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AN IDENTITY ON THE $2m$-TH POWER MEAN VALUE OF THE GENERALIZED GAUSS SUMS

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Abstract. In this paper, using analytic method and the properties of the Legendre's symbol, we prove an exact calculating formula on the $2m$-th power mean value of the generalized quadratic Gauss sums for $m \geq 2$. This solves a conjecture of He and Zhang [*On the $2k$-th power mean value of the generalized quadratic Gauss sums*, Bull. Korean Math. Soc. **48** (2011), no. 1, 9–15].

1. Introduction

Let $q \geq 2$ be an integer and χ be a Dirichlet character modulo q. For any integer n, the classical quadratic Gauss sums $G(n;q)$ and the generalized quadratic Gauss sums $G(n,\chi;q)$ are defined respectively by

$$G(n;q) = \sum_{a=1}^{q} e\left(\frac{na^2}{q}\right),$$

and

$$G(n,\chi;q) = \sum_{a=1}^{q} \chi(a) e\left(\frac{na^2}{q}\right),$$

where $e(x) = e^{2\pi i x}$.

The study of $G(n,\chi;q)$ is important in number theory, since it is a generalization of $G(n,q)$. In [4], Weil proved that if $p \geq 3$ is a prime, then

$$|G(n,\chi;p)| \leq 2\sqrt{p}.$$

In fact, Cochrane and Zheng [2] generalized this result to any integer. That is, for any integer n with $(n,q)=1$, we have

$$|G(n,\chi;q)| \leq 2^{\omega(q)}\sqrt{q},$$

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where $\omega(q)$ is the number of all distinct prime divisors of q.

Beside the upper bound of $G(n,\chi;q)$, the power mean value of $|G(n,\chi;q)|$ had also been studied by some authors. W. Zhang (see [5]) proved that if p is an odd prime and n is an integer with $(n,p)=1$, then

$$\sum_{\chi \mod p} |G(n,\chi;p)|^4 = \begin{cases} (p-1)[3p^2-6p-1+4\left(\frac{n}{p}\right)\sqrt{p}], & \text{if } p\equiv 1 \mod 4;\\ (p-1)(3p^2-6p-1), & \text{if } p\equiv 3 \mod 4 \end{cases}$$

and

$$\sum_{\chi \mod p} |G(n,\chi;p)|^6 = (p-1)(10p^3-25p^2-4p-1), \text{ if } p\equiv 3 \mod 4,$$

where $(\frac{n}{p})$ is the Legendre symbol. For $p\equiv 1 \mod 4$, it is still an open problem to calculate the exact value of $\sum_{\chi \mod p}|G(n,\chi;p)|^6$.

In 2005, W. Zhang and H. Liu [6] proved that if $q\geq 3$ is a square-full number, then for any integer n,k with $(nk,q)=1, k\geq 1$, we have

$$\sum_{\chi \mod q} |G(n,k,\chi;q)|^4 = q\cdot\phi^2(q)\prod_{p|q}(k,p-1)^2\cdot \prod_{\substack{p|q\\(k,p-1)=1}}\frac{\phi(p-1)}{p-1},$$

where $\phi(q)$ is the Euler function and $G(n,k,\chi;q)=\sum_{a=1}^{q}\chi(a)e\left(\frac{na^k}{q}\right)$.

Recently, Y. He and W. Zhang [3] proved the following result.

Let odd number $q>1$ be a square-full number. Then for any integer n with $(n,q)=1$ and $k=3$ or 4, we have the identity

$$\sum_{\chi \mod q} |G(n,\chi;q)|^{2k} = 4^{(k-1)\omega(q)}\cdot q^{k-1}\cdot\phi^2(q).$$

Besides, they conjectured the above identity also holds for $k\geq 5$.

In this paper, we prove this conjecture in the following theorem.

Theorem 1. *Let odd number $q>1$ be a square-full number, $m\geq 2$ be an integer. Then for any integer n with $(n,q)=1$, we have the identity*

$$\sum_{\chi \mod q} |G(n,\chi;q)|^{2m} = 4^{(m-1)\omega(q)}\cdot q^{m-1}\cdot\phi^2(q).$$

2. Proofs

Lemma 1 (See [1, Theorem 9.13]). *If p is an odd prime, then we have*

$$\left(\sum_{a=1}^{p-1}\left(\frac{a}{p}\right)e\left(\frac{a}{p}\right)\right)^2 = \left(\frac{-1}{p}\right)p.$$

Lemma 2. *For any odd prime p, we have*

$$G^2(1;p) = \left(\frac{-1}{p}\right)p\ .$$

Proof. This is a corollary of Lemma 1 in [3]. □

Let $p \geq 3$ be a prime, and let k, n, a be three integers with $1 \leq k \leq n$. Write

$$T_p(n,k,a) = \sum_{\substack{x_1=1 \\ x_1+x_2+\cdots+x_n\equiv a \mod p}}^{p-1} \sum_{x_2=1}^{p-1} \cdots \sum_{x_n=1}^{p-1} \left(\frac{x_1 x_2 \cdots x_k}{p}\right).$$

In order to prove Theorem 1, we need a lemma on the value of $T_p(n,k,a)$.

Lemma 3. *Let $p \geq 3$ be a prime, and let k, n, a be three integers with $1 \leq k \leq n$. Then we have*
(1)

$$T_p(n,k,a) = \begin{cases} (-1)^{n-k} \left(\frac{a}{p}\right) p^{(k-1)/2} \left(\frac{-1}{p}\right)^{(k-1)/2}, & \text{if } 2 \nmid k \text{ and } p \nmid a; \\ 0, & \text{if } 2 \nmid k \text{ and } p \mid a; \\ (-1)^{n+1-k} \left(\frac{-1}{p}\right)^{k/2} p^{(k-2)/2}, & \text{if } 2 \mid k \text{ and } p \nmid a; \\ (-1)^{n-k}(p-1) \left(\frac{-1}{p}\right)^{k/2} p^{(k-2)/2}, & \text{if } 2 \mid k \text{ and } p \mid a, \end{cases}$$

and

$$(2) \qquad \sum_{\substack{x_1=1 \\ x_1+x_2+\cdots+x_n\equiv a \mod p}}^{p-1} \sum_{x_2=1}^{p-1} \cdots \sum_{x_n=1}^{p-1} 1 = \begin{cases} \big((p-1)^n - (-1)^n\big)/p, & \text{if } p \nmid a; \\ \big((p-1)^n + (p-1)(-1)^n\big)/p, & \text{if } p \mid a. \end{cases}$$

Proof. For any $1 \leq k \leq n$, we have

$$T_p(n,k,a) = \sum_{\substack{x_1=1 \\ x_1+x_2+\cdots+x_n\equiv a \mod p}}^{p-1} \cdots\cdots \sum_{x_n=1}^{p-1} \left(\frac{x_1 x_2 \cdots x_k}{p}\right)$$

$$(3) \qquad \begin{aligned} &= \frac{1}{p} \sum_{x_1=1}^{p-1} \cdots\cdots \sum_{x_n=1}^{p-1} \left(\frac{x_1 x_2 \cdots x_k}{p}\right) \sum_{m=1}^{p} e\left(\frac{m(x_1+x_2+\cdots+x_n-a)}{p}\right) \\ &= \frac{1}{p} \sum_{m=1}^{p-1} e\left(\frac{-ma}{p}\right) \left(\sum_{x=1}^{p-1} \left(\frac{x}{p}\right) e\left(\frac{mx}{p}\right)\right)^k \left(\sum_{y=1}^{p-1} e\left(\frac{my}{p}\right)\right)^{n-k} \\ &= \frac{(-1)^{n-k}}{p} \sum_{m=1}^{p-1} \left(\frac{m}{p}\right)^k e\left(\frac{-ma}{p}\right) \left(\sum_{x=1}^{p-1} \left(\frac{x}{p}\right) e\left(\frac{x}{p}\right)\right)^k. \end{aligned}$$

Case 1. $p \mid a$, $2 \nmid k$. Then by (3) we have

$$T_p(n,k,a) = \frac{(-1)^{n-k}}{p} \left(\sum_{x=1}^{p-1} \left(\frac{x}{p}\right) e\left(\frac{x}{p}\right)\right)^k \left(\sum_{m=1}^{p-1} \left(\frac{m}{p}\right)\right) = 0.$$

Case 2. $p \mid a,\ 2 \mid k$. Then by Lemma 1 and (3), we have

$$\begin{aligned} T_p(n,k,a) &= \frac{(-1)^{n-k}}{p} \sum_{m=1}^{p-1} \left(\frac{-1}{p}\right)^{k/2} p^{k/2} \\ &= (-1)^{n-k}(p-1)\left(\frac{-1}{p}\right)^{k/2} p^{(k-2)/2}. \end{aligned}$$

Case 3. $p \nmid a,\ 2 \nmid k$. Then by Lemma 1 and (3), we have

$$\begin{aligned} &T_p(n,k,a) \\ &= \frac{(-1)^{n-k}}{p}\left(\frac{-a}{p}\right) \sum_{m=1}^{p-1}\left(\frac{m}{p}\right) e\left(\frac{m}{p}\right)\left(\sum_{x=1}^{p-1}\left(\frac{x}{p}\right) e\left(\frac{x}{p}\right)\right)^k \\ &= \frac{(-1)^{n-k}}{p}\left(\frac{-a}{p}\right)\left(\frac{-1}{p}\right)^{(k+1)/2} p^{(k+1)/2} \\ &= (-1)^{n-k}\left(\frac{a}{p}\right) p^{(k-1)/2}\left(\frac{-1}{p}\right)^{(k-1)/2}. \end{aligned}$$

Case 4. $p \nmid a,\ 2 \mid k$. Then by Lemma 1 and (3), we have

$$\begin{aligned} &T_p(n,k,a) \\ &= \frac{(-1)^{n-k}}{p}\left(\frac{-1}{p}\right)^{k/2} p^{k/2} \sum_{m=1}^{p-1} e\left(\frac{-ma}{p}\right) \\ &= (-1)^{n+1-k}\left(\frac{-1}{p}\right)^{k/2} p^{(k-2)/2}. \end{aligned}$$

Next, we shall prove the equality (2) in Lemma 3.

$$\begin{aligned} &\sum_{\substack{x_1=1 \\ x_1+x_2+\cdots+x_n\equiv a \bmod p}}^{p-1}\sum_{x_2=1}^{p-1}\cdots\sum_{x_n=1}^{p-1} 1 \\ &= \frac{1}{p}\sum_{x_1=1}^{p-1}\sum_{x_2=1}^{p-1}\cdots\sum_{x_n=1}^{p-1}\sum_{m=1}^{p} e\left(\frac{m(x_1+x_2+\cdots+x_n-a)}{p}\right) \\ &= \frac{1}{p}\sum_{m=1}^{p} e\left(\frac{-ma}{p}\right)\left(\sum_{x=1}^{p-1} e\left(\frac{mx}{p}\right)\right)^n \\ &= \frac{1}{p}\sum_{m=1}^{p-1} e\left(\frac{-ma}{p}\right)(-1)^n + \frac{(p-1)^n}{p}. \end{aligned} \tag{4}$$

If $p \mid a$, then the right side of (4) becomes $\big((p-1)^n + (p-1)(-1)^n\big)/p$; if $p \nmid a$, then the right side of (4) becomes $\big((p-1)^n - (-1)^n\big)/p$.

This completes the proof of Lemma 3. □

Lemma 4. *Let $p \geq 3$ be a prime, $\alpha \geq 2$, a and n be three integers with $1 \leq a \leq p^\alpha - 1$ and $(n,p) = 1$. If $p^{\alpha-1} \parallel a^2 - 1$, we write $a = rp^{\alpha-1} + \varepsilon$, where $1 \leq r \leq p-1$ and $\varepsilon = \pm 1$, then we have*

$$\sum_{b=1}^{p^\alpha}{}' e\left(\frac{nb^2(a^2-1)}{p^\alpha}\right) = \begin{cases} 0\ , & \text{if } p^{\alpha-1} \nmid a^2-1; \\ p^{\alpha-1}\left[\left(\frac{2\varepsilon rn}{p}\right)G(1;p) - 1\right]\ , & \text{if } p^{\alpha-1} \parallel a^2-1; \\ \phi(p^\alpha)\ , & \text{if } p^\alpha \mid a^2-1\ . \end{cases}$$

Proof. See the proof of Lemma 4 of [3]. □

Lemma 5 (See [6, Lemma 6])**.** *Let $m, n \geq 2$ and u be three integers with $(m,n) = 1$ and $(u, mn) = 1$. Then for any character $\chi = \chi_1\chi_2$ with χ_1* mod m *and χ_2* mod n*, we have the identity*

$$G(u,\chi;mn) = \chi_1(n)\chi_2(m)G(un,\chi_1;m)G(um,\chi_2;n).$$

Lemma 6. *Let $p \geq 3$ be a prime, $\alpha \geq 2, m \geq 2$ be two integers. Then for any integer n with $(n,p) = 1$, we have the identity*

$$\sum_{\chi \bmod p^\alpha} |G(n,\chi;p^\alpha)|^{2m} = 4^{(m-1)} \cdot \phi^2(p^\alpha) \cdot p^{(m-1)\alpha}.$$

Proof. By the definition of $G(n,\chi;p^\alpha)$, we have

$$\begin{aligned} |G(n,\chi;p^\alpha)|^2 &= \sum_{a=1}^{p^\alpha}{}'\sum_{b=1}^{p^\alpha}{}' \chi(a)\overline{\chi(b)}e\left(\frac{n(a^2-b^2)}{p^\alpha}\right) \\ &= \sum_{a=1}^{p^\alpha}{}' \chi(a)\sum_{b=1}^{p^\alpha}{}' e\left(\frac{nb^2(a^2-1)}{p^\alpha}\right)\ . \end{aligned}$$

Hence, by this formula we have

$$\begin{aligned} &\sum_{\chi \bmod p^\alpha} |G(n,\chi;p^\alpha)|^{2m} \\ =\ &\sum_{\chi \bmod p^\alpha} \sum_{x_1=1}^{p^\alpha}{}'\sum_{x_2=1}^{p^\alpha}{}' \cdots \sum_{x_m=1}^{p^\alpha}{}' \chi(x_1\cdots x_m)\prod_{i=1}^{m}\left(\sum_{y_i=1}^{p^\alpha}{}' e\left(\frac{ny_i^2(x_i^2-1)}{p^\alpha}\right)\right) \\ =\ &\phi(p^\alpha) \sum_{x_1=1}^{p^\alpha}{}'\sum_{x_2=1}^{p^\alpha}{}' \cdots \sum_{\substack{x_m=1 \\ x_1x_2\cdots x_m \equiv 1 \bmod p^\alpha}}^{p^\alpha}{}' \prod_{i=1}^{m}\left(\sum_{y_i=1}^{p^\alpha}{}' e\left(\frac{ny_i^2(x_i^2-1)}{p^\alpha}\right)\right). \end{aligned}$$

Then by Lemma 4 we have

$$\sum_{\chi \bmod p^\alpha} |G(n,\chi;p^\alpha)|^{2m} = \phi(p^\alpha)\sum_{k=0}^{m}\binom{m}{k}A(m,k), \tag{5}$$

where

$$A(m,k)=\sideset{}{'}\sum_{\substack{x_1=1\\ p^{\alpha-1}\|x_1^2-1}}^{p^\alpha}\cdots\sideset{}{'}\sum_{\substack{x_k=1\\ p^{\alpha-1}\|x_k^2-1\\ x_1x_2\cdots x_m\equiv 1}}^{p^\alpha}\sideset{}{'}\sum_{\substack{x_{k+1}=1\\ p^\alpha|x_{k+1}^2-1\\ \bmod p^\alpha}}^{p^\alpha}\cdots\sideset{}{'}\sum_{\substack{x_m=1\\ p^\alpha|x_m^2-1}}^{p^\alpha}\prod_{i=1}^{m}\left(\sideset{}{'}\sum_{y_i=1}^{p^\alpha}e\left(\frac{ny_i^2(x_i^2-1)}{p^\alpha}\right)\right).$$

Now, in order to prove Lemma 6, we need to calculate $A(m,k)$.

$$\begin{aligned}
&A(m,k)\\
&=\sideset{}{'}\sum_{\substack{x_1=1\\ p^{\alpha-1}\|x_1^2-1}}^{p^\alpha}\cdots\sideset{}{'}\sum_{\substack{x_k=1\\ p^{\alpha-1}\|x_k^2-1\\ x_1x_2\cdots x_m\equiv 1}}^{p^\alpha}\sideset{}{'}\sum_{\substack{x_{k+1}=1\\ p^\alpha|x_{k+1}^2-1\\ \bmod p^\alpha}}^{p^\alpha}\cdots\sideset{}{'}\sum_{\substack{x_m=1\\ p^\alpha|x_m^2-1}}^{p^\alpha}\prod_{i=1}^{m}\left(\sideset{}{'}\sum_{y_i=1}^{p^\alpha}e\left(\frac{ny_i^2(x_i^2-1)}{p^\alpha}\right)\right)\\
&=2\phi(p^\alpha)\sideset{}{'}\sum_{\substack{x_1=1\\ p^{\alpha-1}\|x_1^2-1}}^{p^\alpha}\cdots\sideset{}{'}\sum_{\substack{x_k=1\\ p^{\alpha-1}\|x_k^2-1\\ x_1x_2\cdots x_{m-1}\equiv 1}}^{p^\alpha}\sideset{}{'}\sum_{\substack{x_{k+1}=1\\ p^\alpha|x_{k+1}^2-1\\ \bmod p^\alpha}}^{p^\alpha}\cdots\sideset{}{'}\sum_{\substack{x_{m-1}=1\\ p^\alpha|x_{m-1}^2-1}}^{p^\alpha}\prod_{i=1}^{m-1}\left(\sideset{}{'}\sum_{y_i=1}^{p^\alpha}e\left(\frac{ny_i^2(x_i^2-1)}{p^\alpha}\right)\right)\\
&=2\phi(p^\alpha)A(m-1,k).
\end{aligned}$$

Hence, by induction on m, we have

$$A(m,k)=2^{m-k}\phi^{m-k}(p^\alpha)A(k,k). \tag{6}$$

Next, we shall calculate $A(k,k)$. By the definition, we have

$$A(k,k)=\sideset{}{'}\sum_{\substack{x_1=1\\ p^{\alpha-1}\|x_1^2-1\\ x_1x_2\cdots x_k\equiv 1}}^{p^\alpha}\cdots\sideset{}{'}\sum_{\substack{x_k=1\\ p^{\alpha-1}\|x_k^2-1\\ \bmod p^\alpha}}^{p^\alpha}\prod_{i=1}^{k}\left(\sideset{}{'}\sum_{y_i=1}^{p^\alpha}e\left(\frac{ny_i^2(x_i^2-1)}{p^\alpha}\right)\right).$$

Write $x_i=r_ip^{\alpha-1}+\varepsilon_i(1\le r_i\le p-1,\varepsilon_i=\pm1)$ for $i=1,2,\dots,k$. Then by Lemma 4, we have

$$\begin{aligned}
&A(k,k)\\
&=p^{k(\alpha-1)}\sum_{\substack{r_1=1\\ \varepsilon_1r_1+\varepsilon_2r_2+\cdots+\varepsilon_kr_k\equiv 0\mod p\\ \varepsilon_1\varepsilon_2\cdots\varepsilon_k=1}}^{p-1}\sum_{r_2=1}^{p-1}\cdots\sum_{r_k=1}^{p-1}\prod_{i=1}^{k}\left(\left(\frac{2n\varepsilon_ir_i}{p}\right)G(1;p)-1\right)\\
&=p^{k(\alpha-1)}\sum_{\substack{r_1=1\\ r_1+r_2+\cdots+r_k\equiv 0\mod p\\ \varepsilon_1\varepsilon_2\cdots\varepsilon_k=1}}^{p-1}\sum_{r_2=1}^{p-1}\cdots\sum_{r_k=1}^{p-1}\prod_{i=1}^{k}\left(\left(\frac{2nr_i}{p}\right)G(1;p)-1\right)\\
&=2^{k-1}p^{k(\alpha-1)}\sum_{\substack{r_1=1\\ r_1+r_2+\cdots+r_k\equiv 0\mod p}}^{p-1}\sum_{r_2=1}^{p-1}\cdots\sum_{r_k=1}^{p-1}\prod_{i=1}^{k}\left(\left(\frac{2nr_i}{p}\right)G(1;p)-1\right)
\end{aligned}$$

$$= 2^{k-1}p^{k(\alpha-1)} \cdot \sum_{\substack{r_1=1 \\ r_1+r_2+\cdots+r_k\equiv 0}}^{p-1}\sum_{r_2=1}^{p-1}\cdots\sum_{\substack{r_k=1 \\ \bmod p}}^{p-1}\left((-1)^k + \sum_{j=1}^{k}(-1)^{k-j}\binom{k}{j}\left(\frac{2n}{p}\right)^j G^j(1;p)\left(\frac{r_1r_2\cdots r_j}{p}\right)\right).$$

By Lemma 3, the above equality becomes

$$\begin{aligned} &A(k,k)\\ &= 2^{k-1}p^{k(\alpha-1)}(-1)^k\left(\frac{1}{p}\left((p-1)^k+(p-1)(-1)^k\right)\right.\\ &\quad\left.+\sum_{j=1}^{\lfloor k/2\rfloor}(-1)^{2j}\binom{k}{2j}\left(\frac{2n}{p}\right)^{2j}G^{2j}(1;p)(-1)^{k-2j}\left(\frac{-1}{p}\right)^j p^{j-1}(p-1)\right). \end{aligned}$$

By Lemma 2, we have

$$\begin{aligned} &A(k,k)\\ &= 2^{k-1}p^{k(\alpha-1)-1}\Big((-1)^k(p-1)^k+(p-1)+\sum_{j=1}^{\lfloor k/2\rfloor}\binom{k}{2j}p^{2j}(p-1)\Big)\\ &= 2^{k-1}p^{k(\alpha-1)-1}\Big((-1)^k(p-1)^k+(p-1)\big((p+1)^k+(1-p)^k\big)/2\Big)\\ &= 2^{k-2}p^{k(\alpha-1)-1}\Big((p+1)(1-p)^k+(p-1)(p+1)^k\Big). \end{aligned}$$

Hence, by (6) we have

$$A(m,k) = 2^{m-2}p^{m(\alpha-1)-1}\Big((-1)^k(p+1)(p-1)^m+(p-1)^{m-k+1}(p+1)^k\Big).$$

Finally, by (5) we have

$$\begin{aligned} &\sum_{\chi \bmod p^\alpha}|G(n,\chi;p^\alpha)|^{2m}\\ &= \phi(p^\alpha)\sum_{k=0}^{m}\binom{m}{k}2^{m-2}p^{m(\alpha-1)-1}\\ &\qquad\times\Big((-1)^k(p+1)(p-1)^m+(p-1)^{m-k+1}(p+1)^k\Big)\\ &= \phi(p^\alpha)2^{m-2}p^{m(\alpha-1)-1}(p+1)(p-1)^m\sum_{k=0}^{m}\binom{m}{k}(-1)^k\\ &\quad+\phi(p^\alpha)2^{m-2}p^{m(\alpha-1)-1}\sum_{k=0}^{m}\binom{m}{k}(p-1)^{m-k+1}(p+1)^k\\ &= 0+\phi(p^\alpha)2^{m-2}p^{m(\alpha-1)-1}(p-1)(2p)^m \end{aligned}$$

$$= 4^{m-1}\phi^2(p^\alpha)p^{\alpha(m-1)}.$$

This completes the proof of Lemma 6. □

Proof of Theorem 1. Since q is an odd square-full number, let

$$q = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_{\omega(q)}^{\alpha_{\omega(q)}},$$

we have $\alpha_i \geq 2, i = 1, \ldots, \omega(q)$. For any integer n with $(n, q) = 1$, by Lemma 5 and Lemma 6, we obtain

$$\begin{aligned}\sum_{\chi \mod q} |G(n, \chi; q)|^{2m} &= \prod_{\substack{i=1 \\ p_i^{\alpha_i} \| q}}^{\omega(q)} \sum_{\chi \mod p_i^{\alpha_i}} |G(nq/p_i^{\alpha_i}, \chi; p_i^{\alpha_i})|^{2m} \\ &= \prod_{\substack{i=1 \\ p_i^{\alpha_i} \| q}}^{\omega(q)} \left(4^{m-1} p_i^{\alpha_i(m-1)} \phi^2(p_i^{\alpha_i})\right) \\ &= 4^{(m-1)\omega(q)} \cdot q^{m-1} \cdot \phi^2(q)\ .\end{aligned}$$

This completes the proof of Theorem 1. □

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