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CR-SUBMANIFOLDS OF A LORENTZIAN PARA-SASAKIAN MANIFOLD ENDOWED WITH A QUARTER SYMMETRIC METRIC CONNECTION

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Abstract. We define a quarter symmetric metric connection in a Lorentzian para-Sasakian manifold and study CR-submanifolds of a Lorentzian para-Sasakian manifold endowed with a quarter symmetric metric connection. Moreover, we also obtain integrability conditions of the distributions on CR-submanifolds.

1. Introduction

A. Bejancu introduced the notion of CR-submanifolds of a Kaehler manifold in [3]. Later, CR-submanifolds of Sasakian manifold were studied by M. Kobayashi in [9]. K. Motsumoto introduced the idea of Lorentzian para-Sasakian structure and studied its several properties in [10]. B. Prasad in [12], S. Prasad and R. H. Ojha in [13] studied submanifolds of a Lorentzian para-Sasakian manifolds. U. C. De and Anup Kumar Sengupta studied CR-submanifolds of a Lorentzian para-Sasakian manifold in [7]. In this paper we study CR-submanifolds of a Lorentzian para-Sasakian manifold endowed with a quarter symmetric metric connection.

Let ∇ be a linear connection in an n-dimensional differentiable manifold $\bar{M}$. The torsion tensor T and the curvature tensor R of ∇ are given respectively by [4]

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y],$$
$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

The connection ∇ is symmetric if torsion tensor T vanishes, otherwise it is non-symmetric. The connection ∇ is a metric connection if there is a Riemannian metric g in $\bar{M}$ such that $\nabla g = 0$, otherwise it is non-metric. It is well known that a linear connection is symmetric and metric if and only if it is the Levi-Civita connection.

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In [8], S. Golab introduced the idea of quarter symmetric linear connection. A linear connection ∇ is said to be quarter symmetric connection if its torsion tensor T is of the form

$$T(X, Y) = \eta(Y)\phi X - \eta(X)\phi Y,$$

where η is a 1-form. In [2], the author, J. B. Jun and A. Haseeb studied some properties of hypersurfaces of an almost r-paracontact Riemannian manifold with quarter symmetric metric connection. In [1], the author, C. Ozgur and A. Haseeb studied properties of hypersurfaces of an almost r-paracontact Riemannian manifold with quarter symmetric non-metric connection.

Motivated by the studies of authors in [5], [6], and [11] in this paper we study CR-submanifolds of a Lorentzian para-Sasakian manifold endowed with a quarter symmetric metric connection. We discuss integrability of distributions on CR-submanifolds with a quarter symmetric metric connection. We also consider parallel distributions on CR-submanifolds.

This paper is organized as follows: In Section 2, we give a brief introduction of Lorentzian para-Sasakian manifold. In Section 3, we study CR-submanifolds of an LP-Sasakian manifold with quarter symmetric metric connection. We find the necessary conditions that the induced connection on a CR-submanifold of an LP-Sasakian manifold is also a quarter symmetric metric connection. We also discuss the integrability conditions of distributions on CR-submanifolds.

2. LP-Sasakian manifold

Let $\bar{M}$ be a $(2n+1)$-dimensional almost contact metric manifold with a metric tensor g, a tensor field ϕ of type $(1,1)$, a vector field ξ and a 1-form η which satisfy

$$\phi^2 X = X + \eta(X)\xi, \eta(\xi) = -1, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{2.2}$$

$$g(X, \xi) = \eta(X), \tag{2.3}$$

$$g(\phi X, Y) = g(X, \phi Y) = \psi(X, Y) \tag{2.4}$$

for all vector fields X, Y tangent to $\bar{M}$. Such a manifold is termed as Lorentzian para-contact manifold and the structure (ϕ, η, ξ, g) a Lorentzian para-contact structure [10].

Also in a Lorentzian para-contact structure the following relations hold:

$$\phi\xi = 0,\ \eta(\phi X) = 0,\ \text{rank}(\phi) = n - 1.$$

A Lorentzian para-contact manifold $\bar{M}$ is called Lorentzian para-Sasakian (LP-Sasakian) manifold if [10].

$$(\bar{\bar{\nabla}}_X \phi)(Y) = g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi, \tag{2.5}$$

$$\bar{\bar{\nabla}}_X \xi = \phi X \tag{2.6}$$

for all vector fields X, Y tangent to $\bar{M}$, where $\bar{\bar{\nabla}}$ is the Riemannian connection with respect to g.

3. CR-submanifolds of an LP-Sasakian manifold

Definition 3.1. An m-dimensional Riemannian submanifold M of a Lorentzian para-Sasakian manifold $\bar{M}$ is called a CR-submanifold if ξ is tangent to M and there exists on M a pair of distributions $(D, D^{\perp})$ such that

(i) TM orthogonally decomposes as $D \oplus D^{\perp}$,

(ii) the distribution D_x is invariant under ϕ, that is, $\phi D_x \subset D_x$ for each $x \in M$.

(iii) the distribution $D^{\perp}$ is anti invariant under ϕ, that is, $\phi D_x^{\perp}(M) \subset T_x^{\perp}(M)$ where $T_x M$ and $T_x^{\perp} M$ are tangent and normal spaces of M at $x \in M$.

The distribution D (resp. $D^{\perp}$) is called the horizontal (resp. vertical) distribution. The pair $(D, D^{\perp})$ is called ξ-horizontal (resp. ξ-vertical) if $\xi_x \in D_x$ (resp. $\xi_x \in D_x^{\perp}$) for $x \in M$.

Any vector X tangent to M is given by

$$X = PX + QX, \tag{3.1}$$

where PX and QX belong to the distribution D and $D^{\perp}$ respectively. For any vector field N normal to M, we put

$$\phi N = BN + CN, \tag{3.2}$$

where BN (resp. CN) denotes the tangential (resp. normal) component of ϕN.

We remark that owing to the existence of the 1-form η, we can define a quarter symmetric metric connection $\bar{\nabla}$ as

$$\bar{\nabla}_X Y = \bar{\bar{\nabla}}_X Y + \eta(Y)\phi X - g(\phi X, Y)\xi \tag{3.3}$$

such that $\bar{\nabla}_X g = 0$ for any $X, Y \in TM$.

Inserting (3.3) in (2.5), we have

$$(\bar{\nabla}_X \phi)Y = 0, \tag{3.4}$$

$$\bar{\nabla}_X \xi = 2\phi X. \tag{3.5}$$

We denote by g the metric tensor of $\bar{M}$ as well as that induced on M. Let $\bar{\nabla}$ be the quarter symmetric metric connection on $\bar{M}$ and ∇ be the induced connection on M with respect to unit normal N. Then

Theorem 3.2. (i) *If M is ξ-horizontal, $X, Y \in D$ and D is parallel with respect to ∇, then the connection induced on a CR-submanifold of an LP-Sasakian manifold with a quarter symmetric metric connection is also a quarter symmetric metric connection.*

(ii) *If M is ξ-vertical, $X, Y \in D^{\perp}$ and $D^{\perp}$ is parallel with respect to ∇, then the connection induced on a CR-submanifold of an LP-Sasakian manifold* *with a quarter symmetric metric connection is also a quarter symmetric metric connection.*

(iii) *The Gauss formula with respect to the quarter symmetric metric connection is of the form* $\bar{\nabla}_X Y = \nabla_X Y + h(X, Y)$.

Proof. Let ∇ be the induced connection with respect to unit normal N on a CR-submanifold of an LP-Sasakian manifold from quarter symmetric metric connection $\bar{\nabla}$. Then

$$\bar{\nabla}_X Y = \nabla_X Y + m(X, Y), \tag{3.6}$$

where m is a tensor field of type (0, 2) on the CR-submanifolds M. If $\dot{\nabla}$ is the induced connection on CR-submanifold from Riemannian connection $\bar{\bar{\nabla}}$, then

$$\bar{\bar{\nabla}}_X Y = \dot{\nabla}_X Y + h(X, Y), \tag{3.7}$$

where h is a second fundamental form. By the definition of quarter symmetric metric connection

$$\bar{\nabla}_X Y = \bar{\bar{\nabla}}_X Y + \eta(Y)\phi X - g(\phi X, Y)\xi.$$

Now, using above equations we have

$$\nabla_X Y + m(X, Y) = \dot{\nabla}_X Y + h(X, Y) + \eta(Y)\phi X - g(\phi X, Y)\xi.$$

Using (3.1) the above equation can be written as

$$\begin{aligned} &P\nabla_X Y + Q\nabla_X Y + m(X, Y) \\ &= P\dot{\nabla}_X Y + Q\dot{\nabla}_X Y + h(X, Y) + \eta(Y)\phi PX + \eta(Y)\phi QX \\ &\quad - g(\phi X, Y)P\xi - g(\phi X, Y)Q\xi. \end{aligned} \tag{3.8}$$

Comparing tangential and normal components from both sides, we get

$$h(X, Y) = m(X, Y), \tag{3.9}$$

$$P\nabla_X Y = P\dot{\nabla}_X Y + \eta(Y)\phi PX - g(\phi X, Y)P\xi \tag{3.10}$$

and

$$Q\nabla_X Y = Q\dot{\nabla}_X Y + \eta(Y)\phi QX - g(\phi X, Y)Q\xi. \tag{3.11}$$

Using (3.9), the Gauss formula for a CR-submanifold of an LP-Sasakian manifold with quarter symmetric metric connection is

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y). \tag{3.12}$$

This proves (iii). In view of (3.10), if M is ξ-horizontal, $X, Y \in D$ and D is parallel with respect to ∇, then the connection induced on a CR-submanifold of an LP-Sasakian manifold with a quarter symmetric metric connection is also a quarter symmetric metric connection.

Similarly, using (3.11), if M is ξ-vertical, $X, Y \in D^\perp$ and $D^\perp$ is parallel with respect to ∇, then the connection induced on a CR-submanifold of an LP-Sasakian manifold with a quarter symmetric metric connection is also a quarter symmetric metric connection.

Weingarten formula is given by

$$\bar{\nabla}_X N = -A_N X + \nabla^{\perp}_X N + \eta(N)\phi X \tag{3.13}$$

for any $X, Y \in TM,\ N \in T^{\perp}M$, where h (resp. A_N) is the second fundamental form (resp. tensor) of M in $\bar{M}$ and $\nabla^{\perp}$ denotes the operator of the normal connection. Moreover, we have

$$g(h(X,Y),N) = g(A_N X, Y). \tag{3.14}$$

□

Integrability of distributions

Lemma 3.3. *Let M be a CR-submanifold of an LP-Sasakian manifold $\bar{M}$ with quarter symmetric metric connection. Then*

$$P\nabla_X \phi PY - PA_{\phi QY}X = \phi P\nabla_X Y, \tag{3.15}$$

$$Q\nabla_X \phi PY - QA_{\phi QY}X = Bh(X,Y), \tag{3.16}$$

$$h(X,\phi PY) + \nabla^{\perp}_X \phi QY = \phi Q\nabla_X Y + Ch(X,Y) \tag{3.17}$$

for all $X, Y \in TM$.

Proof. By virtue of (3.1), (3.2), (3.4), (3.12), and (3.13), we can easily get

$$\begin{aligned} &P\nabla_X \phi PY + Q\nabla_X \phi PY + h(X,\phi PY) - PA_{\phi QY}X - QA_{\phi QY}X + \nabla^{\perp}_X \phi QY \\ &= \phi P\nabla_X Y + \phi Q\nabla_X Y + Bh(X,Y) + Ch(X,Y). \end{aligned}$$

Equations (3.15)-(3.17) follows by equating horizontal, vertical and normal components. □

Lemma 3.4. *Let M be a ξ-vertical CR-submanifold of an LP-Sasakian manifold $\bar{M}$ with quarter symmetric metric connection. Then*

$$\phi P[Y,Z] = A_{\phi Y}Z - A_{\phi Z}Y$$

for any $Y, Z \in D^{\perp}$.

Proof. By virtue of (3.4), (3.12) and (3.13) we have for $Y, Z \in D^{\perp}$

$$-A_{\phi Z}Y + \nabla^{\perp}_Y \phi Z = \phi(\nabla_Y Z + h(Y,Z)).$$

Using (3.17), we get

$$\phi P\nabla_Y Z = -A_{\phi Z}Y - Bh(Y,Z).$$

Interchanging Y and Z, we have

$$\phi P\nabla_Z Y = -A_{\phi Y}Z - Bh(Z,Y).$$

On subtracting above two equations, we have

$$\phi P[Y,Z] = A_{\phi Y}Z - A_{\phi Z}Y.$$

□

Thus we have:

Theorem 3.5. *Let M be a CR-submanifold of an LP-Sasakian manifold $\bar{M}$ with quarter symmetric metric connection. Then the distribution $D^{\perp}$ is integrable if and only if*

$$A_{\phi Z}Y = A_{\phi Y}Z$$

for all $Y, Z \in D^{\perp}$.

Proposition 3.6. *Let M be a ξ-vertical CR-submanifold of an LP-Sasakian manifold $\bar{M}$ with quarter symmetric metric connection. Then*

$$\phi Ch(X,Y) = Ch(\phi X, Y) = Ch(X, \phi Y) \tag{3.18}$$

for any $X, Y \in D$.

Proof. From (3.16), we have for $X, Y \in D$

$$Q\nabla_X \phi Y = Bh(X,Y). \tag{3.19}$$

Replacing X by ϕX, we have

$$Q\nabla_{\phi X}\phi Y = Bh(\phi X, Y). \tag{3.20}$$

Using (3.19) and (2.1), we get

$$Q\nabla_X Y = g(\phi X, Y)Q\xi + Bh(\phi X, Y). \tag{3.21}$$

Adding (3.20) and (3.21), we have

$$(Q\nabla_X Y + Q\nabla_{\phi X}\phi Y) \in D.$$

Replacing X by ϕX and Y by ϕY in (3.17), we have

$$-h(\phi X, Y) = \phi Q(\nabla_{\phi X}\phi Y) + Ch(\phi X, \phi Y). \tag{3.22}$$

Also interchanging X and Y in (3.17), we have

$$h(\phi X, Y) = \phi Q(\nabla_Y X) + Ch(X, Y). \tag{3.23}$$

Adding (3.22), (3.23) and using (2.1) and $(Q\nabla_X Y + Q\nabla_{\phi X}\phi Y) \in D$, we get

$$Ch(\phi X, Y) = Ch(X, \phi Y).$$

Again from (3.19) and (2.1), we have

$$Q\nabla_X Y = g(X, \phi Y)Q\xi + 2\eta(X)\eta(Y)Q\xi + Bh(X, \phi Y).$$

Using above equation in (3.17), we get

$$Ch(\phi X, Y) = \phi Ch(X, Y)$$

which completes the proposition. □

Theorem 3.7. *Let M be a ξ-vertical CR-submanifold of an LP-Sasakian manifold $\bar{M}$ with quarter symmetric metric connection. Then the distribution D is integrable if and only if $h(X, \phi Y) = h(Y, \phi X)$ for any $Y, Z \in D$.*

Proof. From (3.17), we get

$$h(X,\phi Y)=\phi Q\nabla_X Y+Ch(X,Y)$$

which gives

$$\phi Q[X,Y]=h(X,\phi Y)-h(Y,\phi X).$$

Thus D is integrable if and only if $h(X,\phi Y)=h(Y,\phi X)$. □

Proposition 3.8. *Let M be a ξ-vertical CR-submanifold of a Lorentzian para-Sasakian manifold $\bar{M}$ with quarter symmetric metric connection. Then the distribution $D^{\perp}$ is parallel with respect to the connection ∇ on M, if and only if, $A_N X\in D^{\perp}$ for each $X\in D^{\perp}$ and $N\in TM^{\perp}$.*

Proof. Let $Y,X\in D^{\perp}$. Then using (3.12) and (3.13), we have

$$-A_{\phi Y}X+\nabla^{\perp}_X\phi Y=\phi(\nabla_X Y+h(X,Y)).$$

Taking inner product with $Z\in D$, we have

$$-g(A_{\phi Y}X,Z)=g(\nabla_X Y,\phi Z).$$

Therefore, $\nabla_X Y=0$ if and only if $A_{\phi Y}X\in D^{\perp}$ for all $X\in D^{\perp}$. From which our assertion follows. □

Definition 3.9. A CR-submanifold M of an LP-Sasakian manifold $\bar{M}$ with quarter symmetric metric connection is said to be totally geodesic if $h(X,Y)=0$ for $X\in D$ and $Y\in D^{\perp}$.

It follows immediately that a CR-submanifold is mixed totally geodesic if and only if $A_N X\in D$ for each $X\in D$ and $N\in T^{\perp}M$.

Let $X\in D$ and $Y\in\phi D^{\perp}$. For a mixed totally geodesic ξ-horizontal CR-submanifold M of an LP-Sasakian manifold $\bar{M}$ with quarter symmetric metric connection. Then from (3.4) we have

$$(\bar{\nabla}_X\phi)N=0.$$

Since $\bar{\nabla}_X\phi N=(\bar{\nabla}_X\phi)N+\phi(\bar{\nabla}_X N)$ so that $\bar{\nabla}_X\phi N=\phi(\bar{\nabla}_X N)$.
Using (3.12) and (3.13) in above equation, we have

$$\nabla_X(\phi N)=-\phi A_N X+\phi\nabla^{\perp}_X N,$$

as $\phi A_N X\in D$, so $\nabla_X\phi N\in D$ if and only if $\phi\nabla^{\perp}_X N=0$.

Thus we have the following theorem.

Theorem 3.10. *Let M be a mixed totally ξ-horizontal CR-submanifold of an LP-Sasakian manifold $\bar{M}$ with quarter symmetric metric connection. Then the normal section $N\in\phi D^{\perp}$ is D-parallel if and only if $\nabla_X(\phi N)\in D$ for $X\in D$.*

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