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A HARDY INEQUALITY ON H-TYPE GROUPS

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Abstract. We prove a Hardy inequality related to Carnot-Carathéodory distance on H-type groups based on a representation formula on such groups.

1. Introduction

The Hardy inequality in $\mathbb{R}^N$ states that, for all $u \in C_0^\infty(\mathbb{R}^N)$ and $N \geq 3$,

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx \geq \frac{(N-2)^2}{4} \int_{\mathbb{R}^N} \frac{u^2}{|x|^2} dx. \tag{1.1}$$

Inequality (1.1) has been generalized to Heisenberg group $\mathbb{H}_n$ and other nilpotent groups by several authors ([5, 6, 7, 8, 10, 11, 21]). Especially, the following Hardy inequality holds for $1 < p < 2n$ and $u \in C_0^\infty(\mathbb{H}_n)$ (see [5], Theorem 3.3),

$$\int_{\mathbb{H}_n} |\nabla_H u|^p \geq \left(\frac{2n-p}{p}\right)^p \int_{\mathbb{H}_n} \frac{|u|^p}{d^p}, \tag{1.2}$$

where d is the Korányi-Folland non-isotropic gauge. We note there is another distance, called Carnot-Carathéodory distance on $\mathbb{H}_n$ and this distance plays an important role in the study of hypoelliptic heat kernel (see [1, 9, 12, 13, 15, 16, 17]). Since $d_{cc}(\xi) \geq d(\xi)$ for all $\xi \in \mathbb{H}_n$ (see [18], Lemma 5.1), inequality (1.2) imply the following

$$\int_{\mathbb{H}_n} |\nabla_H u|^p \geq \left(\frac{2n-p}{p}\right)^p \int_{\mathbb{H}_n} \frac{|u|^p}{d_{cc}^p}. \tag{1.3}$$

However, the constant in (1.3) is not sharp.

The aim of this paper is to give a new proof of Hardy inequalities associated with d_{cc} on H-type groups G, a remarkable class of stratified groups of step two introduced by Kaplan [14]. To do so, we prove a representation formula

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associated with d_{cc}, following the idea of Cohn et al. ([3, 4]) and Zhu ([22]). Furthermore, the constant we obtain seems sharp (see Remark 4.3).

To state our result, we need some notations (for details, see Section 2). We denote by ∇_G the horizontal gradient on G. Let $Q = m+2n$ be the homogenous dimension of G. To this end we have:

Theorem 1.1. *Let $1 < p < Q - \alpha$. There holds, for all $f \in C_0^\infty(G)$,*

$$\int_G \frac{|\nabla_G f|^p}{d_{cc}^\alpha} \geq \left(\frac{Q-p-\alpha}{p}\right)^p \int_G \frac{|f|^p}{d_{cc}^{p+\alpha}}. \tag{1.4}$$

2. Notations and preliminaries

Recall that a H-type group G is a Carnot group of step two in which the group law has the form (see [2])

$$(x,t)\circ(x',t') = \begin{pmatrix} x_i + x_i', \;\; i = 1,2,\ldots,m \\ t_j + t_j' + \frac{1}{2}\langle U^{(j)}x, x'\rangle, \;\; j = 1,2,\ldots,n \end{pmatrix}, \tag{2.1}$$

where the matrices $U^{(1)}, U^{(2)}, \ldots, U^{(n)}$ have the following properties:

(1) $U^{(j)}$ is a $m \times m$ skew symmetric and orthogonal matrix for every $j = 1, 2, \ldots, n$;

(2) $U^{(i)}U^{(j)} + U^{(j)}U^{(i)} = 0$ for every $i, j \in \{1, 2, \ldots, n\}$ with $i \neq j$.

An easy computation shows that the vector fields in the algebra $\mathfrak{g}$ of $G = (\mathbb{R}^{m+n}, \circ)$ that agree at the origin with $\frac{\partial}{\partial x_j}(j = 1, \ldots, m)$ are given by

$$X_j = \frac{\partial}{\partial x_j} + \frac{1}{2}\sum_{k=1}^n \left(\sum_{i=1}^m U_{j,i}^{(k)} x_i\right)\frac{\partial}{\partial t_k},$$

and that $\mathfrak{g}$ is spanned by the left-invariant vector fields $X_1, \ldots, X_m, \frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_n}$. The horizontal gradient is the vector given by

$$\nabla_G = (X_1, \ldots, X_m) = \nabla_x + \frac{1}{2}U^{(1)}x\frac{\partial}{\partial t_1} + \cdots + \frac{1}{2}U^{(n)}x\frac{\partial}{\partial t_n} \tag{2.2}$$

with

$$x = (x_1, \ldots, x_m) \;\text{ and }\; \nabla_x = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_m}\right).$$

We call a curve $\gamma : [a, b] \to G$ a horizontal curve connecting two points $\xi, \eta \in G$ if $\gamma(a) = \xi$, $\gamma(b) = \eta$ and $\dot{\gamma}(s) \in \text{span}\{X_1, \ldots, X_m\}$ for all s. The Carnot-Carathéodory distance between ξ, η is defined as

$$d_{cc}(\xi, \eta) = \inf_\gamma \int_a^b \sqrt{\langle\dot{\gamma}(s), \dot{\gamma}(s)\rangle}ds,$$

where the infimum is taken over all horizontal curves γ connecting ξ and η. It is known that any two points ξ, η on G can be joined by a horizontal curve of finite length and then d_{cc} is a metric on G. With this norm, we can define the metric ball centered at origin e and with radius ρ by

$$B_{cc}(e,\rho) = \{\eta : d_{cc}(e,\eta) < \rho\}$$

and the unit sphere $\Sigma = \partial B_{cc}(e,1)$. For simplicity, we write $d_{cc}(\xi) = d_{cc}(e,\xi)$.

For each real number $\lambda > 0$, there is a dilation naturally associated with the group structure which is usually denoted as $\delta_\lambda(x,t) = (\lambda x, \lambda^2 t)$. The Jacobian determinant of δ_λ is λ^Q, where $Q = m + 2n$ is the homogenous dimension of G. For simplicity, we use the notation $\lambda(x,t) = (\lambda x, \lambda^2 t)$. The Carnot-Carathéodory distance d_{cc} satisfies

$$d_{cc}(\lambda(x,t)) = \lambda d_{cc}(x,t), \quad \lambda > 0.$$

Given any $\xi = (x,t) \in G$, set $x^* = \frac{z}{d_{cc}(x,t)}$, $t^* = \frac{t}{d_{cc}(x,t)^2}$ and $\xi^* = (x^*, t^*)$ if $d_{cc}(x,t) \neq 0$. The polar coordinates on G associated with d_{cc} is the following (cf. [20]):

$$\int_G f(x,t)dxdt = \int_0^\infty \int_\Sigma f(\lambda(x^*,t^*))\lambda^{Q-1} d\sigma d\lambda, \quad f \in L^1(G).$$

Set

$$\mu(\varphi) = \frac{2\varphi - \sin 2\varphi}{2\sin^2\varphi} : (-\pi,\pi) \to \mathbb{R}.$$

Then μ is a diffeomorphism of the interval $(-\pi,\pi)$ onto $\mathbb{R}$ (cf. [1]). We denote by μ^{-1} the inverse function of μ. The Carnot-Carathéodory distance d_{cc} satisfies (cf. [9])

$$d_{cc}(x,t) = \begin{cases} \frac{\varphi}{\sin\varphi}|x|, & x \neq 0,\, t \neq 0, \\ |x|, & t = 0, \\ \sqrt{4\pi|t|}, & x = 0, \end{cases} \tag{2.3}$$

where

$$\varphi = \mu^{-1}\left(\frac{4|t|}{|x|^2}\right), \quad |x|^2 = \sum_{j=1}^m x_j^2 \quad \text{and} \quad |t|^2 = \sum_{j=1}^n t_j^2.$$

Let $\mathbb{Z} = \{(x,t) \in G : x = 0\}$ be the center of G and $\mathbb{Z}' = \{(x,t) \in G : t = 0\}$. Since the function $\frac{4|t|}{|x|^2}$ is of class C^∞ in $G \setminus (\mathbb{Z} \bigcup \mathbb{Z}')$, the Carnot-Carathéodory distance $d_{cc}(x,t)$ is of class C^∞ in $G \setminus (\mathbb{Z} \bigcup \mathbb{Z}')$. Therefore (cf. [20])

$$|\nabla_G d_{cc}(x,t)| = 1, \quad (x,t) \in G \setminus (\mathbb{Z} \bigcup \mathbb{Z}').$$

3. Geodesics in a H-type group

In this section we shall give a parametrization of G using the geodesics. We refer to [19] the analogous parametrization of Heisenberg groups. Recall that the Kokn's sublaplacian on the H-type group G is given by

$$\begin{aligned}\Delta_G = \sum_{j=1}^{m} X_j^2 &= \sum_{j=1}^{m}\left(\frac{\partial}{\partial x_j} + \frac{1}{2}\sum_{k=1}^{n}\left(\sum_{i=1}^{m} U_{j,i}^{(k)} x_i\right)\frac{\partial}{\partial t_k}\right)^2 \\ &= \Delta_x + \frac{1}{4}|x|^2\Delta_t + \sum_{k=1}^{n}\langle U^{(k)}x, \nabla_x\rangle\frac{\partial}{\partial t_k},\end{aligned}$$

where

$$\Delta_x = \sum_{j=1}^{m}\left(\frac{\partial}{\partial x_j}\right)^2, \quad \Delta_t = \sum_{k=1}^{n}\left(\frac{\partial}{\partial t_k}\right)^2.$$

The associated Hamiltonian function $H(x,t,\xi,\theta)$ is of the form

$$\begin{aligned}H(x,t,\xi,\theta) &= \sum_{j=1}^{m}\left(\xi_j + \frac{1}{2}\sum_{k=1}^{n}\left(\sum_{i=1}^{m} U_{j,i}^{(k)} x_i\right)\theta_k\right)^2 \\ &= |\xi|^2 + \frac{1}{4}|x|^2|\theta|^2 + \sum_{k=1}^{n}\langle \theta_k U^{(k)}x, \xi\rangle,\end{aligned}$$

where

$$\xi = (\xi_1,\ldots,\xi_m) = \left(\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_m}\right), \quad \theta = (\theta_1,\ldots,\theta_n) = \left(\frac{\partial}{\partial t_1},\ldots,\frac{\partial}{\partial t_n}\right).$$

It has been proved by Eldredge ([9]) that geodesics in G are solutions of the Hamiltonian system associated with Hamiltonian function $H(x,t,\xi,\theta)$. For simplicity, we introduce the notation $M_\theta = \sum_{k=1}^{n}\theta_k U^{(k)}$. Properties of the matrices $\{U^{(1)}, U^{(2)}, \ldots, U^{(n)}\}$ give us the following results.

Lemma 3.1. *Let I be the identity matric. Put $M_\theta = \sum_{k=1}^{n}\theta_k U^{(k)}$ and suppose $|\theta| \neq 0$:*

(1) $M_\theta^T = -M_\theta, \quad M_\theta^2 = -|\theta|^2 \cdot I, \quad M_\theta^{-1} = -|\theta|^{-2}M_\theta$;

(2) $\exp(sM_\theta) = \cos(s|\theta|)\cdot I + \sin(s|\theta|)\cdot \frac{M_\theta}{|\theta|}$.

Proof. (1) By Theorem 2.1,

$$\begin{aligned}M_\theta^T &= \left(\sum_{k=1}^{n}\theta_k U^{(k)}\right)^T = -\sum_{k=1}^{n}\theta_k U^{(k)} = -M_\theta; \\ M_\theta^2 &= \left(\sum_{k=1}^{n}\theta_k U^{(k)}\right)^2 = \sum_{k=1}^{n}\theta_k^2 (U^{(k)})^2 + \sum_{i<j}\theta_i\theta_j(U^{(i)}U^{(j)} + U^{(j)}U^{(i)}) \\ &= -\sum_{k=1}^{n}\theta_k^2 \cdot I = -|\theta|^2 \cdot I.\end{aligned}$$

So $M_\theta^{-1} = -|\theta|^{-2}M_\theta$.

(2) Since $M_\theta^2 = -|\theta|^2 \cdot I$, we have

$$\begin{aligned}\exp(sM_\theta) &= \sum_{n=0}^{\infty} \frac{(sM_\theta)^n}{n!} = \sum_{n=0}^{\infty} \frac{(sM_\theta)^{2n+1}}{(2n+1)!} + \sum_{k=0}^{\infty} \frac{(sM_\theta)^{2k}}{(2k)!} \\ &= \frac{M_\theta}{|\theta|} \sum_{n=0}^{\infty} (-1)^n \frac{(s|\theta|)^{2n+1}}{(2n+1)!} + \sum_{k=0}^{\infty} (-1)^k \frac{(s|\theta|)^{2k}}{(2k)!} \cdot I \\ &= \cos(s|\theta|) \cdot I + \sin(s|\theta|) \cdot \frac{M_\theta}{|\theta|}.\end{aligned}$$

□

Hamiltonian function $H(x,t,\xi,\theta)$ may be denoted by

$$\begin{aligned}H(x,t,\xi,\theta) &= |\xi|^2 + \frac{1}{4}|x|^2|\theta|^2 + \langle M_\theta x, \xi\rangle \\ &= \langle \xi + \frac{1}{2}M_\theta x, \xi + \frac{1}{2}M_\theta x\rangle = \langle \zeta, \zeta\rangle,\end{aligned}$$

where $\zeta = \xi + \frac{1}{2}M_\theta x$. The corresponding Hamilton's equations for a curve $(x(s), t(s), \xi(s), \theta(s))$ take the form

$$\begin{cases} \dot{x}(s) = \dfrac{\partial H}{\partial \xi} = 2\zeta(s), \\ \dot{\xi}(s) = -\dfrac{\partial H}{\partial x} = M_\theta \zeta(s), \\ \dot{t}_j(s) = \dfrac{\partial H}{\partial \theta_k} = \langle \zeta(s), U^{(j)}x(s)\rangle, \quad j = 1, \dots, n, \\ \dot{\theta}(s) = -\dfrac{\partial H}{\partial t} = 0, \quad i.e., \quad \theta(s) = \theta(0), \end{cases} \tag{3.1}$$

Assume $(x(0), t(0)) = (0,0)$. We solve the Hamiltonian system explicitly bellow. By (3.1), θ is constant, and we take it to be the free parameter. The equations (3.1) imply that:

$$\dot{\zeta}(s) = \dot{\xi}(s) + \frac{1}{2}M_\theta \dot{x}(s) = 2M_\theta \zeta(s).$$

Therefore,

$$\zeta(s) = e^{2sM_\theta}\zeta(0)$$

and by Lemma 3.1,

(3.2)

$$\begin{aligned}x(s) &= 2\int_0^s \zeta(r)dr = M_\theta^{-1}(\zeta(s) - \zeta(0)) = M_\theta^{-1}(I - e^{-2sM_\theta})\zeta(s) \\ &= M_\theta^{-1}(e^{2sM_\theta} - I)\zeta(0) = (\cos(2s|\theta|) - 1) \cdot M_\theta^{-1}\zeta(0) + \sin(2s|\theta|) \cdot \frac{\zeta(0)}{|\theta|} \\ &= (1 - \cos(2s|\theta|)) \cdot \frac{M_\theta \zeta(0)}{|\theta|^2} + \sin(2s|\theta|) \cdot \frac{\zeta(0)}{|\theta|}.\end{aligned}$$

We now describe the t-components of a geodesic curve. By (3.1) and (3.2)

$$\begin{aligned}\dot{t}_k(s) &= \langle \zeta(s), U^{(k)} x(s)\rangle \\ &= \langle \frac{1}{2}\dot{x}(s), U^{(k)} x(s)\rangle \\ &= \langle \frac{1}{2}\dot{x}(s), \frac{1}{2} U^{(k)} M_\theta^{-1}(I - e^{-2sM_\theta})\zeta(s)\rangle \\ &= \langle \frac{1}{2}\dot{x}(s), \frac{1}{2} U^{(k)} M_\theta^{-1}(I - e^{-2sM_\theta})\dot{x}(s)\rangle.\end{aligned} \tag{3.3}$$

A simply calculation gives us

$$\begin{aligned}U^{(k)} M_\theta^{-1}(e^{2sM_\theta} - I) &= U^{(k)} M_\theta^{-1} - U^{(k)} M_\theta^{-1} e^{-2sM_\theta} \\ &= U^{(k)} M_\theta^{-1} - U^{(k)} M_\theta^{-1}(\cos 2s|\theta| - \frac{M_\theta}{|\theta|}\sin 2s|\theta|) \\ &= (1 - \cos 2s|\theta|) U^{(k)} M_\theta^{-1} - \frac{\sin 2s|\theta|}{|\theta|} U^{(k)}.\end{aligned} \tag{3.4}$$

By Lemma 3.1,

$$M^{-1} = -|\theta|^{-2} M_\theta = -|\theta|^{-2} \sum_{i=1}^{n} \theta_i U^{(i)}$$

and we obtain, by (3.4),

$$\begin{aligned}&U^{(k)} M_\theta^{-1}(e^{2sM_\theta} - I) \\ &= (1 - \cos 2s|\theta|) U^{(k)} M_\theta^{-1} - \frac{\sin 2s|\theta|}{|\theta|} U^{(k)} \\ &= -\frac{(1 - \cos 2s|\theta|)}{|\theta|^2} U^{(k)} \sum_{i=1}^{n} \theta_i U^{(i)} - \frac{\sin 2s|\theta|}{|\theta|} U^{(k)} \\ &= \frac{1 - \cos 2s|\theta|}{|\theta|^2}\theta_k \cdot I - \frac{1 - \cos 2s|\theta|}{|\theta|^2} \sum_{i \neq k} \theta_i U^{(k)} U^{(i)} - \frac{\sin 2s|\theta|}{|\theta|} U^{(k)}.\end{aligned} \tag{3.5}$$

Properties of the matrices $\{U^{(1)}, U^{(2)}, \ldots, U^{(n)}\}$ imply $U^{(k)}U^{(i)}$ $(i \neq k)$ is also a $m \times m$ skew symmetric and orthogonal matrix. Therefore we get, for $i \neq k$,

$$\langle U^{(k)} U^{(i)} \dot{x}(s), \dot{x}(s)\rangle = 0$$

and also

$$\langle U^{(k)} \dot{x}(s), \dot{x}(s)\rangle = 0, \quad k = 1, \ldots, n.$$

Thus, by (3.3) and (3.5),

$$\begin{aligned}\dot{t}_k(s) &= \frac{1-\cos 2s|\theta|}{4|\theta|^2}\theta_k\langle \dot{x}(s), \dot{x}(s)\rangle \\ &= \frac{1-\cos 2s|\theta|}{4|\theta|^2}\theta_k\langle e^{2sM_\theta}\zeta(0), e^{2sM_\theta}\zeta(0)\rangle \\ &= \frac{1-\cos 2s|\theta|}{4|\theta|^2}\theta_k\langle \zeta(0), \zeta(0)\rangle.\end{aligned} \tag{3.6}$$

Integrating equations (3.6) we obtain, for $k = 1, \dots, n$,

$$t_k(s) = \frac{\theta_k}{4|\theta|^2}\left(s - \frac{\sin 2s|\theta|}{2|\theta|}\right)\langle \zeta(0), \zeta(0)\rangle.$$

Taking the initial date $(x(0), t(0)) = (0, 0)$ and $(\zeta(0), \theta(0)) = (A, \phi/2) = (A_1, \dots, A_m, \phi_1/2, \dots, \phi_n/2)$, we find the solutions

$$\begin{cases} x(s) = (1-\cos(s|\phi|)) \cdot \dfrac{M_\phi A}{|\phi|^2} + \sin(s|\phi|) \cdot \dfrac{A}{|\phi|}, \\ t(s) = \dfrac{\phi}{2|\phi|} \cdot \dfrac{|\phi|s - \sin s|\phi|}{|\phi|^2}\langle A, A\rangle, \end{cases}$$

where $M_\phi = \sum_{k=1}^n \phi_k U^{(k)}$. Letting $|\phi| \to 0+$, we get the Euclidean geodesics

$$(x(s), t(s)) = (\zeta(0)s, 0)$$

and hence the correct normalization is $\langle \zeta(0), \zeta(0)\rangle = \sum_{i=1}^m A_i^2 = 1$.

Set

$$\Omega = \{(A, \phi, \rho) \in \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R} :\ |A|^2 = \sum_{j=1}^m A_j^2 = 1,\ 0 \le |\phi|\rho \le 2\pi,\ \rho \ge 0\}.$$

and define $\Phi : \Omega \to G$ by $\Phi(A, \phi, \rho) = (x(A, \phi, \rho), t(A, \phi, \rho))$, where

$$\begin{cases} x(A, \phi, \rho) = (1-\cos(|\phi|\rho)) \cdot \dfrac{M_\phi A}{|\phi|^2} + \sin(|\phi|\rho) \cdot \dfrac{A}{|\phi|}, \\ t(A, \phi, \rho) = \dfrac{\phi}{2|\phi|} \cdot \dfrac{|\phi|\rho - \sin(|\phi|\rho)}{|\phi|^2} \end{cases} \tag{3.7}$$

and $M_\phi = \sum_{k=1}^n \phi_k U^{(k)}$. We note the range of Φ is G and the center $\mathbb{Z}$ is just the set of points $\Phi(A, \phi, \rho)$ with $|\phi|\rho = 2\pi$ and the set $\mathbb{Z}'$ is just the set of points $\Phi(A, \phi, \rho)$ with $|\phi| = 0$. Furthermore, if one fixes $\rho > 0$, equations (3.7) parameterize $\partial B_{cc}(e, \rho)$.

By (3.7),

$$|x|^2 = (1-\cos(|\phi|\rho))^2 \cdot \frac{|\phi|^2}{|\phi|^4} + \sin^2(|\phi|\rho) \cdot \frac{1}{|\phi|^2} = \frac{2(1-\cos(|\phi|\rho))}{|\phi|^2}$$

and

$$\mu(\varphi) = \frac{4|t|}{|x|^2} = \frac{|\phi|\rho - \sin(|\phi|\rho)}{1-\cos(|\phi|\rho)} = \frac{|\phi|\rho - \sin(|\phi|\rho)}{2\sin^2(\frac{|\phi|\rho}{2})} = \mu\left(\frac{|\phi|\rho}{2}\right).$$

Therefore,

$$\varphi = \frac{|\phi|\rho}{2}, \quad |\phi|\rho \in [0, 2\pi) \tag{3.8}$$

since μ is a diffeomorphism of the interval $(-\pi, \pi)$ onto $\mathbb{R}$.

4. The proof

For any $\delta > 0$, set $\mathbb{Z}_\delta = (-\delta, \delta)^m \times \mathbb{R}^n$, $\mathbb{Z}'_\delta = \mathbb{R}^m \times (-\delta, \delta)^n$ and $\Sigma_\delta = \Sigma \setminus (\mathbb{Z}_\delta \cup \mathbb{Z}'_\delta)$. Note that $\mathbb{Z}_\delta$ and $\mathbb{Z}'_\delta$ are open sets that contain $\mathbb{Z}$ and $\mathbb{Z}'$, respectively. Now we prove the following representation formula. The proof is similar to that of [3, 4] and [22].

Lemma 4.1. *Let $R_2 > R_1 > 0$ and $f \in C(G) \cap C^1(G \setminus (\mathbb{Z} \bigcup \mathbb{Z}'))$. There holds*

$$\int_{\Sigma_\delta} f(R_2\xi^*)d\sigma - \int_{\Sigma_\delta} f(R_1\xi^*)d\sigma = \int_{R_1}^{R_2} \int_{\Sigma_\delta} \langle \nabla_G f, \nabla_G d_{cc} \rangle d\sigma d\rho. \tag{4.1}$$

Proof. We have, for $\xi^* \in \Sigma_\delta$,

$$\begin{aligned} f(R_2\xi^*)d - f(R_1\xi^*) &= \int_{R_1}^{R_2} \frac{d}{d\rho} f(\rho\xi^*)d\rho \\ &= \int_{R_1}^{R_2} \left(\left\langle \nabla_x f(\xi), \frac{\partial x}{\partial \rho} \right\rangle + \left\langle \nabla_t f(\xi), \frac{\partial t}{\partial \rho} \right\rangle \right) d\rho. \end{aligned}$$

Therefore,

$$\begin{aligned} &\int_{\Sigma_\delta} f(R_2\xi^*)d\sigma - \int_{\Sigma_\delta} f(R_1\xi^*)d\sigma \\ &= \int_{\Sigma_\delta} \int_{R_1}^{R_2} \frac{d}{d\rho} f(\rho\xi^*)d\rho d\sigma \\ &= \int_{\Sigma_\delta} \int_{R_1}^{R_2} \left(\left\langle \nabla_x f(\xi), \frac{\partial x}{\partial \rho} \right\rangle + \left\langle \nabla_t f(\xi), \frac{\partial t}{\partial \rho} \right\rangle \right) d\rho d\sigma, \end{aligned}$$

where $\xi = (x, t) = \rho\xi^*$. By (3.7) and (2.2),

(4.2)

$$\begin{aligned} &\left\langle \nabla_x f(\xi), \frac{\partial x}{\partial \rho} \right\rangle + \left\langle \nabla_t f(\xi), \frac{\partial t}{\partial \rho} \right\rangle \\ &= \left\langle \nabla_x f(\xi),\ \sin(|\phi|\rho) \cdot \frac{M_\phi A}{|\phi|} + \cos(|\phi|\rho)A \right\rangle + \left\langle \nabla_t f(\xi), \frac{\phi}{2|\phi|} \cdot \frac{1 - \cos(|\phi|\rho)}{|\phi|} \right\rangle \\ &= \left\langle \nabla_G f(\xi),\ \sin(|\phi|\rho) \cdot \frac{M_\phi A}{|\phi|} + \cos(|\phi|\rho)A \right\rangle \\ &\quad + \frac{1}{2} \sum_{k=1}^{n} \left(\frac{\phi_k}{|\phi|} \cdot \frac{1 - \cos(|\phi|\rho)}{|\phi|} - \left\langle U^{(k)}x, \sin(|\phi|\rho) \cdot \frac{M_\phi A}{|\phi|} + \cos(|\phi|\rho)A \right\rangle \right) \frac{\partial f}{\partial t_k}. \end{aligned}$$

Again using (3.7), we have, since $U^{(k)}$ and $U^{(k)}U^{(j)}(j \neq k)$ are $m \times m$ skew symmetric matrixes,

(4.3)

$$
\begin{aligned}
&\left\langle U^{(k)}x, \sin(|\phi|\rho)\cdot\frac{M_\phi A}{|\phi|} + \cos(|\phi|\rho)A\right\rangle \\
&= \left\langle (1-\cos(|\phi|\rho))\frac{U^{(k)}M_\phi A}{|\phi|^2} + \sin(|\phi|\rho)\frac{U^{(k)}A}{|\phi|}, \sin(|\phi|\rho)\frac{M_\phi A}{|\phi|} + \cos(|\phi|\rho)A\right\rangle \\
&= \frac{(1-\cos(|\phi|\rho))\sin(|\phi|\rho)}{|\phi|^3}\langle U^{(k)}M_\phi A, M_\phi A\rangle + \frac{(\cos(|\phi|\rho)\sin(|\phi|\rho)}{|\phi|}\langle U^{(k)}A, A\rangle \\
&\quad + \left(\frac{(1-\cos(|\phi|\rho))\cos(|\phi|\rho)}{|\phi|^2} - \frac{(\sin^2(|\phi|\rho)}{|\phi|^2}\right)\langle U^{(k)}M_\phi A, A\rangle \\
&= \frac{\cos(|\phi|\rho)-1}{|\phi|^2}\langle U^{(k)}M_\phi A, A\rangle \\
&= \frac{1-\cos(|\phi|\rho)}{|\phi|^2}\phi_k + \frac{\cos(|\phi|\rho)-1}{|\phi|^2}\sum_{j\neq k}\langle U^{(k)}U^{(j)}A, A\rangle\phi_j \\
&= \frac{1-\cos(|\phi|\rho)}{|\phi|^2}\phi_k.
\end{aligned}
$$

Therefore, we have, by (4.2) and (4.3),

$$
\begin{aligned}
&\int_{\Sigma_\delta} f(R_2\xi^*)d\sigma - \int_{\Sigma_\delta} f(R_1\xi^*)d\sigma \\
&= \int_{\Sigma_\delta}\int_{R_1}^{R_2}\left\langle \nabla_G f(\xi),\ \sin(|\phi|\rho)\cdot\frac{M_\phi A}{|\phi|} + \cos(|\phi|\rho)A\right\rangle d\rho d\sigma.
\end{aligned}
$$

To finish the proof, it is enough to show

$$\nabla_G d_{cc}(x,t) = \sin(|\phi|\rho)\frac{M_\phi A}{|\phi|} + \cos(|\phi|\rho)A, \quad (x,t) \in G \setminus (\mathbb{Z}\bigcup\mathbb{Z}').$$

This is just the following Lemma 4.2. The proof of Lemma 4.1 is now completed. □

Lemma 4.2. *There holds, for $(x,t) \in G \setminus (\mathbb{Z}\bigcup\mathbb{Z}')$,*

$$\nabla_G d_{cc}(x,t) = \sin(|\phi|\rho)\frac{M_\phi A}{|\phi|} + \cos(|\phi|\rho)A.$$

Proof. Recall that if $(x,t) \in G \setminus (\mathbb{Z}\bigcup\mathbb{Z}')$, then

$$d_{cc}(x,t) = \frac{\varphi}{\sin\varphi}|x|, \quad \varphi = \mu^{-1}\left(\frac{4|t|}{|x|^2}\right).$$

Though a simple calculation, we have

$$\mu'(\varphi) = \frac{2\sin\varphi - 2\varphi\cos\varphi}{\sin^3\varphi};$$

$$\begin{aligned}\nabla_x\varphi &= \frac{1}{\mu'(\varphi)}\cdot\frac{-8|t|x}{|x|^4}\\ &= -\frac{4|t|x}{|x|^4}\cdot\frac{\sin^3\varphi}{\sin\varphi-\varphi\cos\varphi};\\ \nabla_t\varphi &= \frac{1}{\mu'(\varphi)}\cdot\frac{-4t}{|x|^2|t|}\\ &= -\frac{2t}{|x|^2|t|}\cdot\frac{\sin^3\varphi}{\sin\varphi-\varphi\cos\varphi};\end{aligned}$$

$$\begin{aligned}\nabla_x d_{cc}(x,t) &= \nabla_x\left(\frac{\varphi}{\sin\varphi}|x|\right)\\ &= \frac{\varphi\cos\varphi-\sin\varphi}{\sin^2\varphi}|x|\nabla_x\varphi+\frac{\varphi}{\sin\varphi}\cdot\frac{x}{|x|}\\ &= -\frac{4\sin\varphi|t|x}{|x|^3}+\frac{\varphi}{\sin\varphi}\cdot\frac{x}{|x|}\\ &= -\mu(\varphi)\sin\varphi\cdot\frac{x}{|x|}+\frac{\varphi}{\sin}\varphi\cdot\frac{x}{|x|}\\ &= \cos\varphi\cdot\frac{x}{|x|};\end{aligned}$$

$$\nabla_t d_{cc}(x,t)=\nabla_t\left(\frac{\varphi}{\sin\varphi}|x|\right)=\frac{\varphi\cos\varphi-\sin\varphi}{\sin^2\varphi}|x|\nabla_t\varphi=\frac{2\sin\varphi}{|x|}\cdot\frac{t}{|t|}.$$

Therefore, we obtain, by (2.2) and (3.7),

$$\begin{aligned}\nabla_G d_{cc}(x,t) &= \nabla_x d_{cc}(x,t)+\frac{1}{2}\sum_{k=1}^{n}U^{(k)}x\frac{\partial d_{cc}(x,t)}{\partial t_k}\\ &= \cos\varphi\cdot\frac{x}{|x|}+\frac{\sin\varphi}{|x|}\cdot\left(\sum_{k=1}^{n}\frac{t_k}{|t|}\cdot U^{(k)}\right)x\\ &= \cos\varphi\cdot\frac{x}{|x|}+\frac{\sin\varphi}{|x|}\cdot\left(\sum_{k=1}^{n}\frac{\phi_k}{|\phi|}\cdot U^{(k)}\right)x\\ &= \cos\varphi\cdot\frac{x}{|x|}+\sin\varphi\cdot\frac{M_\phi x}{|x||\phi|}.\end{aligned}$$

On the other hand, by (3.7),

$$|x|=\sqrt{(1-\cos(|\phi|\rho))^2\cdot\frac{|\phi|^2}{|\phi|^4}+\frac{\sin^2(|\phi|\rho)}{|\phi|^2}}=\sqrt{\frac{2(1-\cos(|\phi|\rho))}{|\phi|^2}}=\frac{2\sin(\frac{|\phi|\rho}{2})}{|\phi|}$$

and

$$\frac{x}{|x|}=\frac{(1-\cos(|\phi|\rho))M_\phi A}{|\phi|\cdot 2\sin(\frac{|\phi|\rho}{2})}+\frac{\sin(|\phi|\rho)}{2\sin(\frac{|\phi|\rho}{2})}\cdot A=\frac{\sin(\frac{|\phi|\rho}{2})}{|\phi|}M_\phi A+\cos\left(\frac{|\phi|\rho}{2}\right)A.$$

Thus,

$$
\begin{aligned}
&\nabla_G d_{cc}(x,t)\\
=\;& \cos\varphi\cdot\frac{x}{|x|}+\sin\varphi\cdot\frac{M_\phi x}{|x||\phi|}\\
=\;& \cos\varphi\cdot\left(\frac{\sin(\frac{|\phi|\rho}{2})}{|\phi|}M_\phi A+\cos\left(\frac{|\phi|\rho}{2}\right)A\right)\\
&+\frac{\sin\varphi}{|\varphi|}\cdot\left(\frac{\sin(\frac{|\phi|\rho}{2})}{|\phi|}M_\phi^2 A+\cos\left(\frac{|\phi|\rho}{2}\right)M_\phi A\right)\\
=\;& \cos\varphi\cdot\left(\frac{\sin(\frac{|\phi|\rho}{2})}{|\phi|}M_\phi A+\cos\left(\frac{|\phi|\rho}{2}\right)A\right)\\
&+\frac{\sin\varphi}{|\varphi|}\cdot\left(-\frac{\sin(\frac{|\phi|\rho}{2})}{|\phi|}|\phi|^2 A+\cos\left(\frac{|\phi|\rho}{2}\right)M_\phi A\right)\ \text{(by Lemma 3.1)}\\
=\;& \cos\left(\varphi+\frac{|\phi|\rho}{2}\right)A+\frac{\sin(\varphi+\frac{|\phi|\rho}{2})}{|\phi|}M_\phi A\\
=\;& \cos(|\phi|\rho)A+\frac{\sin(|\phi|\rho)}{|\phi|}M_\phi A.\quad \text{(by (3.8))}
\end{aligned}
$$

This completes the proof of Lemma 4.2. □

Proof of Theorem 1.1. Let $\epsilon > 0$ and set $f_\epsilon := (|f|^2+\epsilon^2)^{p/2}-\epsilon^p$. f_ϵ has the same support as f and hence $f_\epsilon \in C_0^\infty(G)$. Since $d_{cc}(x,t) \in C(G)\cap C^1(G\setminus(\mathbb{Z}\bigcup\mathbb{Z}'))$, we can replace f with $f_\epsilon d_{cc}^{Q-p-\alpha}$ in Lemma 4.1 and obtain, for $R_2 > R_1 > 0$ and any $\delta > 0$,

$$
\begin{aligned}
&(Q-p-\alpha)\int_{R_1}^{R_2}\int_{\Sigma_\delta} f_\epsilon d_{cc}^{Q-p-\alpha-1}d\sigma d\rho\\
&-R_2^{Q-p-\alpha}\int_{\Sigma_\delta} f_\epsilon(R_2u^*)d\sigma+R_1^{Q-p-\alpha}\int_{\Sigma_\delta} f_\epsilon(R_1u^*)d\sigma\\
=\;& -p\int_{R_1}^{R_2}\int_{\Sigma_\delta}(|f|^2+\epsilon^2)^{(p-2)/2}f\langle\nabla_G f,\nabla_G d_{cc}\rangle d_{cc}^{Q-p-\alpha}d\sigma d\rho\\
\le\;& p\int_{R_1}^{R_2}\int_{\Sigma_\delta}(|f|^2+\epsilon^2)^{(p-2)/2}|f|\cdot|\nabla_G f|d_{cc}^{Q-p-\alpha}d\sigma d\rho\\
\le\;& p\int_{R_1}^{R_2}\int_{\Sigma_\delta}(|f|^2+\epsilon^2)^{(p-1)/2}|\nabla_G f|d_{cc}^{Q-p-\alpha}d\sigma d\rho.
\end{aligned}
$$

Letting $\delta \to 0$ yields

$$
(Q-p-\alpha)\int_{R_1}^{R_2}\int_{\Sigma} f_\epsilon d_{cc}^{Q-p-\alpha-1}d\sigma d\rho
$$

$$- R_2^{Q-p-\alpha}\int_\Sigma f_\epsilon(R_2u^*)d\sigma + R_1^{Q-p-\alpha}\int_\Sigma f_\epsilon(R_1u^*)d\sigma$$
$$\leq p\int_{R_1}^{R_2}\int_\Sigma(|f|^2+\epsilon^2)^{(p-1)/2}|\nabla_G f|d_{cc}^{Q-p-\alpha}d\sigma d\rho.$$

Letting $R_2 \to +\infty$ and $R_1 \to 0+$, we obtain,

$$\begin{aligned}&(Q-p-\alpha)\int_0^\infty\int_\Sigma f_\epsilon d_{cc}^{Q-p-\alpha-1}d\sigma d\rho\\ &\leq p\int_0^\infty\int_\Sigma(|f|^2+\epsilon^2)^{(p-1)/2}|\nabla_G f|d_{cc}^{Q-p-\alpha}d\sigma d\rho\\ &= p\int_G\frac{(|f|^2+\epsilon^2)^{(p-1)/2}\cdot|\nabla_G f|}{d_{cc}^{p+\alpha-1}}.\end{aligned}$$

By dominated convergence, letting $\epsilon \to 0+$ yields

$$(Q-p-\alpha)\int_G\frac{|f|^p}{d_{cc}^{p+\alpha}} \leq p\int_G\frac{|f|^{p-1}\cdot|\nabla_G f|}{d_{cc}^{p+\alpha-1}}.$$

By Hölder's inequality,

$$(Q-p-\alpha)\int_G\frac{|f|^p}{d_{cc}^{p+\alpha}} \leq p\left(\int_G\frac{|f|^p}{d_{cc}^{p+\alpha}}\right)^{\frac{p-1}{p}}\left(\int_G\frac{|\nabla_G f|^p}{d_{cc}^{\alpha}}\right)^{\frac{1}{p}}.$$

Canceling and raising both sides to the power p, we obtain (1.4). □

Remark 4.3. We fail to prove that the constant is sharp in Theorem 1.1. The reason is that we do not know whether the radial function $F_\epsilon(d_{cc}) = g_\epsilon(d_{cc})\phi(d_{cc})$ can be approximated by the functions in $C_0^\infty(G)$ since d_{cc} is not differentiable in $\mathbb{Z}\bigcup\mathbb{Z}'$, where

$$g_\varepsilon(d_{cc}) = \begin{cases}\epsilon^{-(Q-p-\alpha)/p}, & d_{cc}\leq\epsilon;\\ d_{cc}^{-(Q-p-\alpha)/p}, & d_{cc}>\epsilon,\end{cases}$$

and $\phi(\cdot) \in C_0^\infty([0,+\infty))$ is a cutoff function satisfying $\phi(t) = 1$ if $0 \leq t \leq 1$ and $\phi(t) = 0$ if $t \geq 2$. Since d_{cc} is a Lipschitz function, so does F_ϵ. Therefore, for all $\varphi \in C_0^\infty(G)$ (see [20], pages 351–352),

$$\begin{aligned}&\int_G X_jF_\epsilon(d_{cc}(x,t))\varphi(x,t)dxdt\\ &= -\int_G F_\epsilon(d_{cc}(x,t))X_j\varphi(x,t)dxdt,\quad j=1,2,\ldots,m,\end{aligned}$$

which means the distribution derivatives $\{X_jF_\epsilon : j = 1, 2, \ldots, m\}$ exist. It is easy to check that F_ϵ is a differentiable function on $\{(x,t) \in G : x \neq 0, z \neq 0, d_{cc}(x,t) \neq \epsilon\}$. So the distribution derivatives

$$\begin{aligned}&X_jF_\epsilon(d_{cc}(x,t)) = F_\epsilon'\cdot X_jd_{cc}(x,t),\\ &(x,t)\in\{(x,t)\in G : x\neq 0, z\neq 0, d_{cc}(x,t)\neq\epsilon\},\end{aligned}$$

for $j = 1, 2, \ldots, m$. We compute

$$\begin{aligned}\int_G \frac{|\nabla_G F_\varepsilon|^p}{d_{cc}^\alpha} &= |\Sigma| \int_0^2 |F_\epsilon'(\rho)|^p \rho^{Q-1-\alpha} d\rho \\ &= |\Sigma| \int_\epsilon^1 |g_\epsilon'(\rho)|^p \rho^{Q-1-\alpha} d\rho + |\Sigma| \int_1^2 |F_\epsilon'(\rho)|^p \rho^{Q-1-\alpha} d\rho \\ &= -\left(\frac{Q-p-\alpha}{p}\right)^p |\Sigma| \ln \epsilon + |\Sigma| \int_1^2 |F_\epsilon'(\rho)|^p \rho^{Q-1-\alpha} d\rho\end{aligned}$$

and

$$\begin{aligned}\int_G \frac{|F_\varepsilon|^p}{d_{cc}^{p+\alpha}} &= |\Sigma| \int_0^2 |F_\epsilon(\rho)|^p \rho^{Q-1-p-\alpha} d\rho \\ &= |\Sigma| \int_0^\epsilon |g_\epsilon(\rho)|^p \rho^{Q-1-p-\alpha} d\rho + |\Sigma| \int_\epsilon^1 |g_\epsilon(\rho)|^p \rho^{Q-1-p-\alpha} d\rho \\ &\quad + |\Sigma| \int_1^2 |F_\epsilon(\rho)|^p \rho^{Q-1-p-\alpha} d\rho \\ &= \frac{|\Sigma|}{Q-p-\alpha} - |\Sigma| \ln \epsilon + |\Sigma| \int_1^2 |F_\epsilon(\rho)|^p \rho^{Q-1-p-\alpha} d\rho,\end{aligned}$$

where $|\Sigma| = \int_\Sigma d\sigma$. Since $\int_1^2 |F_\epsilon'(\rho)|^p \rho^{Q-1-\alpha} d\rho$ and $\int_1^2 |F_\epsilon(\rho)|^p \rho^{Q-1-p-\alpha} d\rho$ are independent of ϵ, we have

$$\lim_{\varepsilon \to 0} \frac{\int_G \frac{|\nabla_G F_\varepsilon|^p}{d_{cc}^\alpha}}{\int_G \frac{|F_\varepsilon|^p}{d_{cc}^{p+\alpha}}} = \left(\frac{Q-p-\alpha}{p}\right)^p.$$

If the function $F_\varepsilon(d_{cc}) \in \overline{C_0^\infty(G)}$, then the constant in Theorem 1.1 is sharp.

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