bv|^2 \\ &\quad + \lambda V(x)(|u|^2 + |v|^2)) - \lambda \int_{\mathbb{R}^N} G(x,u,v) \\ &= \frac{1}{2}\|(u,v)\|^2_\lambda - \lambda \int_{\mathbb{R}^N} G(x,u,v) \quad \text{for } (u,v) \in E_\lambda, \end{aligned}$$

where $G(x,u,v) = \frac{K(x)}{2^*}(|u|^{2^*} + |v|^{2^*}) + H(u,v)$.

Under the assumptions of Theorem 1, standard arguments [22] indicate that $J_\lambda \in C^1(E_\lambda, \mathbb{R})$ and the critical points of J_λ are weak solutions of (2.1).

Moreover, for convenience, let us recall the definition and some properties of the Krasnoselski genus [5].

The concept of the index theory is most easily explained for an even functional E on some Banach space X with symmetry group $G = Z_2 = \{id, -id\}$.

Define

$$\mathbb{A} = \{A \in X \mid A \text{ is closed and } A = -A\}$$

to be the class of closed symmetric subsets of X.

Definition 2.1. For $A \in \mathbb{A}$, define the Krasnoselskii genus of A, denoted by $i(A)$, as

$$i(A) = \begin{cases} 0, \quad A = \phi, \\ \inf\{m; \exists h \in C^0(A, \mathbb{R}^m/\{0\}), h(-u) = -h(u)\}, \\ \infty, \text{ if } \{m; \exists h \in C^0(A, \mathbb{R}^m/\{0\}), h(-u) = -h(u)\} = \phi. \end{cases}$$

It is easy to see that $i(A) = \infty$, if $0 \in A$.

Moreover, the Krasnoselskii genus has the following properties.

Lemma 2.1. (1) $i(A) \geq 0, \quad i(A) = 0 \iff A = \emptyset$;

(2) (*Normalization*) $i(A) = 1$ *if A contains only a pair symmetric points*;

(3) (*Monotonicity*) *For any $A, B \in \mathbb{A}$, if $A \sqsubset B$, then $i(A) \leq i(B)$*;

(4) (*Subadditivity*) $i(A \cup B) \leq i(A) + i(B), \quad \forall A, B \in \mathbb{A}$;

(5) (*Super-variant*) *For any continuous odd map $\varphi : X \to X$ and set $A \in \mathbb{A}$, there holds $i(A) \leq i(\overline{\varphi A})$*;

(6) (*Continuity*) *For any $A \in \mathbb{A}$, if A is compact, there exists a symmetric neighborhood N of A such that $i(\bar{N}) = i(A)$; Furthermore, if A is compact and $0 \in A$, then $i(A) < \infty$*;

(7) *Suppose X_1 is a m-dimensional subspace of X and S is the surface of unit ball in X, then $i(X_1 \cap S) = m$*;

(8) *Suppose $X = X_1 \oplus X_2$, $\dim X_1 = m$, if $A, B \in \mathbb{A}$ and $i(A) > m$, then $A \cap X_2 \neq \emptyset$.*

3. Technical lemmas

We call a sequence $\{(u_n, v_n)\} \subset E_\lambda$ is a $(PS)_c$ sequence, if

$$J_\lambda(u_n, v_n) \to c, J_\lambda^{'}(u_n, v_n) \to 0 \text{ in } E_\lambda^{-1}.$$

J_λ is said to satisfy the $(PS)_c$ condition, if any $(PS)_c$ sequence contains a convergent subsequence.

Similar to the proof of Lemma 3.1 in [12], the following result can be obtained.

Lemma 3.1. *Assume that the assumptions of Theorem 3 hold and $\{(u_n, v_n)\}$ is a $(PS)_c$ sequence for J_λ. Then, $c \geq 0$ and $\{(u_n, v_n)\}$ is bounded in E_λ.*

Proof. By (H_3), we have

$$\begin{aligned} & J_\lambda(u_n, v_n) - \frac{1}{\theta} J_\lambda^{'}(u_n, v_n)(u_n, v_n) \\ = & (\frac{1}{2} - \frac{1}{\theta})\|(u_n, v_n)\|_\lambda^2 + (\frac{1}{\theta} - \frac{1}{2^*})\lambda \int_{\mathbb{R}^N} K(x)(|u_n|^{2^*} + |v_n|^{2^*}) \\ & + \lambda \int_{\mathbb{R}^N} (\frac{1}{\theta}(u_n H_u(u_n, v_n) + v_n H_v(u_n, v_n)) - H(u_n, v_n)) \\ \geq & (\frac{1}{2} - \frac{1}{\theta})\|(u_n, v_n)\|_\lambda^2 \geq 0. \end{aligned}$$

So $\{(u_n, v_n)\}$ is bounded in E_λ and $c \geq 0$. The proof is completed. □

By Lemma 3.1, $(PS)_c$ sequence $\{(u_n, v_n)\}$ is bounded in E_λ. So we can assume $(u_n, v_n) \rightharpoonup (u, v)$ in E_λ. By Remark 2.1, passing to a subsequence, $u_n \to u$, $v_n \to v$ in $L^p_{loc}(\mathbb{R}^N)$ for any $p \in [2, 2^*)$ and $u_n \to u, v_n \to v$ a.e. in $\mathbb{R}^N$. It is standard that (u, v) is a critical point of J_λ, namely a weak solution of (2.1).

Lemma 3.2. *Let $s \in [2, 2^*)$. Then there is a subsequence $\{(u_{n_j}, v_{n_j})\}$ such that for any $\varepsilon > 0$, there exists $r_\varepsilon > 0$ with*

$$\limsup_{i\to\infty} \int_{B_i \setminus B_r} |u_{n_i}|^s + |v_{n_i}|^s \le \epsilon \quad \textit{for all } r \ge r_\epsilon,$$

where $B_r := \{x \in \mathbb{R}^N : |x| \le r\}$.

Proof. From $u_n \to u$ and $v_n \to v$ in $L^s_{loc}(\mathbb{R}^N)$, we have

$$\int_{B_i} |u_n|^s + |v_n|^s \to \int_{B_i} |u|^s + |v|^s \quad \text{as } n \to \infty.$$

Thus, there exists $\tilde{n}_i \in \mathbb{N}$ such that

$$\int_{B_i} (|u_{n_j}|^s - |u|^s) + (|v_{n_j}|^s - |v|^s) < \frac{1}{i} \quad \text{for all } n_j = \tilde{n}_i + j,\ j = 1, 2, \ldots.$$

Let $\tilde{n}_{i+1} \ge \tilde{n}_i$ and for $n_i = \tilde{n}_i + i$, we have

$$\int_{B_i} (|u_{n_i}|^s - |u|^s) + (|v_{n_i}|^s - |v|^s) < \frac{1}{i} \quad \text{for all } n_i = \tilde{n}_i + i,\ i = 1, 2, \ldots.$$

Observe that there exists $r_\epsilon > 0$ satisfying

$$\int_{\mathbb{R}^N \setminus B_r} |u|^s + |v|^s < \frac{\varepsilon}{3} \quad \text{for all } r \ge r_\varepsilon.$$

Therefore,

$$\begin{aligned}
\int_{B_i \setminus B_r} |u_{n_i}|^s + |v_{n_i}|^s &= \int_{B_i} |u_{n_i}|^s - |u|^s + \int_{B_i \setminus B_r} |u|^s + \int_{B_r} |u|^s - |u_{n_i}|^s \\
&\le \frac{1}{i} + \int_{\mathbb{R}^N \setminus B_r} |u|^s + \int_{B_r} |u|^s - |u_{n_i}|^s \\
&\le \varepsilon \quad \text{as } i \to \infty.
\end{aligned}$$

The proof is completed. □

Let $\eta \in C^\infty(\mathbb{R}^+)$ which satisfy $0 \le \eta(t) \le 1, t \ge 0$. $\eta(t) = 1$, if $t \le 1$ and $\eta(t) = 0$, if $t \ge 2$. Define $\widetilde{u}_j(x) = \eta(2|x|/j)u(x)$, $\widetilde{v}_j(x) = \eta(2|x|/j)v(x)$, then $\widetilde{u}_j \to u$ in $E_{1,A}$ and $\widetilde{v}_j \to v$ in $E_{1,B}$.

Lemma 3.3.

$$\lim_{j\to\infty} \left| \int_{\mathbb{R}^N} \left(H_u(u_{n_j}, v_{n_j}) - H_u(u_{n_j} - \widetilde{u}_j, v_{n_j} - \widetilde{v}_j) - H_u(\widetilde{u}_j, \widetilde{v}_j) \right) \varphi \right| = 0, \tag{3.1}$$

$$\lim_{j\to\infty} \left| \int_{\mathbb{R}^N} \left(H_v(u_{n_j}, v_{n_j}) - H_v(u_{n_j} - \widetilde{u}_j, v_{n_j} - \widetilde{v}_j) - H_v(\widetilde{u}_j, \widetilde{v}_j) \right) \psi \right| = 0 \tag{3.2}$$

uniformly in $(\varphi,\psi)\in E_1$ *with* $\|(\varphi,\psi)\|_{E_1}\le 1$.

Proof. Note that $\widetilde{u}_j\to u$ in $E_{1,A}$ and $u_j\rightharpoonup u$ in $E_{1,A}$, the local compactness of Sobolev embedding implies that, for any $r>0$,

$$\lim_{j\to\infty}\left|\int_{B_r}\big(H_u(u_{n_j},v_{n_j})-H_u(u_{n_j}-\widetilde{u}_j,v_{n_j}-\widetilde{v}_j)-H_u(\widetilde{u}_j,\widetilde{v}_j)\big)\varphi\right|=0$$

uniformly in $\|\varphi\|_{1,A}\le 1$. For any $\varepsilon>0$, there exists $r_\varepsilon>0$ such that

$$\limsup_{i\to\infty}\int_{B_i\setminus B_r}|\widetilde{u}_i|^s\le\int_{\mathbb{R}^N\setminus B_r}|u|^s\le\varepsilon\quad\text{for all } r\ge r_\varepsilon.$$

It follows from Lemma 3.2 (for $s=2,\alpha+1,\beta+1$) that

$$\begin{aligned}
&\lim_{j\to\infty}\left|\int_{\mathbb{R}^N}\big(H_u(u_{n_j},v_{n_j})-H_u(u_{n_j}-\widetilde{u}_j,v_{n_j}-\widetilde{v}_j)-H_u(\widetilde{u}_j,\widetilde{v}_j)\big)\varphi\right|\\
&=\lim_{j\to\infty}\left|\int_{B_j\setminus B_r}\big(H_u(u_{n_j},v_{n_j})-H_u(u_{n_j}-\widetilde{u}_j,v_{n_j}-\widetilde{v}_j)-H_u(\widetilde{u}_j,\widetilde{v}_j)\big)\varphi\right|\\
&\le c_1\limsup_{j\to\infty}\int_{B_j\setminus B_r}(|u_{n_j}|+|v_{n_j}|+|\widetilde{u}_j|+|\widetilde{v}_j|)|\varphi|\\
&\quad+c_2\limsup_{j\to\infty}\int_{B_j\setminus B_r}(|u_{n_j}|^\alpha+|v_{n_j}|^\beta+|\widetilde{u}_j|^\alpha+|\widetilde{v}_j|^\beta)|\varphi|\\
&\le c_1\limsup_{j\to\infty}(\|u_{n_j}\|_{L^2(B_j\setminus B_r)}+\|v_{n_j}\|_{L^2(B_j\setminus B_r)}\\
&\quad+\|\widetilde{u}_j\|_{L^2(B_j\setminus B_r)}+\|\widetilde{v}_j\|_{L^2(B_j\setminus B_r)})\|\varphi\|_{L^2}\\
&\quad+c_2\bigg(\limsup_{j\to\infty}(\|u_{n_j}\|^\alpha_{L^{\alpha+1}(B_j\setminus B_r)}+\|\widetilde{u}_j\|^\alpha_{L^{\alpha+1}(B_j\setminus B_r)})\|\varphi\|_{L^{\alpha+1}}\\
&\quad+\limsup_{j\to\infty}(\|v_{n_j}\|^\beta_{L^{\beta+1}(B_j\setminus B_r)}+\|\widetilde{v}_j\|^\beta_{L^{\beta+1}(B_j\setminus B_r)})\|\varphi\|_{L^{\beta+1}}\bigg)\\
&\le c_3\varepsilon^{\frac12}+c_4\varepsilon^{\frac{\alpha}{\alpha+1}}+c_5\varepsilon^{\frac{\beta}{\beta+1}},
\end{aligned}$$

which implies the conclusion is correct. Similarly, we have that

$$\lim_{j\to\infty}\left|\int_{\mathbb{R}^N}\big(H_v(u_{n_j},v_{n_j})-H_v(u_{n_j}-\widetilde{u}_j,v_{n_j}-\widetilde{v}_j)-H_v(\widetilde{u}_j,\widetilde{v}_j)\big)\psi\right|=0.$$

The proof is completed. □

Lemma 3.4. *One has along a subsequence*

$$J_\lambda(u_n-\widetilde{u}_n,v_n-\widetilde{v}_n)\to c-J_\lambda(u,v),$$
$$J'_\lambda(u_n-\widetilde{u}_n,v_n-\widetilde{v}_n)\to 0\ \textit{in}\ E_\lambda^{-1}.$$

Proof. Since $\widetilde{u}_j\to u$ in $E_{1,A}$, $\widetilde{v}_j\to v$ in $E_{1,B}$ and $(u_j,v_j)\rightharpoonup(u,v)$ in E_λ, one has

$$J_\lambda(u_n-\widetilde{u}_n,v_n-\widetilde{v}_n)$$

$$
\begin{aligned}
&= J_\lambda(u_n, v_n) - J_\lambda(\widetilde{u}_n, \widetilde{v}_n) \\
&\quad + \frac{\lambda}{2^*}\int_{\mathbb{R}^N} K(x)(|u_n|^{2^*} - |u_n - \widetilde{u}_n|^{2^*} - |\widetilde{u}_n|^{2^*}) \\
&\quad + \frac{\lambda}{2^*}\int_{\mathbb{R}^N} K(x)(|v_n|^{2^*} - |v_n - \widetilde{v}_n|^{2^*} - |\widetilde{v}_n|^{2^*}) \\
&\quad + \lambda\int_{\mathbb{R}^N} H(u_n, v_n) - H(u_n - v_n, \widetilde{u}_n - \widetilde{v}_n) - H(\widetilde{u}_n, \widetilde{v}_n) + o(1).
\end{aligned}
$$

Along the lines in proving the Brezis-Lieb lemma, it is easy to check that

$$
\lim_{n\to\infty}\int_{\mathbb{R}^N} K(x)(|u_n|^{2^*} - |u_n - \widetilde{u}_n|^{2^*} - |\widetilde{u}_n|^{2^*}) = 0,
$$

$$
\lim_{n\to\infty}\int_{\mathbb{R}^N} K(x)(|v_n|^{2^*} - |v_n - \widetilde{u}_n|^{2^*} - |\widetilde{v}_n|^{2^*}) = 0
$$

and

$$
\lim_{n\to\infty}\int_{\mathbb{R}^N} H(u_n, v_n) - H(u_n - v_n, \widetilde{u}_n - \widetilde{v}_n) - H(\widetilde{u}_n, \widetilde{v}_n) = 0.
$$

Note that $J_\lambda(u_n, v_n) \to c$ and $J_\lambda(\widetilde{u}_n, \widetilde{v}_n) \to J_\lambda(u, v)$, we get

$$
J_\lambda(u_n - \widetilde{u}_n, v_n - \widetilde{v}_n) \to c - J_\lambda(u, v).
$$

For any $(\varphi, \psi) \in E_\lambda$, we have

$$
\begin{aligned}
&J'_\lambda(u_n - \widetilde{u}_n, v_n - \widetilde{v}_n)(\varphi, \psi) \\
&= J'_\lambda(u_n, v_n)(\varphi, \psi) - J'_\lambda(\widetilde{u}_n, \widetilde{v}_n)(\varphi, \psi) \\
&\quad + \lambda\int_{\mathbb{R}^N} K(x)(|u_n|^{2^*-2}u_n - |u_n - \widetilde{u}_n|^{2^*-2}(u_n - \widetilde{u}_n) - |\widetilde{u}_n|^{2^*-2}\widetilde{u}_n)\varphi \\
&\quad + \lambda\int_{\mathbb{R}^N} K(x)(|v_n|^{2^*-2}v_n - |v_n - \widetilde{v}_n|^{2^*-2}(v_n - \widetilde{v}_n) - |\widetilde{v}_n|^{2^*-2}\widetilde{v}_n)\psi \\
&\quad + \lambda\int_{\mathbb{R}^N} (H_u(u_n, v_n) - H_u(u_n - \widetilde{u}_n, v_n - \widetilde{v}_n) - H_u(\widetilde{u}_n, \widetilde{v}_n))\,\varphi \\
&\quad + \lambda\int_{\mathbb{R}^N} (H_v(u_n, v_n) - H_v(u_n - \widetilde{u}_n, v_n - \widetilde{v}_n) - H_v(\widetilde{u}_n, \widetilde{v}_n))\,\psi.
\end{aligned}
$$

It is standard to check that

$$
\lim_{n\to\infty}\int_{\mathbb{R}^N} K(x)(|u_n|^{2^*-2}u_n - |u_n - \widetilde{u}_n|^{2^*-2}(u_n - \widetilde{u}_n) - |\widetilde{u}_n|^{2^*-2}\widetilde{u}_n)\varphi = 0
$$

and

$$
\lim_{n\to\infty}\int_{\mathbb{R}^N} K(x)(|v_n|^{2^*-2}v_n - |v_n - \widetilde{v}_n|^{2^*-2}(v_n - \widetilde{v}_n) - |\widetilde{v}_n|^{2^*-2}\widetilde{v}_n)\psi = 0
$$

uniformly in $\|(\varphi, \psi)\|_\lambda \le 1$. Therefore, the conclusion required holds by Lemma 3.3. The proof is completed. □

Let $u_n^1 = u_n - \widetilde{u}_n, v_n^1 = v_n - \widetilde{v}_n$, then $u_n - u = u_n^1 + (\widetilde{u}_n - u), v_n - v = v_n^1 + (\widetilde{v}_n - v)$. So $(u_n, v_n) \to (u, v)$ in E_λ if and only if $(u_n^1, v_n^1) \to (0, 0)$ in E_λ.

Note that

$$\begin{aligned}
& J_\lambda(u_n^1, v_n^1) - \frac{1}{2} J_\lambda'(u_n^1, v_n^1)(u_n^1, v_n^1) \\
= & (\frac{1}{2} - \frac{1}{2^*})\lambda \int_{\mathbb{R}^N} K(x)(|u_n^1|^{2^*} + |v_n^1|^{2^*}) \\
& + \lambda \int_{\mathbb{R}^N} (\frac{1}{2}(u_n^1 H_u(u_n^1, v_n^1) + v_n^1 H_v(u_n^1, v_n^1)) - H(u_n^1, v_n^1)) \\
\geq & \frac{\lambda}{N} K_{\min} \int_{\mathbb{R}^N} (|u_n^1|^{2^*} + |v_n^1|^{2^*}),
\end{aligned}$$

where $K_{\min} = \inf_{x \in \mathbb{R}^N} K(x) > 0$. Hence by Lemma 3.4, we get

$$\|u_n^1\|_{2^*}^{2^*} + \|v_n^1\|_{2^*}^{2^*} \leq \frac{N(c - J_\lambda(u, v))}{\lambda K_{\min}} + o(1). \tag{3.3}$$

Now, we determine the energy level of the functional J_λ below which the (PS) condition holds.

Let $V_b(x) = \max\{V(x), b\}$, where b is the positive constant in the assumption (V_0). Since the set ν^b has finite measure and $u_n^1, v_n^1 \to 0$ in $L_{loc}^2(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^N} V(x)(|u_n^1|^2 + |v_n^1|^2) = \int_{\mathbb{R}^N} V_b(x)(|u_n^1|^2 + |v_n^1|^2) + o(1).$$

By (H_2) and (H_3), there exists $C_b > 0$ such that

$$\begin{aligned}
& \int_{\mathbb{R}^N} K(x)(|u|^{2^*} + |v|^{2^*}) + uH_u(u, v) + vH_v(u, v) \\
& \leq b(\|u\|_2^2 + \|v\|_2^2) + C_b(\|u\|_{2^*}^{2^*} + \|v\|_{2^*}^{2^*}).
\end{aligned}$$

Let S be the best Sobolev constant satisfying

$$S\|u\|_{2^*}^2 \leq \int_{\mathbb{R}^N} |\nabla u|^2 \quad \text{for all } u \in H^1(\mathbb{R}^N).$$

Lemma 3.5. *Under the assumptions of Theorem 3, there is a constant $\alpha_0 > 0$ (independent of λ) such that, for any $(PS)_c$ sequence $\{(u_n, v_n)\} \subset E_\lambda$ for J_λ with $(u_n, v_n) \rightharpoonup (u, v)$, either $(u_n, v_n) \to (u, v)$ or $c - J_\lambda(u, v) \geq \alpha_0 \lambda^{1-\frac{N}{2}}$.*

Proof. Assume that $(u_n, v_n) \nrightarrow (u, v)$, we have

$$\liminf_{n\to\infty} \|(u_n^1, v_n^1)\|_\lambda > 0 \quad \text{and} \quad c - J_\lambda(u, v) > 0.$$

By the Sobolev inequality, we obtain

$$\begin{aligned}
& S(\|u_n^1\|_{2^*}^2 + \|v_n^1\|_{2^*}^2) \\
\leq & \int_{\mathbb{R}^N} (|\nabla|u_n^1||^2 + |\nabla|v_n^1||^2 + \lambda V(x)(|u_n^1|^2 + |v_n^1|^2)) - \int_{\mathbb{R}^N} \lambda V(x)(|u_n^1|^2 + |v_n^1|^2) \\
\leq & \int_{\mathbb{R}^N} |\nabla u_n^1 + iAu_n^1|^2 + |\nabla v_n^1 + iBv_n^1|^2 + \lambda V(x)(|u_n^1|^2 + |v_n^1|^2)
\end{aligned}$$

$$
\begin{aligned}
&- \int_{\mathbb{R}^N} \lambda V(x)(|u_n^1|^2 + |v_n^1|^2) \\
&= \lambda \int_{\mathbb{R}^N} K(x)(|u_n^1|^{2^*} + |v_n^1|^{2^*}) + u_n^1 H_u(u_n^1, v_n^1) + v_n^1 H_v(u_n^1, v_n^1) \\
&\quad - \int_{\mathbb{R}^N} \lambda V_b(x)(|u_n^1|^2 + |v_n^1|^2) + o(1) \\
&\le \lambda b(\|u_n^1\|_2^2 + \|v_n^1\|_2^2) + C_b\lambda(\|u_n^1\|_{2^*}^{2^*} + \|v_n^1\|_{2^*}^{2^*}) - \lambda b(\|u_n^1\|_2^2 + \|v_n^1\|_2^2) + o(1) \\
&= \lambda C_b(\|u_n^1\|_{2^*}^{2^*} + \|v_n^1\|_{2^*}^{2^*}) + o(1).
\end{aligned}
$$

Meanwhile, it is easy to show that

$$\liminf_{n\to\infty} \|u_n^1\|_{2^*}^{2^*} + \|v_n^1\|_{2^*}^{2^*} > 0.$$

Thus, by (3.3), we get

$$
\begin{aligned}
S &\le \lambda C_b(\|u_n^1\|_{2^*}^{2^*} + \|v_n^1\|_{2^*}^{2^*})^{\frac{2^*-2}{2^*}} + o(1) \\
&\le \lambda C_b(\frac{N(c - J_\lambda(u,v))}{\lambda K_{\min}})^{\frac{2}{N}} + o(1) \\
&= \lambda^{1-\frac{2}{N}} C_b(\frac{N}{K_{\min}})^{\frac{2}{N}} (c - J_\lambda(u,v))^{\frac{2}{N}} + o(1).
\end{aligned}
$$

Therefore, $\alpha_0 \lambda^{1-\frac{N}{2}} \le c - J_\lambda(u,v) + o(1)$, where $\alpha_0 = S^{\frac{N}{2}} C_b^{-\frac{N}{2}} N^{-1} K_{\min}$. The proof is completed. □

Since $H^1(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$ is not compact, J_λ does not satisfy the $(PS)_c$ condition for all $c > 0$. But Lemma 3.5 implies that J_λ satisfies the following local $(PS)_c$ condition.

Lemma 3.6. *Under the assumptions of Theorem 3, there is a constant* $\alpha_0 > 0$ (*independent of* λ) *such that, if a sequence* $\{(u_n, v_n)\} \subset E_\lambda$ *satisfies*

$$J_\lambda(u_n, v_n) \to c < \alpha_0 \lambda^{1-\frac{N}{2}} \text{ and } J_\lambda^{'}(u_n, v_n) \to 0 \text{ in } E_\lambda^{-1},$$

then, $\{(u_n, v_n)\}$ *is relatively compact in* E_λ.

Next, we consider $\lambda \ge 1$. By the assumptions of Theorem 3, we can see that J_λ has the mountain-pass structure.

Lemma 3.7. *Under the assumptions of Theorem 3, there exist* $\alpha_\lambda, \rho_\lambda > 0$ *such that*

$$J_\lambda(u,v) > 0,\ 0 < \|(u,v)\|_\lambda < \rho_\lambda;\ J_\lambda(u,v) \ge \alpha_\lambda,\ \text{if } \|(u,v)\|_\lambda = \rho_\lambda.$$

Proof. By (H_1)-(H_3), for $\delta \le (4\lambda C_2)^{-1}$, there exists C_δ such that

$$\int_{\mathbb{R}^N} G(x,u,v) \le \delta(\|u\|_2^2 + \|v\|_2^2) + C_\delta(\|u\|_{2^*}^{2^*} + \|v\|_{2^*}^{2^*}).$$

Thus

$$J_\lambda(u,v) \ge \frac{1}{2}\|(u,v)\|_\lambda^2 - \lambda\delta(\|u\|_2^2 + \|v\|_2^2) - \lambda C_\delta(\|u\|_{2^*}^{2^*} + \|v\|_{2^*}^{2^*}).$$

Observe that $\|u\|_2^2 + \|v\|_2^2 \leq C_2\|(u,v)\|_\lambda^2$, we have

$$J_\lambda(u,v) \geq \frac{1}{4}\|(u,v)\|_\lambda^2 - \lambda C_\delta(\|u\|_{2^*}^{2^*} + \|v\|_{2^*}^{2^*}),$$

which implies that the conclusions required hold. The proof is completed. □

Lemma 3.8. *Under the assumptions of Theorem 3, for any finite dimensional subspace $F \subset E_\lambda$, we get*

$$J_\lambda(u,v) \to -\infty \text{ as } (u,v) \in F,\ \|(u,v)\|_\lambda \to \infty.$$

Proof. By the assumptions of Theorem 3, one has

$$J_\lambda(u,v) \leq \frac{1}{2}\|(u,v)\|_\lambda^2 - \lambda a_0(|u|_p^p + |v|_q^q) \quad \text{for all } (u,v) \in E_\lambda.$$

Since all norms in a finite dimensional space are equivalent and $p, q > 2$, we easily obtain the desired conclusion. □

Define the functional

$$\Phi_\lambda(u,v) = \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u + i\sqrt{\lambda}Au|^2 + |\nabla v + i\sqrt{\lambda}Bv|^2 + \lambda V(x)(|u|^2 + |v|^2)) - a_0\lambda\int_{\mathbb{R}^N}|u|^p + |v|^q.$$

It is obvious that $\Phi_\lambda \in C^1(E_\lambda)$ and $J_\lambda(u,v) \leq \Phi_\lambda(u,v), (u,v) \in E_\lambda$.

Note that

$$\inf\left\{\int_{\mathbb{R}^N}|\nabla\phi|^2 : \phi \in C_0^\infty(\mathbb{R}^N,\mathbb{R}), \|\phi\|_p = 1\right\} = 0$$

and

$$\inf\left\{\int_{\mathbb{R}^N}|\nabla\psi|^2 : \phi \in C_0^\infty(\mathbb{R}^N,\mathbb{R}), \|\psi\|_q = 1\right\} = 0.$$

For any $\delta > 0$, there exist $\phi_\delta, \psi_\delta \in C_0^\infty(\mathbb{R}^N,\mathbb{R})$, with $\|\phi_\delta\|_p = \|\psi_\delta\|_q = 1$, and $\mathrm{supp}\phi_\delta$, $\mathrm{supp}\psi_\delta \subset B_{r_\delta}(0)$ such that $\|\nabla\phi_\delta\|_2^2$, $\|\nabla\psi_\delta\|_2^2 < \delta$.

Let $e_\lambda(x) = (\phi_\delta(\sqrt{\lambda}x), \psi_\delta(\sqrt{\lambda}x))$, then, $\mathrm{supp}e_\lambda \subset B_{\lambda^{-\frac{1}{2}}r_\delta}(0)$.

For $t \geq 0$, we have

$$\begin{aligned}\Phi_\lambda(te_\lambda) &= \frac{t^2}{2}\|e_\lambda\|_\lambda^2 - a_0\lambda t^p\int_{\mathbb{R}^N}|\phi_\delta(\sqrt{\lambda}x)|^p - a_0\lambda t^q\int_{\mathbb{R}^N}|\psi_\delta(\sqrt{\lambda}x)|^q \\ &= \lambda^{1-\frac{N}{2}}I_\lambda(t\phi_\delta, t\psi_\delta),\end{aligned}$$

where

$$\begin{aligned}I_\lambda(u,v) = \frac{1}{2}\int_{\mathbb{R}^N}&(|\nabla_A u|^2 + |\nabla_A v|^2 + (|A(\lambda^{-\frac{1}{2}}x)|^2 + V(\lambda^{-\frac{1}{2}}x))(|u|^2 + |v|^2)) \\ &- a_0\int_{\mathbb{R}^N}(|u|^p + |v|^q).\end{aligned}$$

Obviously, we get

$$\begin{aligned}&\max_{t\geq 0} I_\lambda(t\phi_\delta, t\psi_\delta)\\ &\leq \frac{p-2}{2p(pa_0)^{\frac{2}{p-2}}}\left\{\int_{\mathbb{R}^N}|\nabla_A\phi_\delta|^2+(|A(\lambda^{-\frac12}x)|^2+V(\lambda^{-\frac12}x))|\phi_\delta|^2\right\}\\ &\quad+\frac{q-2}{2q(qa_0)^{\frac{2}{q-2}}}\left\{\int_{\mathbb{R}^N}|\nabla_A\psi_\delta|^2+(|A(\lambda^{-\frac12}x)|^2+V(\lambda^{-\frac12}x))|\psi_\delta|^2\right\}.\end{aligned}$$

Recall that $A(0)=0$, $B(0)=0$, $V(0)=0$, suppϕ_δ, supp$\psi_\delta\subset B_{r_\delta}(0)$. Therefore, there exists $\Lambda_\delta>0$ such that, for all $\lambda\geq\Lambda_\sigma$, we get

$$\max_{t\geq 0} J_\lambda(t\phi_\delta, t\psi_\delta)\leq\left(\frac{p-2}{2p(pa_0)^{\frac{2}{p-2}}}(5\delta)^{\frac{p}{p-2}}+\frac{q-2}{2q(qa_0)^{\frac{2}{q-2}}}(5\delta)^{\frac{q}{q-2}}\right)\lambda^{1-\frac{N}{2}}. \tag{3.4}$$

Lemma 3.9. *Under the assumptions of Theorem 3, for any $\sigma>0$, there exists $\Lambda_\sigma>0$ such that for each $\lambda\geq\Lambda_\sigma$, there is $\overline{e}_\lambda\in E_\lambda$ with $\|\overline{e}_\lambda\|_\lambda>\rho_\lambda$, $J_\lambda(\overline{e}_\lambda)\leq 0$ and*

$$\max_{t\geq 0} J_\lambda(t\overline{e}_\lambda)\leq\sigma\lambda^{1-\frac{N}{2}},$$

where ρ_λ is defined form Lemma 3.7.

Proof. This proof is similar to Lemma 4.3 in [12], it can be easily obtained. □

For any $m\in N$, we can choose m functions $\phi_\delta^i\in C_0^\infty(\mathbb{R}^N)$ such that supp$\phi_\delta^i\cap$ supp$\phi_\delta^j=\emptyset$, $i\neq j$, $\|\phi_\delta^i\|_p=1$ and $\|\nabla\phi_\delta^i\|_2^2<\delta$. Similarly, one can also get m functions $\psi_\delta^i\in C_0^\infty(\mathbb{R}^N)$ with supp$\psi_\delta^i\cap$supp$\psi_\delta^j=\emptyset$, $i\neq j$, $\|\psi_\delta^i\|_q=1$ and $\|\nabla\psi_\delta^i\|_2^2<\delta$. Let $r_\delta^m>0$ be such that supp$(\phi_\delta^i,\psi_\delta^i)\subset B^i_{r_\delta^m}(0)$ for $i=1,2,\ldots,m$.

Set $e_\lambda^i(x)=(\phi_\delta^i(\sqrt{\lambda}x),\psi_\delta^i(\sqrt{\lambda}x))=(f_\lambda^i,g_\lambda^i)$, $i=1,2,\ldots,m$, then, supp$e_\lambda^i(x)\subset B_{\lambda^{-\frac12}r_\delta^m}(0)$.

Let $F_{\lambda\delta}^m=\text{span}\{e_\lambda^1,e_\lambda^2,\ldots,e_\lambda^m\}$. For each

$$(u,v)=\sum_{i=1}^m k_ie_\lambda^i\in F_{\lambda\delta}^m,$$

we get

$$\int_{\mathbb{R}^N}(|\nabla_Au|^2+|\nabla_Bv|^2)=\sum_{i=1}^m|k_i|^2(\int_{\mathbb{R}^N}|\nabla_Af_\lambda^i|^2+\int_{\mathbb{R}^N}|\nabla_Bg_\lambda^i|^2),$$

$$\int_{\mathbb{R}^N}V(x)(|u|^2+|v|^2)=\sum_{i=1}^m|k_i|^2(\int_{\mathbb{R}^N}V(x)|f_\lambda^i|^2+\int_{\mathbb{R}^N}V(x)|g_\lambda^i|^2),$$

$$\frac{1}{2^*}\int_{\mathbb{R}^N}K(x)(|u|^{2^*}+|v|^{2^*})=\frac{1}{2^*}\sum_{i=1}^m|k_i|^{2^*}(\int_{\mathbb{R}^N}K(x)|f_\lambda^i|^{2^*}+\int_{\mathbb{R}^N}K(x)|g_\lambda^i|^{2^*})$$

and

$$\int_{\mathbb{R}^N} H(u,v) = \sum_{i=1}^{m} \int_{\mathbb{R}^N} H(k_i f_\lambda^i, k_i g_\lambda^i).$$

Therefore

$$J_\lambda(u,v) = \sum_{i=1}^{m} J_\lambda(k_i e_\lambda^i)$$

and

$$J_\lambda(k_i e_\lambda^i) \le \phi_\lambda(k_i e_\lambda^i).$$

Set $\beta_\delta := \max\{\|(\phi_\delta^i, \psi_\delta^i)\|_2^2 : \ i = 1, 2, \ldots, m\}$ and choose some $A_{m\delta} > 0$ such that

$$V(\lambda^{\frac{-1}{2}} x) \le \frac{\delta}{\beta_\delta} \quad \text{for all } |x| \le r_\delta^m \text{ and } \lambda \ge \Lambda_{m\delta}.$$

Similar to the proof mentioned above, we can acquire the following inequality

$$\max_{(u,v)\in F_{\lambda\delta}^m} J_\lambda(u,v) \le \left(\frac{m(p-2)}{2p(pa_0)^{\frac{2}{p-2}}} (5\delta)^{\frac{p}{p-2}} + \frac{m(q-2)}{2q(qa_0)^{\frac{2}{q-2}}} (5\delta)^{\frac{q}{q-2}} \right) \lambda^{\frac{2-N}{2}}. \tag{3.5}$$

By using the estimate, we can get the following result.

Lemma 3.10. *Under the assumptions of Lemma 3.7, for any $m \in \mathbb{N}$ and $\sigma > 0$, there exists $\Lambda_{m\sigma} > 0$ such that for each $\lambda \ge \Lambda_{m\delta}$, there exists a m-dimensional subspace F satisfying*

$$\max_{(u,v)\in F} J_\lambda(u,v) \le \sigma \lambda^{\frac{2-N}{2}}.$$

Proof. Choose $\sigma > 0$ so small that

$$\left(\frac{m(p-2)}{2p(pa_0)^{\frac{2}{p-2}}} (5\delta)^{\frac{p}{p-2}} + \frac{m(q-2)}{2q(qa_0)^{\frac{2}{q-2}}} (5\delta)^{\frac{q}{q-2}} \right) \le \sigma$$

and take $F = F_{\lambda\delta}^m$. By (3.5), we get the conclusion as required. □

4. Proof of the main results

Proof of Theorem 1. By Lemma 3.9, for any $0 < \sigma < \alpha_0$, there exists $\Lambda_\sigma > 0$, such that for each $\lambda \ge \Lambda_\sigma$, we have $c_\lambda \le \sigma \lambda^{1-\frac{N}{2}}$, where

$$c_\lambda = \inf_{\gamma\in\Gamma_\lambda} \max_{t\ge 0} J_\lambda(\gamma(t)),$$
$$\Gamma_\lambda = \{\gamma \in C([0,1], E_\lambda) : \gamma(0) = 0, \gamma(1) = \overline{e}_\lambda\}.$$

In virtue of Lemma 3.5, J_λ satisfies the $(PS)_{c_\lambda}$ condition. Hence, by the mountain pass theorem, there exists $(u_\lambda, v_\lambda) \in E_\lambda$ satisfying $J_\lambda'(u_\lambda, v_\lambda) = 0$ and $J_\lambda(u_\lambda, v_\lambda) = c_\lambda$. Therefore, (u_λ, v_λ) is a weak solution of (2.1). Moreover, it is well known that (u_λ, v_λ) is a least energy solution of (2.1).

Note that $J_\lambda(u_\lambda, v_\lambda) \le \sigma \lambda^{1-\frac{N}{2}}$, $J_\lambda'(u_\lambda, v_\lambda) = 0$ and

$$J_\lambda(u_\lambda, v_\lambda) = J_\lambda(u_\lambda, v_\lambda) - \frac{1}{\theta} J_\lambda'(u_\lambda, v_\lambda)(u_\lambda, v_\lambda)$$

$$= (\frac{1}{2} - \frac{1}{\theta})\|(u_\lambda, v_\lambda)\|_\lambda^2 + (\frac{1}{\theta} - \frac{1}{2^*})\lambda \int_{\mathbb{R}^N} K(x)(|u_\lambda|^{2^*} + |v_\lambda|^{2^*})$$
$$+ \lambda \int_{\mathbb{R}^N} (\frac{1}{\theta}(u_\lambda H_u(u_\lambda, v_\lambda) + v_\lambda H_v(u_\lambda, v_\lambda)) - H(u_\lambda, v_\lambda))$$
$$\geq (\frac{1}{2} - \frac{1}{\theta})\|(u_\lambda, v_\lambda)\|_\lambda^2.$$

So, the diamagnetic inequality implies that

$$\frac{\theta - 2}{2\theta} \int_{\mathbb{R}^N} (|\nabla|u_\lambda||^2 + |\nabla|v_\lambda||^2 + \lambda V(x)(|u_\lambda|^2 + |v_\lambda|^2)) \leq \sigma\lambda^{1-\frac{N}{2}}.$$

The proof is completed. □

Proof of Theorem 2. By Lemma 3.10, for any $m \in \mathbb{N}$ and $\sigma \in (0, \alpha_0)$, there exists $\Lambda_{m\sigma}$ such that for $\lambda \geq \Lambda_{m\sigma}$, we can choose a m-dimensional subspace F with $\max \Phi_\lambda(F) \leq \sigma\lambda^{1-\frac{N}{2}}$. By Lemma 3.8, there is $R > 0$ (depending on λ and m) such that $\Phi_\lambda(u) \leq 0$ for all $u \in F|B_R$.

Denote the set of all symmetric (in the sense that $-\Omega = \Omega$) and closed subsets of E_λ by Σ. For each $\Omega \in \Sigma$, let gen(Ω) be the Krasnoselski genus and set

$$i(A) := \min_{h \in \Gamma_m} (h(\Omega) \cap \partial B_{\rho_\lambda}),$$

where Γ_m is the set of all odd homeomorphisms $h \in C(E_\lambda, E_\lambda)$ and ρ_λ is the number of Lemma 3.7. Then i is a version of Benci's pseudoindex [5]. Let

$$c_{\lambda_j} = \inf_{i(\Omega) \geq j} \sup_{u \in \Omega} J_\lambda(u), \ 1 \leq j \leq m.$$

Since $J_\lambda(u) \geq \alpha_\lambda$ for all $u \in \partial B_{\rho_\lambda}$ (see Lemma 3.7) and $i(F) = \dim F = m$, we have

$$\alpha_\lambda \leq c_{\lambda_1} \leq c_{\lambda_2} \leq \cdots \leq c_{\lambda_m} \leq \sup_{(u,v) \in F^m_{\lambda\sigma}} J_\lambda(u, v) \leq \sigma\lambda^{1-\frac{N}{2}}.$$

In connection with Lemma 3.6, we know that J_λ satisfies the $(PS)_{c_{\lambda_j}}$ condition at all levels c_{λ_j}. By the critical point theory, all c_{λ_j} are critical levels and J_λ has at least m pairs of non-trivial critical points. Finally, as in the proof of Theorem 1, we easily get these solutions are the least energy solutions. □

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